Available online at www.sciencedirect.com
SCIENCE DIRECT®

Games and Economic Behavior 56 (2006) 61–86

GAMES and Economic Behavior

www.elsevier.com/locate/geb

Indifference or indecisiveness? Choice-theoretic foundations of incomplete preferences

Kfir Eliaz *, Efe A. Ok

Department of Economics, New York University, 269 Mercer Street, New York, NY 10003, USA

Received 22 June 2004

Available online 20 March 2006

Abstract

A commonly held belief in economics is that an individual's preferences that are revealed by her choices must be complete. This paper takes issue with this position by showing that one may be able to distinguish between indifference and indecisiveness of an agent upon observing her choice behavior. We relax the standard Weak Axiom of Revealed Preferences (WARP) and show that a potent theory of individual choice (with and without risk) can be founded on this weaker axiom when it is coupled with some other standard postulates. The most notable features here are that an agent is allowed to be indifferent between certain alternatives and indecisive about others. Moreover, an outside observer can identify which of these actually occur upon examining the (observable) choice behavior of the decision maker. As an application, we also show how our theory may be able to cope with the classical preference reversal phenomenon.

JEL classification: D71; C78

1. Introduction

One of the most widely debated topics in the theory of individual decision making concerns the notion of incomplete preferences, which, by definition, leaves room for an individual to remain indecisive on occasion. Especially recently, such preference structures have been studied from a variety of angles. In particular, several authors have made a formal connection between incomplete preferences and multi-objective decision making under certainty and risk, and others have studied choice models in which an agent may not be able to compare some alternatives

* Corresponding author.
E-mail addresses: kfir.eliaz@nyu.edu (K. Eliaz), efe.ok@nyu.edu (E.A. Ok).

0899-8256/$ – see front matter
doi:10.1016/j.geb.2005.06.007

due to the presence of uncertainty. In turn, some authors have brought out arguments based on the incompleteness of preferences that "explain" some of the experimentally observed anomalies (such as the endowment effect and/or status quo bias), and still others have considered economic models, ranging from industrial organization to political economy, in which incompleteness of individual preferences plays a key role.[1]

It is, however, fair to say that there remains a foundational problem with the notion of incomplete preferences. A well-founded tradition that was pioneered by Samuelson (1938) identifies "choice" as a more basic notion than "preference," largely because the former is observable and the latter is not. Adopting this point of view, then, one seems to be forced to accept that "incompleteness" of one's preferences are unobservable, and hence while intuitively meaningful, such preference structures lack choice-theoretic foundations. Put succinctly, the primary objective of the present paper is to take issue with this position, and argue formally that incomplete preferences can be "revealed" much the same way complete preferences can.[2]

Our point of departure stems from the fact that revealed preference theory is itself based on certain assumptions, and relaxations of these assumptions may well give rise to behavior which is rationalized upon maximizing an incomplete preference relation. This point is, of course, recognized by many economists and philosophers. For instance, in his important work on conflict resolution, Levi (1986, p. 84), who uses the phrase "unresolved conflict" synonymously with "incomplete preferences," argues as follows:

> "... Those who insist on revealed preference models of evaluation impose stronger restrictions on rational agents than those who allow for decision making on unresolved conflict require. These stronger restrictions are as controversial as the insistence that rational agents have resolved their conflicts by the moment of choice. Far from showing that the proposals made here are misleading exercises in terminological reform, doctrines of revealed preference exploit their terminological practices to beg the question against those who wish to recognize the rationality of decision making under unresolved conflict."

The upshot here is that, contrary to what might at first appear, it is not the essence of the revealed preference exercise that forces the induced preferences to be complete, but this is rather a consequence of the assumptions imposed on one's choice behavior.

Let us illustrate this point by means of a concrete example. Consider a parent, call her Mrs. Watson, who faces the problem of choosing a movie to rent from a video store for her

[1] The two classical references on the theory of incomplete preferences are Aumann (1962) and Bewley (1986). The formal connection between multi-objective decision making and incomplete preferences are studied by Seidenfeld et al. (1995), Ok (2002), Dubra et al. (2004), and Sagi (2005), among others. On the other hand, while Mandler (2005), Masatlioglu and Ok (2005), and Sagi (2005) study the endowment effect and reference-dependent choice by using incomplete preferences, Roemer (1999) and Levy (2004) use incomplete preferences to tackle equilibrium existence problems in certain multidimensional voting models. Some applications to industrial organization are given in Bade (2005).

