LINEAR ORDERS, DISCRETE, DENSE, AND CONTINUOUS

THEOREM 1: If <A,<> is a linear order in which every cut determines a jump (= discreet and continuous), then either <A,<> is a finite linear order, or <A,<> is isomorphic to one of $\mathbf{Z}$, $\mathbf{Z}^+$ or $\mathbf{Z}^-$**.**

PROOF: Let $a,b \in A$. Since every cut in A determines a jump, there is no transition or gap between a and b. This means that there are finitely many elements between a and b. This means that A is at most countable.

Pick any element a of A. $\{b \in A: b \leq a\}$ is finite or countable.
If $\{b \in A: b \leq a\}$is finite, map the minimum onto 1, its direct successor onto 2, etc. Since between any element of A and the minimum there are finitely many elements, every element of A comes into the mapping, and the result is a bijection between A and $\mathbf{Z}^+$ (if A is countable) or a finite initial stretch of $\mathbf{Z}^+$ (if A is finite)**.**
If $\{b \in A: b \leq a\}$ is countable, look at $\{b \in A: a \leq b\}$. If that is finite, map the maximum on -1, its direct predecessor onto -2, etc. Again, since any element of A is a finite predecessor of this maximum, every element of A comes into the mapping. The result will be a bijection between A and $\mathbf{Z}^-$**.**
If $\{b \in A: b \leq a\}$ is countable and $\{b \in A: a \leq b\}$ is also countable, map a onto 0, its predecessors onto -1,-2, etc, and its successors onto 1,2,..etc. Again, each element of A is a finite predecessor or a finite successor of a, hence it comes in the mapping. The result is a bijection between A and $\mathbf{Z}$. This exhausts all the possibilities. ◄

This means then that $\mathbf{Z}$ exhaustively covers the cases of linear orders with only jumps. Since this includes all finite linear orders, linear orders without jumps (i.e. dense linear orders) are by necessity infinite. The following theorem says that the countable case is exhaustively covered by $\mathbf{Q}$:

THEOREM 2: **CANTOR'S THEOREM**

Consider the following four substructures of the rational numbers <**Q**,<> (with the order the obvious restriction):
<**[0,1]**,<>, <**[0,1)**,<>, <**(0,1]**,<>, <**(0,1)**,<>.
Every countable dense linear order is isomorphic to one of these four structures.

i.e.: The countable dense linear orders with end points are isomorphic to **[0,1]**.
The countable dense linear orders without end points are isomorphic to **(0,1)**.
The countable dense linear orders with a minimum but no maximum are isomorphic to **[0,1)**.
The countable dense linear orders with a maximum but no minimum are isomorphic to **(0,1]**.

PROOF:
Let $<A,<>$ and $<B,<>$ be countable dense linear orders without endpoints.
Let $A = a_0,a_1,...,a_n,...$ be an enumeration of A, and $B = b_0,b_1,...,b_n,...$ be an emumeration of B. Since A and B are countable, such enumerations exist.

We define a sequence $f_0,f_1,....f_n,...$ as follows:

1. $f_0 = \{<a_0,b_0>\}$

Note that f_0 is, trivially, a finite one-one function that preserves the order.

2. If n>0 and n is odd, then $f_n = f_{n-1} \cup \{<a,b>\}$ where:

  2a: a is the first element in enumeration *A* such that $a \notin \mathrm{dom}(f_{n-1})$.

Since $\mathrm{dom}(f_{n-1})$ is finite, and A countable there always is a first element in *A* not in $\mathrm{dom}(f_{n-1})$.

  2b: 2b1: If for every $x \in \mathrm{dom}(f_{n-1})$: $a < x$, then b is the first element in enumeration *B* such that $b\notin \mathrm{ran}(f_{n-1})$ and for every $y \in \mathrm{ran}(f_{n-1})$: $b < y$.

Since $\mathrm{ran}(f_{n-1})$ is finite, and B countable, and since B has no endpoints, there always is a first element in *B* not in $\mathrm{ran}(f_{n-1})$ and before every element in $\mathrm{ran}(f_{n-1})$.

  2b: 2b2: If for every $x \in \mathrm{dom}(f_{n-1})$: $x < a$, then b is the first element in enumeration *B* such that $b\notin \mathrm{ran}(f_{n-1})$ and for every $y \in \mathrm{ran}(f_{n-1})$: $y < b$.

Since $\mathrm{ran}(f_{n-1})$ is finite, and B countable, and since B has no endpoints, there always is a first element in *B* not in $\mathrm{ran}(f_{n-1})$ and after every element in $\mathrm{ran}(f_{n-1})$.

Since A is linear, if $a \notin \mathrm{dom}(f_{n-1})$ and a is not before every x in $\mathrm{dom}(f_{n-1})$ and not after every x in $\mathrm{dom}(f_{n-1})$, then for some $x_1,x_2 \in \mathrm{dom}(f_{n-1})$: $x_1 < a < x_2$. Since $\mathrm{dom}(f_{n-1})$ is finite this means that for some $x_1,x_2 \in \mathrm{dom}(f_{n-1})$: $x_1 < a < x_2$ and for no $x_3 \in \mathrm{dom}(f_{n-1})$: $x_1 < x_3 < a$ and for no $x_3 \in \mathrm{dom}(f_{n-1})$: $a < x_3 < x_2$.

