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Social aggregators

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Abstract. This paper proposes a general framework for analyzing a class of functions called social aggregators, which map profiles of linear orders to a set of binary relations. This class of aggregators includes aggregators that yield a preference relation (social welfare functions) and those which yield a choice of an alternative (social choice functions). Equipped with this framework, I identify a property called Preference Reversal (*PR*) such that any Pareto efficient aggregator having this property must be dictatorial. This allows me to state a general impossibility theorem, which includes Arrow's Theorem and the Gibbard Satterthwaite Theorem as two special examples. Furthermore, I show that monotonicity and *IIA* are closely linked, by demonstrating that both are actually special cases of *PR* in specific environments.

1 Introduction

The theory of social choice and welfare has extensively studied the aggregation of individual preferences or rankings. Two seminal results in this line of inquiry are Arrow's Theorem (Arrow [1]) and the Gibbard-Satterthwaite Theorem (Gibbard [9] and Satterthwaite [14], henceforth, *GS*). These two theorems investigate two particular types of "social aggregators": A social welfare function (*SWF*), which aggregates "private" rankings into a single

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"social" preference relation (Arrow), and a social choice function (*SCF*), which aggregates "private" rankings into a single "social" choice (*GS*). Arrow and *GS* introduce two separate frameworks, each geared at the particular aggregator the authors consider, and two separate axioms, monotonicity and *IIA*, establishing the following result: Any Pareto efficient *SCF* satisfying monotonicity must be dictatorial, and any Pareto efficient *SWF* satisfying *IIA* must be dictatorial as well.

Given the results of Arrow and *GS*, the question arises whether other types of social aggregators with desirable properties must also be dictatorial. Individual rankings of alternatives can be aggregated in several different ways. For example, given a list of teachers' ranking of students according to their ability, we may be interested in a "fair" allocation of the students into two universities, where by "fair" we mean that the students of one university will not be ranked higher than the students in the other universities according to the majority of the teachers. Such an aggregator does not yield a preference relation or a set of alternatives, and therefore does not fit into the frameworks of Arrow or *GS*. To analyze such non-standard social aggregators one must use a framework which accommodates a broad class of aggregators.

This paper proposes a general framework for analyzing a class of social aggregators, which includes *SCF*'s and *SWF*'s. Equipped with this framework, I identify a property called Preference Reversal (*PR*) such that any Pareto efficient aggregator having this property must be dictatorial. This allows me to state a general impossibility theorem, which includes Arrow's Theorem and the *GS* Theorem as two special examples. Furthermore, I show that monotonicity and *IIA* are closely linked, by demonstrating that both are actually special cases of *PR* in specific environments.

The first step towards a unified approach to aggregating individual rankings is to speak of a single social aggregator (instead of one function which chooses an alternative and one function which chooses a preference ordering). In this paper a social aggregator assigns a *binary relation* to every profile of rankings or preferences. Thus, a function which assigns a single alternative to every preference profile is equivalent to a social aggregator whose range consists of binary relations having the following property: For every profile there is a single alternative a that relates to every other alternative, but no alternative distinct from a relates to any other alternative (I denote this class of relations by $\mathbf{R}_0$). This binary relation can be interpreted as follows: a relates to b if and only if a is the single chosen alternative. Similarly, a function which assigns a preference ordering for every preference profile is equivalent to a binary relation which satisfies completeness and transitivity.

Given that we can speak of a single aggregator for both theorems, the next step is to show that monotonicity and *IIA* can be replaced with a single property: Preference Reversal (*PR*). This is shown in the two main results of this paper:

1. If there are at least three alternatives, then a monotonic social aggregator satisfies *PR* if the range of binary relations is $\mathbf{R}_0$.
2. If there are at least three alternatives, then for binary relation satisfying existence of a first best and acyclicality, any social aggregator that satisfies Pareto efficiency and *IIA* must also satisfy *PR*.

From the above results follows a single impossibility theorem: If there are at least three alternatives, then for binary relations satisfying acyclicality and existence of a first best, any social aggregator which satisfies Pareto efficiency and *PR* must be dictatorial. This theorem together with the two results quoted above imply that Arrow's Theorem and the GS Theorem are both special cases of a single theorem. Note also that this single theorem relies on a social choice function which maps preference profiles to binary relations which need not be complete or asymmetric. Thus, it may be possible to show that impossibility results for correspondences (see Barberá et al. [5] and Benoît [6]) are also special cases of this one theorem.

2 Related literature

Ever since the two theorems of Arrow and *GS* have been written there has been a growing feeling in the profession that these two theorems are closely connected. Early attempts to connect the two theorems relied on Muller-Satterthwaite [11] who showed that strongly monotonic social choice functions must be dictatorial (see Moulin [10] and Peleg [12]).

The last couple of years have seen a revival of interest in the two classical theorems of Arrow and *GS*. The recent comeback of Arrow and *GS* was largely inspired by Geanakoplos [8], who found an ingenious procedure to simplify the proof of Arrow's Theorem. Roughly speaking, Geanakoplos's insight was that a dictator can be identified by making a series of changes in a profile of identical preferences. Following Geanakoplos [8], Benoît [7] showed that Geanakoplos's procedure can be adapted to provide a simple and elegant proof of *GS*. Most recently Reny [13] demonstrated that Geanakoplos's idea can be used to construct a single step-by-step procedure with which one can prove both theorems. Our proof of Theorem 1 relies on Reny's adaptation of Geanakoplos's procedure.