[2] Two other papers that examine the choice-theoretic foundations of incomplete preferences are Danan (2003) and Mandler (2005). Both of these papers distinguish between *revealed* and *psychological* preferences, and assume that revealed preferences (those that emanate from pairwise choice situations) are, per force, complete. While Danan (2003) studies ways of recovering psychological preferences from revealed preferences, Mandler (2005) shows that incompleteness of psychological preferences causes intransitivity of revealed preferences, but not "irrational" choice behavior. Danan (2003) is, therefore, closer to the present work from a conceptual viewpoint, but the choice models we study have little in common.

two children, Alice and Tom. There are exactly three movies under consideration, x, y and z, and the strict preferences of Alice and Tom are summarized as follows.

Alice	Tom
y	x
z	y
x	z

(So, for instance, Alice strictly prefers y to movie z, and z to x.) Since the kids' preferences are similar with respect to y and z (i.e. they coincide on $\{y, z\}$), it is plain that the parent will not choose to rent z when y is available. Since the preferences of Alice and Tom with respect to x and y are orthogonal to each other, however, there is no obvious way of choosing between x and y for Mrs. Watson. Assuming that she can rent at that moment only one movie, it is quite conceivable that she would settle her choice problem by, say, flipping a coin between x and y, which Alice and Tom would likely to consent since this seems to be the "fair" resolution of the present conflict of interests. So, it is quite plausible to say that the "choice set" of Mrs. Watson for the choice problem $\{x, y, z\}$ is $\{x, y\}$ (at least in the sense that both x and y can be chosen with positive probability, but z cannot). But suppose, when Mrs. Watson was just preparing to toss a coin to settle the matter, the store owner informs the family that the movie y is rented out while there are still some copies of x and z available. What will Mrs. Watson do in this case? Since the preferences of Alice and Tom are negatively correlated between x and z, it seems quite rational that she would reason exactly as before, and decide to settle the matter again by flip of a coin. This means that the "choice set" of Mrs. Watson for the choice problem $\{x, z\}$ is $\{x, z\}$.

It seems quite difficult to argue that there is something clearly wrong in the way Mrs. Watson deals with her two choice problems here. Yet, according to the standard theory of choice, her behavior is "irrational," for indeed it violates the *weak axiom of revealed preference* (WARP). Since her set of choices from $\{x, z\}$ is $\{x, z\}$, the standard theory would view Mrs. Watson as indifferent between x and z. Therefore, because x is a choice from $\{x, y, z\}$, the movie z should also be a potential choice for her in the choice problem $\{x, y, z\}$. (Or, put differently, given her choices from $\{x, y, z\}$, WARP would require that Mrs. Watson choose only x from $\{x, z\}$, much to the protest of Alice.) Consequently, Mrs. Watson's choice behavior cannot be viewed as stemming from the maximization of a complete preference relation (and hence of a utility function). However, and this is the main point of this paper, it can be viewed as arising from the maximization of an incomplete preference relation. Indeed, all that Mrs. Watson does here is to "choose" the Pareto optimal alternatives from the feasible sets she is presented; she chooses $\{x, z\}$ from $\{x, z\}$ because she cannot compare the alternatives x and z on the basis of the Pareto principle, and she chooses $\{x, y\}$ from $\{x, y, z\}$ because she observes that z is Pareto dominated by y, while x and y are incomparable.

Behind the banality of the parable of Mrs. Watson is the observation that WARP is too demanding, at least for choice problems that involve multiple objectives. Indeed, while we framed this example in the garb of a social choice problem, the basic idea, that is, the difficulty with WARP, is valid in any individual choice problem in which the choice alternatives have a multitude of attributes (as in *multicriteria decision-making*).[3]

[3] For instance, think of x, y and z here as applications for a particular fellowship, and consider an agent who evaluates these applications on the basis of two criteria (say, the amount of preparation of the applicant, and the originality of the proposal). Suppose that z is dominated by y with respect to both criteria, while there is no dominance between x and y,

Of course, it is not reasonable to expect that people would always conform with the basic postulates of the standard theory of choice. This theory is quite normative in nature—it is dubbed hence the theory of "rational" choice—and applies, in a context-independent manner, to *all* choice situations. This is the strength of the theory, for it provides a unified framework for the economic analysis at large. But it is also its weakness, for it culminates in a decision making paradigm with only limited explanatory power. The question is, can we modify the standard theory in such a way that an individual is allowed to maximize a (possibly) *incomplete* preference relation, thereby letting her to remain indecisive at times.

In this paper we provide an affirmative answer to this question. Our analysis starts with introducing a natural weakening of WARP, which we call the *Weak Axiom of Revealed Non-Inferiority* (WARNI). Our main results (Theorems 2 and 3) establish that WARNI is a necessary and sufficient condition for rationalizing choice behavior with a possibly incomplete preference relation. The resulting choice theory, thus, allows a "rational individual" to remain *indecisive* at times.