  2b: 2b3: If for some $x_1,x_2 \in \mathrm{dom}(f_{n-1})$: $x_1 < a < x_2$ and for no $x_3 \in \mathrm{dom}(f_{n-1})$: $x_1 < x_3 < a$ and for no $x_3 \in \mathrm{dom}(f_{n-1})$: $a < x_3 < x_2$, then b is the first element in *B* such that $b \notin \mathrm{ran}(f_{n-1})$ and $f_{n-1}(x_1) < b < f_{n-1}(x_2)$.

Since $\mathrm{ran}(f_{n-1})$ is finite, and B countable, and since B is dense, there always is a first element in *B* not in $\mathrm{ran}(f_{n-1})$ and between $f_{n-1}(x_1)$ and $f_{n-1}(x_2)$.

Note that, by the construction, if f_{n-1} is a finite one-one function that preserves the order, then so is f_n. We add to f_{n-1} one pair $<a,b>$, where, by the construction, $<a,b>$ is well defined, $a \notin \mathrm{dom}(f_{n-1})$, $b \notin \mathrm{ran}(f_{n-1})$. This means that, by the construction, if f_{n-1} is a function, so is f_n; if f_{n-1} is one-one, so is f_n, and if f_{n-1} preserves the order, so does f_n.

3. If $n>0$ and n is even, then $f_n = f_{n-1} \cup \{<a,b>\}$ where:

3a: b is the first element in enumeration *B* such that $b \notin ran(f_{n-1})$.

Since $ran(f_{n-1})$ is finite, and B countable there always is a first element in*B* not in $ran(f_{n-1})$.

3b: 3b1: If for every $y \in ran(f_{n-1})$: $b < y$, then a is the first element in enumeration *A* such that $a \notin dom(f_{n-1})$ and for every $x \in dom(f_{n-1})$: $a < x$.

Since $dom(f_{n-1})$ is finite, and A countable, and since A has no endpoints, there always is a first element in *A* not in $dom(f_{n-1})$ and before every element in $dom(f_{n-1})$.

3b: 3b2: If for every $y \in ran(f_{n-1})$: $y < b$, then a is the first element in enumeration *A* such that $a \notin dom(f_{n-1})$ and for every $x \in dom(f_{n-1})$: $x < a$.

Since $dom(f_{n-1})$ is finite, and A countable, and since A has no endpoints, there always is a first element in *A* not in $dom(f_{n-1})$ and after every element in $dom(f_{n-1})$.

Since B is linear, if $b \notin ran(f_{n-1})$ and b is not before every y in $ran(f_{n-1})$ and not after every y in $ran(f_{n-1})$, then for some $y_1,y_2 \in ran(f_{n-1})$: $y_1 < b < y_2$. Since $ran(f_{n-1})$ is finite this means that for some $y_1,y_2 \in ran(f_{n-1})$: $y_1 < b < y_2$ and for no $y_3 \in ran(f_{n-1})$: $y_1 < y_3 < b$ and for no $y_3 \in ran(f_{n-1})$: $b < y_3 < y_2$.

3b: 3b3: If for some $y_1,y_2 \in ran(f_{n-1})$: $y_1 < b < y_2$ and for no $y_3 \in ran(f_{n-1})$: $y_1 < y_3 < b$ and for no $y_3 \in ran(f_{n-1})$: $b < y_3 < y_2$, then a is the first element in *A* such that $a \notin dom(f_{n-1})$ and $f_{n-1}^{-1}(y_1) < a < f_{n-1}^{-1}(y_2)$.

Since $dom(f_{n-1})$ is finite, and A countable, and since A is dense, there always is a first element in *A* not in $dom(f_{n-1})$ and between $f_{n-1}^{-1}(y_1)$ and $f_{n-1}^{-1}(y_2)$.

Note that, by the construction, if f_{n-1} is a finite one-one function that preserves the order, then so is f_n. We add to f_{n-1} one pair $<a,b>$, where, by the construction, $<a,b>$ is well defined, $a \notin dom(f_{n-1})$, $b \notin ran(f_{n-1})$. This means that, by the construction, if f_{n-1} is a function, so is f_n; if f_{n-1} is one-one, so is f_n, and if f_{n-1} preserves the order, so does f_n.

FACT 1: For every n: f_n is a finite one-one function which preserves the order.

PROOF: The induction steps are given in the construction.

FACT 2: For every $a \in A$ there is an n such that $a \in dom(f_n)$.
For every $b \in B$ there is an n such that $b \in ran(f_n)$.

PROOF: This follows from the zig-zag construction.
If $a \in A$ and for some k, $a \notin dom(f_k)$, then for some m, a is the m-th element of *A* not in $dom(f_k)$. By the construction, this means that $a \in dom(f_{k+2m})$, either because it is chosen as the argument for some b before that, or, if not, because at that stage it is the first element in the enumeration *A* which isn't in the domain of the previous function.
The very same argument applies to any $b \in B$.