Another line of recent research, which includes Sen [15] and Svensson [16], uses a different approach to provide a simple direct proof of *GS*. This approach, which builds on a technique that was introduced in Barberà [3] and in Barberá and Peleg [4], relies on induction on the number of individuals.

Despite the works cited above there still remains a gap between the two theorems in the sense that the two are treated separately. That is, all previous works have used *two separate frameworks* to prove *two separate theorems*: (1) If a function that assigns a single alternative to a profile of preferences (a social choice function, *SCF*) satisfies monotonicity and Pareto efficiency, then this function must be dictatorial (Gibbard-Satterthwaite), and (2) If a function that assigns a preference ordering to a profile of preferences (a social welfare function, *SWF*) satisfies Independence of Irrelevant Alternatives (*IIA*) and Pareto efficiency, then this function must be dictatorial (Arrow). Thus, although Reny uses the same method of proof for both theorems, each theorem is based on two separate sets of definitions and two distinct pairs of properties: Pareto efficiency and monotonicity for Arrow and Pareto efficiency and *IIA* for *GS*.

An exception to the above is a recent paper by Barberà [2]. This paper proves a general theorem on preference aggregation, which implies, as corollaries, the two theorems of Arrow and *GS*, as well as other well known results in social choice theory. In that sense, the paper by Barberà emphasizes, as my paper does, the fact there is an underlying meta-theorem from which Arrow and *GS* are obtained as special cases. However, the approach taken by Barberà is completely different than the one that I take in this paper. In particular, Barberà does not rely on the Geanakoplos procedure. Instead, Barberà builds on his earlier works cited above to show that if one agent can affect the outcome of the aggregation process "locally", i.e., at some profile, then he must be able to affect the outcome "globally", i.e., at any profile. For further details, the reader is encouraged to consult this paper.

3 The framework

Let A denote a finite set of at least three alternatives. Consider a set N of $n \geq 3$ individuals. Let $\succ = (\succ_1, \ldots, \succ_n)$, a profile of n strict linear orders on A, and let P denote the set of all such profiles. We interpret $\succ$ to be the profile of preferences of the members of N. Let $\mathbf{R}$ denote a set of binary relations on A. Note that $\mathbf{R}$ need not be complete. An element in $\mathbf{R}$ will be denoted by R. Given some $R \in \mathbf{R}$ and $a, b \in A$ the notation aRb means that a relates to b according to the binary relation R. A function $F : P \to \mathbf{R}$ will be called a social aggregator. Given a pair of alternatives $a, b \in A$ and a preference profile $\succ \in P$ I will write $aF(\succ)b$ for "a relates to b when the preference profile is $\succ$", and $a\urcorner F(\succ)b$ for "a does not relate to b when the preference profile is $\succ$". I will refer to the binary relation $F(\succ)$ between any two alternatives a and b as the "social relation" between those two alternatives.

3.1 Properties of binary relations

Special examples of social aggregators can be obtained by imposing restrictions on the set of binary relations $\mathbf{R}$. Consider the following properties of binary relation sets:

(1) *Acyclicality* (*AC*): For all $R \in \mathbf{R}$ and for every three alternatives a, b and c in A, if aRb and $c\urcorner Rb$, then $c\urcorner Ra$.
(2) *Completeness* (*C*): For all $R \in \mathbf{R}$ and for every pair of alternatives $a, b \in A$, either aRb or bRa or both.

(3) *Existence of a best alternative* (*BA*): For all $R \in \mathbf{R}$ there exists an alternative $a \in A$ such that aRb for every $b \in A \backslash \{a\}$.

To see the relation between social aggregators and *SCF* and *SWF* we need to specify the domain $\mathbf{R}$. If $\mathbf{R}$ satisfies *AC*, *BA* and *C*, then the social aggregator $F : P \to R$ is equivalent to a *SWF*. Consider next the domain $\mathbf{R}_0$ consisting of the binary relations $R_a = \{(a, b) : b \in A\}$. That is, at R_a, a is the unique best alternative and the other alternatives are not related. Note that $\mathbf{R}_0$ satisfies *BA* and *AC*. The social aggregator $F : P \to \mathbf{R}_0$ is equivalent to a *SCF*. Thus, the class of aggregators which I consider includes the two types of aggregators studied by Arrow and *GS*.