Be that as it may, this theory too is based on "choices" of the individuals, and it may at first seem like it is impossible to determine if an individual, who deems two alternatives, say x and y, equally choosable, is indecisive or indifferent between them. In fact, the literature on revealed preference theory has examined various weakenings of WARP—as we discuss later, some of these are indeed closely related to WARNI—and derived complete preference relations that allow for intransitive indifference. We instead derive transitive (incomplete) preference relations, but, of course, the non-comparable part of this relation may well be non-transitive. It may thus seem at first glance as if our approach is in essence a conceptual relabeling of earlier findings, one that interprets what was viewed as indifference in that literature as indecisiveness. This is incorrect. Such a relabeling would not allow for *any* indifference (other than what is implied by reflexivity), just like the classical revealed preference literature does not allow for *any* incompleteness. Our objective here is, instead, to identify when a choice behavior can be rationalized by a transitive preference relation which realistically allows for an agent to exhibit *both* indifference *and* indecisiveness. After all, a major advantage of the theory proposed here is it enabling one to distinguish between these two cases upon "observing" the choices of a decision maker.

In the final part of our analysis, we extend the choice domain under consideration to include lotteries, and complete our axiomatic framework by adding a property which is a straightforward reflection of the classical independence axiom. This allows us to utilize the expected multi-utility theorem of Dubra et al. (2004) in order to characterize the class of choice correspondences that are generated by the maximization of a *set* of continuous utility functions. This gives a rational choice-theoretic foundation for the multi-objective decision paradigm, and thus yields a unified choice theory. The descriptive power of this theory is superior to the standard single-objective model, and its predictive power, while obviously weaker than the standard paradigm, is still quite strong.[4]

The final arbiter of which sort of a choice theory is most useful should, of course, be the applications (and empirical investigations that would arise thereof). Consequently, we consider here one such application, concerning the famous "preference reversal phenomenon." We show that

and x and z. Consequently, the choice between x and y, and between x and z may well be settled by some form of (subjective) randomization, but the choice between y and z is clear. This individual choice behavior is formally identical to that Mrs. Watson.

[4] To the best of our knowledge, the implications of weakening WARP in the case of risky choice problems are not at all studied in the literature, nor is there any paper that derives multiple utility functions by means of a revealed preference exercise.

decision-makers who are rational in the sense of maximizing an incomplete preference relation (hence satisfying WARNI but not WARP) may well exhibit this phenomenon. This is promising, for our approach yields a unified theory that applies to all choice situations, as opposed to a specialized theory that is particularly designed to "explain" the preference reversal phenomenon.

2. Revealed preference theory revisited

2.1. Choice correspondences

Take an arbitrary nonempty set X to act as the universal set of all mutually exclusive alternatives. By a *choice field* on X we mean a subset Ω_X of $2^X\backslash\{\emptyset\}$ which includes all singleton sets and which is closed under taking finite unions. We view a given choice field on X as the class of all possible choice problems.[5] Formally speaking, we refer to any pair (X, Ω_X) as a *choice space*, provided that X is a set with $|X| \geqslant 3$, and Ω_X is a choice field on X. If Ω_X equals the class of all nonempty finite subsets of X, then we say that (X, Ω_X) is a *finite choice space*. If X is a metric space, and Ω_X is the set of all nonempty compact subsets of X, then we say that (X, Ω_X) is a *compact choice space*.

For any given choice space (X, Ω_X), a *choice correspondence* on Ω_X is defined as any correspondence $c : \Omega_X \rightrightarrows X$ such that $\emptyset \neq c(S) \subseteq S$ for every $S \in \Omega_X$.[6] This definition envisages that, given a choice problem $S \in \Omega_X$, the set $c(S)$ of choices of the individual may contain more than one alternative. While this is a standard formulation, the literature is rather elusive as to what it means to "choose" more than one alternative in a given choice problem.[7] This issue is crucial in that c represents an "observable" entity, and hence it must be clarified what it means to "observe" an agent choosing a subset $c(S)$ from a feasible set S.[8]

For concreteness, we suggest thinking about $c(S)$ as follows:

- the final choice of the agent is one of the members of $c(S)$;
- the decision maker might finalize her choice by (subjectively) randomizing in between the members of $c(S)$ (as in "I don't know/care, I'd flip a coin, I guess").

With respect to this viewpoint, if $\{x\} = c(S)$, we understand that, unless the preferences of the agent changes, the probability that an outside observer would see the agent choosing x is one. Similarly, if $\{x, y\} = c(S)$, then if the agent was confronted with the same choice situation re-

[5] By definition, any choice field on X includes all nonempty finite subsets of X. This is a useful requirement, as most experimental studies on individual decision making involve discrete choice situations.