Now we define:

$$f = \cup\{f_n: n \geq 0\}$$

FACT 3: f is an isomorphism between <A,<> and <B,<>.

PROOF:
-f is, of course, by definition a relation between A and B.
-Since each f_{n+1} is a function extending f_n, f is a function.
-By definition, $dom(f) = \cup\{dom(f_n): n \geq 0\}$. By fact 2, this is A.
Thus f is a function from A into B.
-By definition, $ran(f) = \cup\{ran(f_n): n \geq 0\}$. By fact 2, this is B.
Thus f is a function from A onto B.
-Since each f_n is a one-one function, f is a one-one function. If $a_1,a_2 \in dom(f)$ and $f(a_1)=f(a_2)$, then for some n: $a_1,a_2 \in dom(f_n)$, and hence $f_n(a_1)=f_n(a_2)$,. But then, since f_n is one-one, $a_1=a_2$.
Thus f is a bijection between A and B.
-Since each f_n preserves the order, f preserves the order. If $a_1 < a_2$, then, since for some n, $a_1,a_2 \in dom(f_n)$, by construction $f_n(a_1) < f_n(a_2)$. But then $f(a_1) < f(a_2)$. ◄

We have now proved that all countable dense linear orders without endpoints are isomorphic, and hence they are indeed isomorphic to **(0,1)**.

Now let, <A,<> be a countable dense linear order with a begin point a. <A-{0},<> is a dense linear order without endpoints, and hence isomorphic to **(0,1)**. Let f be the isomorphism. Obviously, $f \cup \{\langle a,0\rangle\}$ is an isomorhism between <A,<> and **[0,1).** Similarly, if <A,<> is a countable dense linear order with an endpoint b, we extend the isomorphism between A-{b} and **(0,1)** to an isomorphism between <A,<> and **(0,1]** by adding <b,1>, and in the same way, we get an isomorphism between <A,<> with begin point a and end point b and **[0,1]** by adding <a,0> and <b,1>.
Since, obviously, these are all the countable dense linear orders the theorem follows. ◄

We have so far dealt with linear orders with only jumps and with countable linear orders without jumps.
What about linear orders with only gaps, and linear orders with only transitions.
Concerning the first, it is easy to see that they do not exist. Namely, let A be any linear order and let $a \in A$, but not an endpoint. Then
$\langle\{b \in A: b \leq a\},\{b \in A: a < b\}\rangle$ determines a jump or a transition. This means that it can't be the case that every cut in A determines a gap.

This means that for linear orders without jumps (i.e. dense linear orders) we can find only two possible kinds: with gaps and transitions, or with only transitions.
The countable cases are cases with gaps and transitions. Gaps in **Q** can be shown by looking at irrational numbers: let $r \in \mathbf{R} - \mathbf{Q}$:
$\mathbf{Q} = \{q \in \mathbf{Q}: q < r\} \cup \{q \in \mathbf{Q}: r < q\}$, and $\langle\{q \in \mathbf{Q}: q < r\},\{q \in \mathbf{Q}: r < q\}\rangle$ is a cut in **Q** which determines a gap.

It follows that linear orders in which every cut determines a transition can only be non-countable. Intuitively we get the set of real numbers **R** by for every cut in **Q** that determines a gap, filling up the gap with an irrational number. This turns the gap into a transition (because you need to extend either T_1 or T_2 of the original cut which determined a gap with the element added to get a partition). As it turns out, there is a way of doing this, and in fact only one way of doing this.

Let <B,<> be a dense linear order without endpoints, and let $A \subseteq B$.

A lies **dense in** B iff for every $b_1,b_2 \in B$: there is an $a \in A$: $b_1 < a < b_2$.

THEOREM 3: Let <B,<> be a linear order in which every cut determines a transition, and A a **countable** subset of B which lies dense in B. Then <B,<> is isomorphic to one of the **real** intervals **(0,1), [0,1), (0,1],[0,1].**

PROOF: we won't prove this here, but the proof is analogous to things we will later prove for completions of Boolean algebras. If you are a linear order in which every cut determines a transition and **Q** lies dense in you, then every transition can be reconstructed as the bounds of a cut of rational intervals, and you are merely the result of adding those bounds only where they are lacking (when there are gaps). Such a structure is called a

completion of the structure <**Q**,<>. Since each incomplete structure has (up to isomorphism) one and only completion, and the completions of isomorphic incomplete structures are isomorphic, the result follows: The **real** intervals **(0,1), [0,1), (0,1], [0,1]** are the four completions of the **rational** intervals **(0,1), [0,1), (0,1],[0,1].**

Of course, since **R** is a linear order in which every cut determines a transition, and **R** has no endpoints and **Q** lies dense in **R, R** is isomorphic to the real interval **(0,1)**.
Not every linear order in which every cut determines a transition is isomorphic to one of these four. There must be a countable subset which is dense in in.