3.2 Properties of aggregators

Social aggregators may have many different properties. Most of the literature has concentrated on the following properties:

- *PAR* (Pareto efficiency) - For every pair of alternatives a and b in A, if every individual i satisfies $a \succ_i b$, then either $aF(\succ)b$ and $b\neg F(\succ)a$, or a and b are not related according to $F(\succ)$.
- *IIA* (Independence of Irrelevant Alternatives) - If whenever the ranking of a versus b is unchanged for each $i = 1, \ldots, n$ when individual i's ranking changes from $\succ_i$ to $\succ_i'$, then the relation of a versus b is the same according to both $F(\succ)$ and $F(\succ')$.
- *MON* (Monotonicity)- If a pair of profiles $\succ, \succ' \in P$ satisfies $aF(\succ)x$ for all $x \in A \backslash \{a\}$, and $a \succ_i' b$ if $a \succ_i b$ for every individual i and for every alternative $b \in A$, then $aF(\succ')x$ for all $x \in A \backslash \{a\}$.
- *D* (dictatorship) - $\exists i \in N$ such that $\forall \succ \in P$ and $\forall a, b \in A$, $b\neg F(\succ)a$ whenever $a \succ_i b$.

There is some sense in which a society may be interested in using aggregators that satisfy the first three properties, but not the fourth one. Since these properties have been discussed extensively in the literature I will not discuss their interpretations here.

Note that our definition of dictatorship differs from the standard definition when we do not restrict the domain $\mathbf{R}$ and the class of aggregators. However, if we restrict $\mathbf{R}$ and $F : P \to \mathbf{R}$ such that the social aggregator becomes equivalent to a *SCF* or to a *SWF*, then the two definitions coincide. To see why, consider first the standard definition of a dictatorial *SCF*: a *SCF* is said to be dictatorial if there is an individual i such that for any $\succ \in P$, an alternative a is chosen if and only if $a \succ_i x$ for every $x \in A \backslash \{a\}$. Recall that for $\mathbf{R} = \mathbf{R}_0$, a social aggregator $F : P \to \mathbf{R}_0$ is equivalent to a *SCF*. We shall say that $F : P \to \mathbf{R}_0$ satisfies D^{SCF} if $\exists i \in N$ such that $\forall \succ \in P$ we have $F(\succ) = R_a$ if and only if $a \succ_i x$ for every $x \in A \backslash \{a\}$.

Lemma 1. *A social aggregator $F : P \to \mathbf{R}_0$ satisfies D^{SCF} if and only if it satisfies D.*

Proof. Clearly, if F satisfies D^{SCF}, then it also satisfies D. Assume next that F satisfies D and let i be the dictator. Consider first some $\succ \in P$ for which $a \succ_i x$ for every $x \in A \backslash \{a\}$. From our choice of domain there must be some $y \in A$ such that $F(\succ) = R_y$. By dictatorship, no $x \in A \backslash \{a\}$ can socially relate to a. Therefore, $F(\succ) = R_a$. Consider next some $\succ \in P$ for which $F(\succ) = R_a$. By D, individual i cannot rank b above a. Therefore, $a \succ_i b$. ■

Consider next the standard definition of a dictatorial *SWF*: A *SWF* is dictatorial if there is an individual i such that $\forall \succ \in P$ and $\forall a, b \in A$, the *SWF* ranks a above b whenever $a \succ_i b$. Recall that a social aggregator $F : P \to \mathbf{R}$ is equivalent to a *SWF* whenever the domain $\mathbf{R}$ satisfies AC, BA and C. The aggregator F is said to satisfy D^{SWF} if there is an individual $i \in N$ such that for every $\succ \in P$ and for every $a, b \in A$, $aF(\succ)b$ and $b \neg F(\succ)a$ whenever $a \succ_i b$.

3.3 Preference reversal

Consider two preference profiles $\succ, \succ'$ in P. Suppose a social choice function F^C chooses the alternative a for $\succ$ but it chooses b for $\succ'$. Suppose also that for $\succ$ the social welfare function F^W ranks a above b, but for $\succ'$ it reverses the ranking. It seems reasonable to conclude that at least one individual raises b above a when we move from $\succ$ to $\succ'$:

$$\exists i \text{ such that } a \succ_i b \text{ and } b \succ'_i a \tag{$*$}$$

However, even if we assume that F^C is monotonic and F^W satisfies *IIA*, condition (*) does not *immediately* follow: One must provide a proof that indeed this condition holds. I therefore introduce a property of social aggregators called Preference Reversal, such that if F^C and F^W were to satisfy this property, then condition (*) would hold.

Definition 1. *A social aggregator F satisfies Preference Reversal* (*PR*) *if for every pair of alternatives a and b in A, if $aF(\succ)b$, $b \neg F(\succ)a$ but $bF(\succ')a$, then there must be an individual i that satisfies $a \succ_i b$ and $b \succ'_i a$.*

Preference Reversal means that if the "social relation" between any two alternatives has been reversed, then someone must have exhibited the same reversal in his preferences. Thus, it is a property which is relatively easy to describe in words. In the context of a social choice (where F is a *SCF*), *PR* means that whenever we replace one alternative with another, then someone must have changed his preferences by preferring the latter to the former. In the context of a social preference relation (where F is a *SWF*), PR means that whenever the social preference between a pair of alternatives has been reversed, then there must be someone who exhibited the same preference reversal.

4 A single theorem

This section presents the main results which imply that Arrow's Theorem and the *GS* Theorem are both special cases of a single theorem.