[6] *Notation*: Given two nonempty sets A and B, by a *correspondence* Γ that maps A into B, we mean a function that maps A into 2^B, and denote this situation as $\Gamma : A \rightrightarrows B$. Notice that, in this terminology, a correspondence may be empty-valued.

[7] For instance, Kreps (1988, p. 12) interprets this as "... If the agent is offered his choice of anything in the set S, he says that any member of $c(S)$ will do just fine." Alternatively, Sen (1993, p. 499) says that "... it may be useful to interpret $c(S)$ as the set of "choosable" elements—the alternatives that can be chosen." It is not even clear if these two descriptions correspond to the same interpretation. In particular, Kreps' phrasing seems more in line with deeming the elements in a choice set as indifferent to one another, whereas that of Sen appears to allow for indecisiveness of the individuals as well.

[8] Notice that even the interpretation of $\{x\} = c(S)$ is at issue here. If we have observed an agent (say, in an experiment) choosing x from S, all we can conclude is that $x \in c(S)$.

peatedly (at least as a thought experiment), we would observe her choosing x some of the times, and y at others.[9]

In the traditional theory of revealed preference, one reads the statement "$x \in c(S)$" as "x is revealed to be at least as good as all other alternatives in S," and thus infers from an observation like "$\{x, y\} \subseteq c(S)$" that x and y are equally desirable for the decision maker. Under this interpretation, therefore, it is not conceivable that an individual may actually be "indecisive" in any choice situation. By contrast, the viewpoint suggested in the previous paragraph duly allows for an agent to be indecisive between choosing x and y, and hence suggests that "$x \in c(S)$" should more prudently be viewed as revealing that "x is not strictly worse than any alternative in S." According to this perspective, which is also adopted by Nehring (1997) and Sen (1997), the statement "$\{x, y\} \subseteq c(S)$" is rather thought of as x and y being "undominated" in S (in a sense that will be made precise shortly), thereby allowing for these alternatives to be *incomparable* for the agent in question. As noted in the Introduction, the primary objective of this paper is, in fact, to develop a revealed preference theory which is in line with precisely this latter perspective.

2.2. *Incomparable pairs*

Fix an arbitrary choice space (X, Ω_X), and consider an individual with a choice correspondence c on Ω_X. Clearly, if $\{x\} = c\{x, y\}$, then, as outside observers we would have little doubt that this individual strictly prefers x over y. But what if we observe $\{x, y\} = c\{x, y\}$, that is, we realize that the agent in question subjectively randomizes between x and y? Then, she may either be indifferent between x and y, or find these alternatives incomparable (and hence is indecisive between them). Intuitively, if the former was the case, and we somehow believed that the individual is in some loose sense "rational," then we would expect the agent to treat x and y identically in all choice situations that involve these alternatives.

To make this point a bit more precise, let us agree to write $S_{y,-x}$ for the set $(S \cup \{y\})\backslash\{x\}$, for any $S \in \Omega_X$ with $x \in S$ and $y \notin S$. In words, $S_{y,-x}$ is the feasible set that is obtained from S by swapping x with y.[10] If the agent in question views the appeal of x and y as equivalent (which is what we intuitively mean as she being indifferent between these outcomes), then, for any $S \in \Omega_X$ with $x \in S$ and $y \notin S$, it would be sensible to expect that $c(S)$ and $c(S_{y,-x})$ would look identical except that the role of x in $c(S)$ would be played by y in $c(S_{y,-x})$. In particular, x would be deemed "choosable" from S iff y would be deemed "choosable" from $S_{y,-x}$, and whatever is found "choosable" in S other than x would remain "choosable" in $S_{y,-x}$ and vice versa. If this is not the case for some feasible set S (with $x \in S$ and $y \notin S$), then there would be good grounds to doubt that the agent was indifferent between x and y even though we have observed $\{x, y\} = c\{x, y\}$. This discussion prompts the following definition.

Definition. Let (X, Ω_X) be a choice space, and c a choice correspondence on Ω_X. Let

$$\mathcal{P}(c) := \big\{(x, y) \in X \times X\colon\ x \neq y \text{ and } \{x, y\} = c\{x, y\}\big\}.$$

[9] It is important to allow for only *subjective* randomizations here. After all, if we observed the agent, say, flipping a coin between x and y, we could never know if 50-50 mixing between x and y is the *unique* best choice for the agent.

[10] In this intuitive discussion we assume implicitly that $S_{y,-x}$ belongs to Ω_X, but this is in general not true, unless S is finite. For this reason, in the formalism developed below we shall use only *finite* feasible sets in defining incomparable pairs, thereby avoiding this technical problem.