Proposition 1. *A monotonic social aggregator $F : P \to \mathbf{R}_0$ satisfies PR.*

Proof. Consider some monotonic social aggregator $F : P \to \mathbf{R}_0$. Let $\succ, \succ' \in P$ be a pair of preference profiles with the following property: There is a pair of distinct alternatives, a and b, that satisfy $aF(\succ)x$ for all $x \in A \backslash \{a\}$ and $bF(\succ')y$ for all $y \in A \backslash \{b\}$. Assume the following:

$$\nexists\, i \in N \text{ satisfying } a \succ_i b \text{ and } b \succ_i' a \qquad (A)$$

Consider a profile $\succ'' \in P$ with the property that for all i,

If $a \succ_i b$, then $a \succ_i'' b \succ_i'' x$ for all $x \in A \backslash \{a, b\}$
If $b \succ_i a$, then $b \succ_i'' x$ and $a \succ_i'' x$ for all $x \in A \backslash \{a, b\}$ and $\succ_i''|_{\{a,b\}} = \succ_i'|_{\{a,b\}}$

Note that $\succ''$ is well defined since by (A), it is never the case that $a \succ_i b$ and $b \succ_i' a$ for some $i \in N$.

When we move from $\succ$ to $\succ''$ none of the individuals lower a in their ranking. Thus, by *MON*, $aF(\succ'')x$ for all $x \in A \backslash \{a\}$. Note also that when we move from $\succ'$ to $\succ''$none of the individuals lower b in their ranking. Thus, by *MON*, $bF(\succ'')x$ for all $x \in A \backslash \{b\}$. It follows that $aF(\succ'')b$ and $bF(\succ'')a$, in contradiction to the definition of $\mathbf{R}_0$. ■

Proposition 2. *If* **R** *satisfies AC, BA and C, then any social aggregator that satisfies* PAR *and IIA also satisfies PR.*

Proof. Let **R** satisfy *AC*, *BA* and *C*. Let F be a social aggregator that satisfies *PAR* and *IIA*. Assume F violates *PR*. Then there exist a pair of profiles $\succ$ and $\succ^*$and a pair of elements a and b that satisfy the following:

(P1) $aF(\succ)b$ and $b \neg F(\succ)a$
(P2) $bF(\succ^*)a$
(P3) $\nexists\, i \in N$ who satisfies $a \succ_i b$ and $b \succ_i^* a$

If the social relation between a and b changes when the preference profile changes from $\succ$ to $\succ^*$, then by *IIA* there must be at least one individual $j \in N$ who changes his ranking of a versus b when his preference relation changes from $\succ_j$ to $\succ_j^*$. By (P3), j must satisfy:

$$b \succ_j a \text{ but } a \succ_j^* b \qquad (1)$$

If $aF(\succ)b$ but $b \succ_j a$, then by *PAR* there must be some other individual k who satisfies $a \succ_k b$. By (P3), this individual must satisfy $a \succ_k^* b$. Thus,

$$a \succ_k b \text{ and } a \succ_k^* b \qquad (2)$$

As both j and k rank a above b in the profile $\succ^*$, then from *PAR* in order for $bF(\succ^*)a$ there must be yet another individual $l \notin \{j, k\}$ who satisfies $b \succ_l^* a$. By (P3) this individual must satisfy $b \succ_l a$. Thus,

$$b \succ_l a \text{ and } b \succ_l^* a \qquad (3)$$

From (3)–(5) we can represent the relative rankings of a versus b by individuals j, k and l according to $\succ$ and $\succ^*$:

$$\begin{array}{cccccc} \succ_j & \succ_k & \succ_\ell & \succ_j^* & \succ_k^* & \succ_\ell^* \\ b & a & b & a & a & b \\ a & b & a & b & b & a \\ aF(\succ)b & \text{and} & b\urcorner F(\succ)a & & bF(\succ^* a) & \end{array}$$

Since A has at least three elements, $\exists c \in A\backslash\{a,b\}$. By *IIA*, as long as no individual changes his ranking of a versus b, the social ranking of those two elements remains the same regardless of how individuals change their ranking of a versus c and b versus c. Consider then the profile $\succ^{**}$ in which every individual's ranking of a versus b is exactly the same as in $\succ^*$. The only possible difference between $\succ^{**}$ and $\succ^*$ is that $\succ^{**}$ satisfies the following properties:

$(P^{**}1)$ If $a \succ_i^* b$ and $a \succ_i b$ then $a \succ_i^{**} c \succ_i^{**} b$
$(P^{**}2)$ If $a \succ_i^* b$ but $b \succ_i a$ then $a \succ_i^{**} b \succ_i^{**} c$
$(P^{**}3)$ If $b \succ_i^* a$, then $b \succ_i^{**} a \succ_i^{**} c$

where $i = 1, \ldots, n$ (note that by (P3), if $b \succ_i^* a$ then $b \succ_i a$). We can therefore depict the rankings of the alternatives a , b and c by individuals j, k and l as follows:

$$\begin{array}{ccc} \succ_j^{**} & \succ_k^{**} & \succ_l^{**} \\ a & a & b \\ b & c & a \\ c & b & c \end{array}$$

By $(P^{**}1) - (P^{**}3)$, every individual i satisfies $a \succ_i^{**} c$. Thus, by *PAR* and *C*: $aF(\succ^{**})c$ and $c\urcorner F(\succ^{**})a$. Since $bF(\succ^*)a$, then by *IIA*: $bF(\succ^{**})a$. We claim that the following must be true:

$$bF(\succ^{**})aF(\succ^{**})c \text{ and } c\urcorner F(\succ^{**})b \tag{4}$$

To see why, note that $bF(\succ^{**})a$ and $c\urcorner F(\succ^{**})a$. Hence, by *AC*, $c\urcorner F(\succ^{**})b$ and by *C*, $bF(\succ^{**})c$.

Consider next a profile $\succ''$ with the following properties:

$(P''1)$ $c \succ_i'' a \succ_i'' b$ for any i who satisfies $a \succ_i b$.
$(P''2)$ $b \succ_i'' c \succ_i'' a$ for any i who satisfies $b \succ_i a$.

We can therefore depict the rankings of the alternatives a, b and c by individuals j, k and l as follows:

$$\begin{array}{ccc} \succ_j'' & \succ_k'' & \succ_l'' \\ b & c & b \\ c & a & c \\ a & b & a \end{array}$$

Note that $c \succ_i'' a$ for $i = 1, \ldots, n$, which implies, by C and *PAR*,

$$cF(\succ'')a \text{ and } a\urcorner F(\succ'')c \tag{5}$$

Note that the ranking of a versus b is unchanged for each $i = 1, \ldots, n$ when individual i's ranking changes from $\succ_i$ to $\succ_i''$. Thus, by *IIA*: $aF(\succ'')b$ and $b\urcorner F(\succ'')a$. Because $cF(\succ'')a$ by (5), and because we have just concluded that $b\urcorner F(\succ'')a$, it must be, by *AC*, that $b\urcorner F(\succ'')c$. Hence, by *C* we have that $cF(\succ'')b$. Because the ranking of b and c is the same under $\succ_i^{**}$ and $\succ_i''$, this contradicts (4). ■

Theorem 1. *Let R satisfy BA and AC. If a social aggregator $F : P \rightarrow \mathbf{R}$ satisfies PAR and PR, then it must be dictatorial.*

Proof. We adjust Reny's method of proof [1] (see Reny [13]) to our framework.

Step 1. Consider any two distinct alternatives $a, b \in A$ and a profile of rankings $\succ^0$ in which a is ranked highest and b lowest for every $i \in N$. By *BA* and *PAR*,

$$aF\left(\succ^0\right)x \text{ and } x\urcorner F\left(\succ^0\right)a \text{ for every } x \in A \backslash \{a\} \tag{6}$$

Consider now changing individual 1's ranking by raising b in it one position at a time. By *PR*, so long as b is below a in 1's ranking, b cannot socially relate to a. Since no individual has changed his ranking of a and any $c \in A \backslash \{a, b\}$, c cannot socially relate to a . By *BA*, a must still socially relate to any other alternative. Once b rises above a in 1's ranking, then by *PR*, no alternative, but perhaps b, relates to a. If b remains unrelated to a, then begin the same process with individual 2 and 3, etc. until for some individual k, b relates to a when b rises above a in k's ranking (by *BA* and *PAR*, there must be such an individual k). Let $\succ^{(1.1)}$ denote the preference profile in which b reaches the second position in k 's ranking, and let $\succ^{(1.2)}$ denote the profile in which b rises to the top of his ranking. The two profiles are depicted in Fig. 1.1 and 1.2 below.

By construction, $aF(\succ^{(1.1)})b$ and $b\urcorner F(\succ^{(1.1)})a$. In addition, for every $c \in A \backslash \{a, b\}$, the ranking of a and c by each individual $i \in N$ is the same in $\succ^0$ and $\succ^{(1.1)}$. Thus, by *PR* and (6),

$$\forall x \in A \backslash \{a\}, x\urcorner F(\succ^{(1.1)})a \tag{7}$$

By *BA*,

$$\forall x \in A \backslash \{a\}, aF(\succ^{(1.1)})x \tag{8}$$

We now show that under $\succ^{(1.2)}$,

$$bF(1.2)y \text{ for every } y \in A \backslash \{b\} \tag{9}$$

and

$$c\urcorner F(1.2)b \text{ for every } c \in A \backslash \{a, b\} \tag{10}$$

[1] Which is inspired by Geanakoplos [8].

$$\begin{array}{cccccccc}
\succ_1^{(1.1)} & \cdots & \succ_{k-1}^{(1.1)} & \succ_k^{(1.1)} & \succ_{k+1}^{(1.1)} & \cdots & \succ_n^{(1.1)} \\
b & \cdots & b & a & a & \cdots & a \\
a & \cdots & a & b & \vdots & \cdots & \vdots \\
\vdots & & \vdots & \vdots & b & & b
\end{array}$$

Fig. 1.1. The prefereance profile $\succ^{(1.1)}$

$$\begin{array}{cccccccc}
\succ_1^{(1.2)} & \cdots & \succ_{k-1}^{(1.2)} & \succ_k^{(1.2)} & \succ_{k+1}^{(1.2)} & \cdots & \succ_n^{(1.2)} \\
b & \cdots & b & b & a & \cdots & a \\
a & \cdots & a & a & \vdots & \cdots & \vdots \\
\vdots & & \vdots & \vdots & b & & b
\end{array}$$