We say that $(x, y) \in X \times X$ is a *c-incomparable pair* if $(x, y) \in \mathcal{P}(c)$, and there exists a finite set $S \in \Omega_X$ with $x \in S$, $y \notin S$, such that at least one of the following is true:

(a) $x \in c(S)$ but $y \notin c(S_{y,-x})$,
(b) $x \notin c(S)$ but $y \in c(S_{y,-x})$,
(c) $c(S)\backslash\{x\} \neq c(S_{y,-x})\backslash\{y\}$.

The set of all c-incomparable pairs is denoted by $\mathcal{I}(c)$.

The following example illustrates this definition.

Example 1 *(Mrs. Watson's Problem).* The choice space that corresponds to the parable of Mrs. Watson considered in the Introduction is $(X, 2^X\backslash\{\emptyset\})$, where $X := \{x, y, z\}$. The choice correspondence c of Mrs. Watson satisfies:

$$c\{x, y\} = \{x, y\}, \quad c\{x, z\} = \{x, z\}, \quad c\{y, z\} = \{y\} \quad \text{and} \quad c\{x, y, z\} = \{x, y\}. \tag{1}$$

It is readily checked here that both (x, y) and (x, z) are c-incomparable pairs, while (y, z) is not. Indeed, while $(x, y) \in \mathcal{P}(c)$, c does not treat the alternatives x and y as "equivalent" across other choice problems. For instance, where $S := \{x, z\}$, we have $c(S)\backslash\{x\} = \{z\} \neq \emptyset = c(S_{y,-x})\backslash\{y\}$. Similarly, $(x, z) \in \mathcal{I}(c)$ because $x \in c(T)$ but $z \notin c(T_{z,-x})$, where $T := \{x, y\}$.

In passing, we note that the notion of incomparability that we have defined above has only formal content at present. For, it is not known what sort of rationale guides an arbitrary choice correspondence c in general, and if c comes out as a result of a maximization of a preference relation in particular. Consequently, it would be premature at this stage to view it as truly capturing the pairs of alternatives that the agent is indecisive about. The latter is, after all, a property of her "unobservable" preferences. However, we will show below that, under a certain rationality requirement, the set $\mathcal{I}(c)$, which is "observable," consists precisely of these pairs.

2.3. Regular preference relations

As is usual, we denote in this paper the symmetric and asymmetric parts of a given reflexive binary relation $\succsim$ on a given nonempty set X by $\sim$ and $\succ$, respectively. (Of course, $\succsim$ is the disjoint union of $\succ$ and $\sim$.) The incomparable part of such a relation is denoted as $\bowtie$. That is, $\bowtie$ is the irreflexive binary relation on X defined by $x \bowtie y$ iff neither $x \succsim y$ nor $y \succsim x$.[11, 12]

A reflexive and transitive (but not necessarily complete) binary relation on X is called a *preference relation* on X. For any preference relation $\succsim$ on X and any nonempty subset S of X, the set of all $\succsim$-*maximal* elements of S is denoted by $\max(S, \succsim)$, that is,

$$\max(S, \succsim) := \{x \in S\colon\ y \succ x \text{ for no } y \in S\}.$$

If S is finite, this set is necessarily nonempty. If S is infinite, however, it may well be empty, unless additional requirements are imposed on S and $\succsim$.

[11] Formally speaking, $\sim := \succsim \cap \succsim^{-1}$, $\succ := \succsim \backslash \sim$ and $\bowtie := (X \times X)\backslash(\succsim \cup \succsim^{-1})$. (Here $\succsim^{-1}$ is the inverse of $\succsim$, that is, $\succsim^{-1} := \{(y, x)\colon\ (x, y) \in \succsim\}$.)

[12] *Terminology*: Let R be a binary relation on a nonempty set A, that is, $\emptyset \neq R \subseteq A \times A$. As is standard in economic theory, we write aRb to denote $(a, b) \in R$. We say that R is *reflexive* if aRa for all $a \in A$, that it is *transitive* if, for any $a, b, c \in A$, $aRbRc$ implies aRc, and that it is *complete*, if, for any $a, b \in A$, either aRb or bRa holds.

Let (X, Ω_X) be a choice space, and c a choice correspondence on Ω_X. Suppose we know that c is *rationalized* by a preference relation $\succsim$ on X, that is, $c = \max(\cdot, \succsim)$, but we are not sure about the exact nature of $\succsim$. If $\{x, y\} = c\{x, y\}$, can we discern whether the individual, whose choice behavior is modeled by c views x and y as indifferent or incomparable? The answer is no, not at this level of generality. For instance, if $c(S) := S$ for all $S \in \Omega_X$, then we have

$$\max\left(\cdot, \succsim^1\right) = c = \max\left(\cdot, \succsim^2\right)$$

where $\succsim^1 := X \times X$ and $\succsim^2 := \{(x, x)\colon\ x \in X\}$. That is, the choice behavior of the subject agent can be thought of as arising from the maximization of a preference relation that deems all alternatives indifferent, or from that of a preference relation that renders all outcomes incomparable. It is simply impossible to distinguish between indifference and indecisiveness by observing c in this case.

It follows that one has to know more about the preference relation that rationalizes a choice correspondence, if, upon observing her choice behavior, we were to identify when the subject agent is indifferent between two alternatives and when she is indecisive. We found that the following property is crucial for this purpose.

Definition. A preference relation on a nonempty set X is said to be *regular* if, for any $x, y \in X$ with $x \bowtie y$, there is a $z \in X$ such that either $x \bowtie z$ and y and z are strictly ordered by $\succ$, or $y \bowtie z$ and x and z are strictly ordered by $\succ$.[13]

Regularity of a preference relation requires a mild amount of richness from the strict part of a preference relation. Indeed, unless $\succsim$ is the trivial preference relation on X that declares everything indifferent, a preference relation $\succsim$ is regular only if $\succ \neq \emptyset$.

Example 2 *(Regular preference relations).* (1) Every complete preference relation is regular.
(2) A preference relation on a doubleton set is regular iff it is complete.
(3) Let X be a tripleton set and $\succsim$ an *incomplete* preference relation on X. Then $\succsim$ is regular iff $\succ \neq \emptyset$.
(4) Let $n \in \mathbb{N}$ and suppose X is an open convex subset of $\mathbb{R}^n$. Let $u_i : X \to \mathbb{R}$ be any strictly increasing and continuous function, $i = 1, 2$, and define the binary relation $\succsim$ on X by $x \succsim y$ iff $u_i(x) \geqslant u_i(y)$ for each $i = 1, 2$. Then $\succsim$ is a regular preference relation on X.

The regularity property seems quite sensible for incomplete preferences. In particular, if the incompleteness of a decision maker's preferences arises due to the multiplicity of the criteria that she uses in evaluating her choice options (as in Example 2(4)), then this property is in the nature of things. To illustrate, suppose the decision maker cannot rank the alternatives x and y, and z is another alternative which is superior to y in all relevant attributes. But if z is only slightly superior to y in each attribute, then in those attributes that y is dominated by x, the alternative z is also dominated by x. In this case we would expect $x \bowtie z \succ y$ to hold. For instance, if z equals the alternative y plus "ε dollars," the situation $x \bowtie z \succ y$ would naturally obtain for small enough $\varepsilon > 0$. Thus the preference relation of this agent is regular in an obvious manner.[14]

[13] When we say that y and z are strictly ordered by $\succ$, we mean either $y \succ z$ or $z \succ y$. Formally speaking, $\succsim$ is regular iff, for any $x, y \in X$ with $x \bowtie y$, there is a $z \in X$ such that the intersections of $\{(x, z), (y, z)\}$ with $\bowtie$ and with $\succ \cup \succ^{-1}$ are singleton sets.

[14] This might even tempt one to consider, instead of regularity, the following property: For all $x, y \in X$ with $x \bowtie y$, there is a $z \in X$ with $x \bowtie z \succ y$. Unfortunately, while duly sensible in many economic environments, this property

2.4. Choice and complete preference relations

For any given choice space (X, Ω_X), the classical individual choice theory qualifies a choice correspondence c on Ω_X to be "rational" if c satisfies the so-called *weak axiom of revealed preference*: If both x and y are in the feasible sets S and T, and we have $x \in c(S)$ and $y \in c(T)$, then $y \in c(S)$. With an eye on its relaxation to come, we wish to phrase this axiom slightly differently.

Weak Axiom of Revealed Preference (WARP). For any $S \in \Omega_X$ and $y \in S$, if *there exists* an $x \in c(S)$ such that $y \in c(T)$ for some $T \in \Omega_X$ with $x \in T$, then $y \in c(S)$.

The idea is that if $x \in T$ and $y \in c(T)$, one may conclude that y is at least as good as x for the subject individual, and thus—this is what WARP says—whenever x is chosen from a set S that contains y, so must y. Evidently, an individual who makes her choices through the maximization of a complete preference relation would always satisfy this property. Conversely, if the choices of a decision maker always abide by WARP, then we can think of her "as if" she maximizes such a relation. This is the fundamental theorem of revealed preference theory.[15]

Theorem 1. *Let (X, Ω_X) be a choice space, and c any choice correspondence on Ω_X. Then, c satisfies WARP if, and only if, there exists a* (*unique*) *complete preference relation $\succsim$ on X such that $c = \max(\cdot, \succsim)$.*

This is a textbook example of the use of revealed preference theory in order to "derive" a *complete* preference relation from the choice behavior of an individual. It is thus of central importance for the foundations of rational decision making. However, it should nevertheless be noted that the significance of this result is based on the appeal of WARP, descriptive or normative, and this is, in turn, contingent upon how one interprets a choice correspondence. If one interprets multiple choices that emanate from a feasible set as we do, then she is forced to view the statement $x \in c(S)$ as revealing that x is not strictly inferior to any alternative in S (but it is not necessarily superior to them). And, in that case, the explanatory power of WARP becomes duly suspect. This viewpoint is implicitly taken, for instance, in the example of Mrs. Watson in Section 1, whose choice behavior violates WARP, even though her decisions can well be explained within the "maximizing paradigm."