Fig. 1.2. The prefereance profile $\succ^{(1.2)}$

Note first, that by construction, $bF(\succ^{(1.2)})a$. Note next that for every $c \in A\backslash\{a,b\}$, the ranking of a and c by each individual $i \in N$ is the same in $\succ^{(1.1)}$ and $\succ^{(1.2)}$. This means that by *PR*, $c\neg F(\succ^{(1.2)})a$ for $c \in A\backslash\{a,b\}$. We therefore have that $bF(\succ^{(1.2)})a$ and $c\neg F(\succ^{(1.2)})a$. Hence, by *AC*, $c\neg F(\succ^{(1.2)})b$.

It remains to show that $bF(\succ^{(1.2)})c$. We shall make use of the following observation:

Observation 1. *Let R be a binary relation on A that satisfies AC. Then for any three alternatives $x, y, z \in A$: If xRy and yRz, then xRz.*

Proof. Assume that xRy, yRz but $x\neg Rz$. By *AC*, $x\neg Ry$, a contradiction.

Assume $b\neg F(\succ^{(1.2)})c$. Since $c\neg F(\succ^{(1.2)})a$ for every $c \in A\backslash\{a,b\}$, alternative a must, by *BA*, relate to every other alternative. In particular, $aF(\succ^{(1.2)})c$. Since $bF(\succ^{(1.2)})a$ we have, by Observation 1, that $bF(\succ^{(1.2)})c$, a contradiction. It follows that (9) must hold.

Step 2. Consider now the profiles $\succ^{(2.1)}$ and $\succ^{(2.2)}$ depicted in Figs. 2.1 and 2.2 respectively.

$\succ^{(2.1)}$ is derived from $\succ^{(1.1)}$ (and $\succ^{(2.2)}$ is derived from $\succ^{(1.2)}$) by moving alternative a to the bottom of individual i's ranking for $i < k$ and moving it to the second last position in i's ranking for $i > k$. We now show that these changes do not affect the social relation associated with the original preference profiles.

First note that no $i \in N$ has changed his ranking of b and any other alternative when the preference profile changed from $\succ^{(1.2)}$ to $\succ^{(2.2)}$. Thus, by *PR*,

$$c\neg F\left(\succ^{(2.2)}\right)b \text{ for every } c \in A\backslash\{a,b\} \tag{11}$$

Assume that $\exists y^* \in A\backslash\{b\}$ such that $b\neg F\left(\succ^{(2.2)}\right)y^*$. By *BA* and (11), $aF(\succ^{(2.2)})x$ for all $x \in A\backslash\{a\}$. In particular, $aF(\succ^{(2.2)})y^*$ and $aF(\succ^{(2.2)})b$. We therefore have that $aF(\succ^{(2.2)})y^*$ and $b\neg F\left(\succ^{(2.2)}\right)y^*$. Hence, by *AC*,

$$
\begin{array}{ccccccc}
\succ_1^{(2.1)} & \cdots & \succ_{k-1}^{(2.1)} & \succ_k^{(2.1)} & \succ_{k+1}^{(2.1)} & \cdots & \succ_n^{(2.1)} \\
b & \cdots & b & a & \vdots & \cdots & \vdots \\
\vdots & \cdots & \vdots & b & a & \cdots & a \\
a & & a & \vdots & b & & b
\end{array}
$$

Fig. 2.1. The prefereance profile $\succ^{(2.1)}$

$$
\begin{array}{ccccccc}
\succ_1^{(2.2)} & \cdots & \succ_{k-1}^{(2.2)} & \succ_k^{(2.2)} & \succ_{k+1}^{(2.2)} & \cdots & \succ_n^{(2.2)} \\
b & \cdots & b & b & \vdots & \cdots & \vdots \\
\vdots & \cdots & \vdots & a & a & \cdots & a \\
a & & a & \vdots & b & & b
\end{array}
$$

Fig. 2.2. The prefereance profile $\succ^{(2.2)}$

$b\urcorner F\big(\succ^{(2.2)}\big)a$. By *PR*, there must be some $i \in N$ that satisfies $a \succ_i^{(2.2)} b$ and $b \succ_i^{(1.2)} a$, a contradiction. Therefore,

$$bF(\succ^{(2.2)})y \text{ for all } y \in A\backslash\{b\} \tag{12}$$

Next, we show that

$$\forall x \in A\backslash\{a\}, aF(\succ^{(2.1)})x \text{ and } x\urcorner F(\succ^{(2.1)})a \tag{13}$$

First note that no $i \in N$ changed his ranking of a and b when his preference ordering changed from $\succ^{(1.1)}$ to $\succ^{(2.1)}$.Therefore, by *PR*, $b\urcorner F(\succ^{(2.1)})a$. Note also that the ranking of b and each $c \in A\backslash\{a, b\}$ is the same under $\succ^{(2.2)}$ and $\succ^{(2.1)}$. By *PR*, this implies

$$\forall c \in A\backslash\{a, b\}, c\urcorner F(\succ^{(2.1)})b \tag{14}$$

From (14) and our conclusion that $b\urcorner F(\succ^{(2.1)})a$, it follows that according to $F(\succ^{(2.1)})$ alternative a must, by *BA*, relate to every other alternative. In particular, $aF(\succ^{(2.1)})b$. From (14) and the acyclicality of $\mathbf{R}$, it follows that every alternative c distinct from a and b must satisfy $c\urcorner F(\succ^{(2.1)})a$. This proves that (13) must hold.

Step 3. Let $\succ^{(3)}$ be the profile depicted in Figure 3, where c is some element in $A\backslash\{a, b\}$.

$$
\begin{array}{ccccccc}
\succ_1^{(3)} & \cdots & \succ_{k-1}^{(3)} & \succ_k^{(3)} & \succ_{k+1}^{(3)} & \cdots & \succ_n^{(3)} \\
\cdot & \cdots & \cdot & a & \cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot & c & \cdot & \cdots & \cdot \\
\cdot & & \cdot & b & \cdot & & \cdot \\
c & & c & \cdot & c & & c \\
b & & b & \cdot & a & & a \\
a & & a & \cdot & b & & b
\end{array}
$$

Fig. 3. The preference profile $\succ^{(3)}$

We claim that

$$\forall x \in A\backslash\{a\}, aF(\succ^{(3)})x \text{ and } x\neg F\left(\succ^{(3)}\right)a \tag{15}$$

To see why, assume there exists some $x^* \in A$ that satisfies $x^*F(\succ^{(3)})a$. Then by *PR*, there must be some individual j for whom $a \succ_j^{(2.1)} x^*$ but $x^* \succ_j^{(3)} a$. However, no $i \in N$ has changed his ranking of a versus any other alternative when the preference profile changed from $\succ^{(2.1)}$ to $\succ^{(3)}$. Therefore, no $x \in A\backslash\{a\}$ satisfies $xF(\succ^{(3)})a$. Hence, by *BA*, $aF(\succ^{(3)})x$ for all $x \in A\backslash\{a\}$.

Step 4. Consider the profile of rankings $\succ^{(4)}$ depicted in Fig. 4.

$\succ_1^{(4)}$	$\cdots$	$\succ_{k-1}^{(4)}$	$\succ_k^{(4)}$	$\succ_{k+1}^{(4)}$	$\cdots$	$\succ_n^{(4)}$
$\cdot$	$\cdots$	$\cdot$	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	c	$\cdot$	$\cdots$	$\cdot$
$\cdot$		$\cdot$	b	$\cdot$		$\cdot$
c		c	$\cdot$	c		c
b		b	$\cdot$	b		b
a		a	$\cdot$	a		a

Fig. 4. The preference profile $\succ^{(4)}$

This profile is derived from $\succ^{(3)}$ by interchanging the ranking of alternatives a and b for individuals $i > k$. Note that no individual has changed his ranking of a versus any alternative $x \in A\backslash\{b\}$ when the preference profile changed from $\succ^{(3)}$ to $\succ^{(4)}$. Hence, by *PR*,

$$x\neg F(\succ^{(4)})a \text{ for all } x \in A\backslash\{a, b\} \tag{16}$$

We would like to show that $aF(\succ^{(4)})x$ for all $x \in A\backslash\{a\}$ and that $b\neg F(\succ^{(4)})a$. Assume that $bF(\succ^{(4)})a$. By (16), $c\neg F(\succ^{(4)})a$. Hence, by *AC*, $c\neg F(\succ^{(4)})b$. Because alternative c is ranked above b in every individual's ranking in Figure 4, it follows from *PAR* that $b\neg F(\succ^{(4)})c$. By *BA*, $F(\succ^{(4)})$ must satisfy that some element socially relates to every other distinct element. Since b does not relate to c, and since every x distinct from a and b does not relate to a, it must be the case that $aF(\succ^{(4)})y$ for all $y \in A\backslash\{a\}$. But then we have that $bF(\succ^{(4)})a$ and $aF(\succ^{(4)})c$, which by Observation 1 implies that $bF(\succ^{(4)})c$, a contradiction. Hence, $b\neg F(\succ^{(4)})a$ and by *BA*,

$$aF(\succ^{(4)})y \text{ for all } y \in A\backslash\{a\}$$

Step 5. Consider next an arbitrary profile of rankings $\succ$ in which individual k ranks a above b. Note that this profile can be obtained from $\succ^{(4)}$ without reducing the ranking of a vs. b in any individual's ranking. Hence, by *PR*, $b\neg F(\succ^*)a$. We may therefore conclude that whenever individual k ranks a above b, the social aggregator cannot relate b to a. Because the choice of a and b was arbitrary, we have shown that for each alternative $a \in A$, there is a dictator for a. Since there cannot be distinct dictators for distinct alternatives, there must be a single dictator for all alternatives. ∎

To see that Arrow's Theorem is a special case of Theorem 1 note that it can be rewritten as follows:

Arrow's Theorem. *Let $F : P \to \mathbf{R}$ where $\mathbf{R}$ is a class of complete and transitive binary relations. If F satisfies PAR and IIA, then F is dictatorial.*

By Proposition 2, F must satisfy *PR* and by Theorem 1 it must be dictatorial.