We submit, therefore, that WARP is not an unquestionable rationality tenet, especially in contexts of coalitional decision-making, social choice, and individual choice problems that involve multiple objectives. After all, in such contexts, even the venerable Pareto choice correspondence (that maps a given set of allocations to the set of Pareto optimal ones) fails to satisfy this axiom. Furthermore, we will later see that in choice problems that involve uncertainty, too, genuine incomparabilities may arise, thereby forcing an individual to use a dominance-based choice rule. For instance, an agent whose preference relation coincides with stochastic dominance (of any order) would necessarily violate the WARP, while it seems unwarranted to qualify such a decision maker as "irrational."

---

is unacceptably more demanding than regularity in general. After all, it cannot possibly be satisfied by an incomplete preference relation defined on a finite set.

[15] The modern theory of revealed preference is founded by the seminal contributions of Arrow (1959), Richter (1966) and Hansson (1968). For an introduction to this topic, see Suzumura (1983) and/or Moulin (1985).

Remark 1. Sen (1971) has suggested to decompose WARP into the following two distinct properties.

(α)-axiom. For any $S, T \in \Omega_X$, if $x \in T \subseteq S$ and $x \in c(S)$, then $x \in c(T)$.

(β)-axiom. For any $S, T \in \Omega_X$ with $T \subseteq S$, if $x, y \in c(T)$ and $x \in c(S)$, then $y \in c(S)$.

It is an easy exercise to show that a choice correspondence on Ω_X satisfies these two properties simultaneously iff it satisfies WARP. However, the appeal of the (α) and (β)-axioms are not quite alike. The (α)-axiom, which was originally proposed by Chernoff (1954), is normatively desirable, and hence quite suiting for a theory of *rational* choice.[16] As the example of Mrs. Watson demonstrates, however, the appeal of the (β)-axiom is ambiguous (especially when one interprets multiple choices as we advanced in Section 2.1). While the choice correspondences considered in these examples satisfy the (α)-axiom, they fail to satisfy the (β)-axiom.

2.5. Choice and incomplete preference relations

The examples considered so far lead us to propose a particular weakening of WARP which accords well with interpreting the statement “$x \in c(S)$” as “x is not revealed inferior to any alternative in S.” We aim to show that this property paves the way towards a rational choice theory that incorporates dominance-based decision-making, thereby improving the explanatory power of the standard theory of individual choice.

The statement of WARP suggests readily the following weaker property.

Weak Axiom of Revealed Non-Inferiority (WARNI). For any $S \in \Omega_X$ and $y \in S$, if *for every* $x \in c(S)$ there exists a $T \in \Omega_X$ with $y \in c(T)$ and $x \in T$, then $y \in c(S)$.

This property appears to be a suitable adaptation of WARP for a choice model that distinguishes between something being *superior* and it being *non-inferior* for a decision maker. Suppose we interpret the situation $x \in T$ and $y \in c(T)$ as “y is revealed not to be inferior to x.” As noted above, under this interpretation WARP loses some of its appeal, for even if y may not be inferior to *some* x chosen from S, it may be inferior to some other alternative z chosen from S, and in this case we would not expect y to be chosen from S. But, so goes WARNI, if y is revealed not to be inferior to *all* of the alternatives chosen from S, then y must be chosen from S as well. It is easily checked that the choice behavior of Mrs. Watson (Section 1), and more generally the Pareto choice correspondence in a social choice context, would satisfy this rationality property.[17]

Replacing WARP with WARNI yields the following counterpart of Theorem 1.[18]

[16] In fact, it is commonly observed in experiments that subjects who violate the (α)-axiom often change their behavior in order to conform with it when their violation of this axiom is pointed out to them. Yet, from a descriptive viewpoint, there are of course several reasons for questioning the (α)-axiom. (See, Sen 1993, 1997 and Kalai et al., 2002.)

[17] During the second revision of this paper we became aware that one of the weakenings of WARP that are considered in Bandyopadhay and Sengupta (1993) is identical to WARNI in the context of finite choice spaces (other than a minor technical wrinkle). However, both the analysis and the conclusions drawn from WARNI here are drastically different from those of that paper, even under the assumption of the finiteness of choice spaces. (See Remark 2(2).)