I now turn to show that the *GS* Theorem is a special case of Theorem 1. We first need to introduce the following pair of properties.

Definition 2. *A social aggregator F is strategy-proof if for every individual i, every $\succ \in P$, and every $\succ_i'$ in $\succ$, whenever a pair of distinct alternatives $a, b \in A$ satisfy*:

$$aF(\succ)x \text{ for all } x \in A\backslash\{a\}, \text{ but } bF(\succ_i', \succ_{-i})y \text{ for all } y \in A\backslash\{b\}$$

then $a \succ_i b$ and $b \succ_i' a$.

Definition 3. *A social aggregator F is onto if for every $a \in A$ there exists a profile $\succ \in P$ that satisfies $aF(\succ)x$ for all $x \in A\backslash\{a\}$.*

To embed the *GS* Theorem in our framework, this theorem can be rewritten as follows:

GS Theorem. *If a social aggregator $F : P \to \mathbf{R}_0$ is strategy-proof and onto, then F is dictatorial.*

It can be shown (see Reny [13]) that if F is strategy-proof and onto, then F is Pareto efficient and monotonic. By Proposition 1, F must also satisfy *PR*. Thus, by Theorem 1, F is dictatorial.

5 Concluding remarks

This paper shows that the two theorems of Arrow and Gibbard-Satterthwaite are actually special cases of *one single theorem*: There is a single set of definitions and a single pair of properties, Pareto efficiency and Preference Reversal, that lead to the dictatorship result. Thus, this paper makes the following contributions:

1. **A single framework –** One may think of different ways in which preference profiles may be aggregated. *SCF*'s and *SWF*'s are two examples. The question arises as to whether there is a class of aggregators that must be dictatorial if and only if they satisfy some conditions. This paper proposes a general class of aggregators, which includes both *SCF*'s and *SWF*'s as special cases. This is done by introducing a mapping from preference profiles to *binary relations*. Thus, I show that aggregating profiles to a single alternative and aggregating profiles to a single ordering are in some sense "equivalent" operations.

2. **A direct "meta-theorem"** – previous work has tried to connect Arrow and GS by showing that both theorems can be proven with the help of Muller and Satterthwaite's (MS) finding that strong monotonicity implies dictatorship. However, strong monotonicity is not a property of *SWF*'s. To prove that Arrow's Theorem is a corollary of MS (see Moulin [10] pp. 52–56) requires one to introduce the notion of a *SCF* (and also the notion of a "blocking coalition"), which is a different aggregator than the one to which the theorem relates. That is, there is no direct link from properties of the aggregator itself (in this case, a *SWF*) to the dictatorship result. In this paper I show that the single "meta-theorem", from which Arrow and GS follow, can be proven directly by relying only on the aggregator and its properties.
3. **Relating *IIA* and monotonicity** – Since both Arrow and GS require Pareto efficiency it seems that *IIA* and monotonicity are closely related. This paper demonstrates that both are special cases of a single property termed Preference Reversal.

References

[1] Arrow KJ (1950) A difficulty in the theory of social welfare. J Polit Econ 58: 328–346
[2] Barberà S (2001) A theorem on preference aggregation. Mimeo, Universitat Autònoma de Barcelona
[3] Barberà S (1983) Strategy-proofness and pivotal voters: a direct proof of the Gibbard-Satterthwaite theorem. Int Econ Rev 24: 413–417
[4] Barberà S, Peleg, B (1990) Strategy-proof voting schemes with continuous preferences. Soc Choice Welfare 7: 31–38
[5] Barberà S, Dutta B, Sen A (2001) Strategy-proof social choice correspondences. J Econ Theory 101: 374–394
[6] Benoît JP (2002) Strategic manipulation in voting games when lotteries and ties are permitted. J Econ Theory 102: 421–436
[7] Benoît JP (2000) The Gibbard-Satterthwaite theorem: a simple proof. Econ Letters 69: 319–322
[8] Geanakoplos J (2001) Three brief proofs of Arrow's impossibility theorem. Econ Theory (forthcoming)
[9] Gibbard A (1973) Manipulation of voting schemes: a general result. Econometrica 41: 587–601
[10] Moulin H (1983) The strategy of social choice. North Holland, Amsterdam
[11] Muller E, Satterthwaite MA (1977) The equivalence of strong positive association and strategy-proofness. J Econ Theory 14: 412–418
[12] Peleg R (1984) Game theoretical analysis of voting in committees. Cambridge University press, Cambridge
[13] Reny PJ (2001) Arrow's Theorem and the Gibbard-Satterthwaite theorem: a unified approach. Econ Letters 70: 99–105
[14] Satterthwaite MA (1975) Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. J Econ Theory 10: 187–217
[15] Sen A (2001) Another direct proof of the Gibbard-Satterthwaite theorem. Econ Letters 70: 381–385
[16] Svensson LG (1999) The proof of Gibbard-Satterthwaite theorem revisited. Mimeo, Lund University