[18] The proofs of all theorems reported below appear in Appendix A.

Theorem 2. *Let (X, Ω_X) be a choice space, and c any choice correspondence on Ω_X. If c satisfies WARNI, then there exists a unique (not necessarily complete) regular preference relation $\succsim$ on X such that $c = \max(\cdot, \succsim)$. Conversely, if Ω_X includes all countable subsets of X and $c = \max(\cdot, \succsim)$ for some preference relation $\succsim$ on X, then c satisfies WARNI.*

In the case of finite choice spaces, we can state this result in a more compact form.

Corollary 1. *Let (X, Ω_X) be a finite choice space and $c : \Omega_X \rightrightarrows X$. Then, c is a choice correspondence on Ω_X that satisfies WARNI if, and only if, there exists a (unique but not necessarily complete) regular preference relation $\succsim$ on X such that $c = \max(\cdot, \succsim)$.*

Despite its elementary nature, Theorem 2 provides a choice-theoretic foundation for incomplete preferences much the same way Theorem 1 "derives" complete preferences from choice behavior. In particular, in the context of finite choice spaces, Corollary 1 shows that relaxing WARP to WARNI in Theorem 1 has the effect of relaxing the completeness property of the rationalizing preference relations to their regularity. Therefore, choice correspondences that satisfy WARNI live well within the confines of the "maximizing paradigm," albeit by means of possibly incomplete (yet regular) preference relations.

While various weakenings of WARP have been examined in the literature on revealed preference theory, a great majority of the related studies have sought to identify the implications of this for the transitivity of the rationalizing preference relations. The incompleteness of preferences is not at all a theme that emerges in this literature, presumably because of the commonly held belief that "choices" that we observe, per force, inform us about what an agent ranks higher than the other feasible alternatives. [19] Theorem 2 could be viewed as a formal response to this belief.

Remark 2. (1) One can show that WARNI implies the (α)-axiom. (See Lemma 1 in Appendix A.) Thus, Theorems 1 and 2 show together that, given the normatively appealing (α)-axiom, the demarcation between the completeness and incompleteness of preferences of a rational agent lies in the comparative plausibility of the (β)-axiom and WARNI.

(2) The use of regularity in Theorem 2 is crucial both conceptually and mathematically. The former point will become transparent in the next subsection. To clarify the latter point, we note that Theorem 2 remains valid if we dropped the phrase "unique regular" in its statement. But that version is hardly interesting. After all, it is an easy exercise to show that if c satisfies WARNI, then $c = \max(\cdot, \succsim^*)$ for the (generally irregular) preference relation $\succsim^*$ on X defined by $x \succ^* y$ iff $\{x\} = c\{x, y\}$, and $x \sim^* y$ iff $x = y$. This point also clarifies the connection between the present work and the part of revealed preference theory that examines the implications of weakening WARP with regards to the *transitivity* of the rationalizing binary relations. Indeed, another way of stating the previous claim is this: If c satisfies WARNI, then $c = \max(\cdot, \succsim_*)$ for the complete binary relation $\succsim_*$ on X defined by $x \succ_* y$ iff $\{x\} = c\{x, y\}$, and $x \sim_* y$ iff

[19] Nehring (1997) stands as an exception in this literature. That paper indeed takes incompleteness of preferences seriously, but focuses on those choice correspondences that may be rationalized only by non-binary relations. In this sense, Nehring's main plot is to weaken WARP to a further extent than we do here, so, for better or worse, his approach exhibits a less parsimonious departure from the standard theory of choice. By contrast, our objective here is to understand exactly which axiom of the standard theory is responsible for the deduced "completeness" of rationalizing preference relations, and how one may weaken this axiom to retain maximizing behavior, albeit using incomplete preference relations.

$\{x, y\} = c\{x, y\}$.[20] The difference between $\succsim^*$ and $\succsim_*$ is purely academic. $\succsim^*$ does not allow for any indifference and $\succsim_*$ rules out any incomparability. Put differently, what $\succsim^*$ calls incomparability, $\succsim_*$ calls indifference, and vice versa. $\succsim^*$ is transitive, but since there is no indifference relative to this relation, this is equivalent to saying that $\succ^*$ is transitive, whereas $\succsim_*$ is a non-transitive relation with $\succ_*$ being transitive.

What makes Theorem 2 interesting is that the rationalizing preference relation that it derives has non-trivial indifference and incomparability parts, and yet it is transitive. Moreover, we shall see in the next subsection that one may in fact identify the difference between these two parts in observable terms.

Remark 3. The countability requirement used in the second part of Theorem 2 stems from a technicality. Indeed, in the context of an arbitrary choice space, a choice correspondence which is rationalized by a regular preference relation need not satisfy WARNI. To give an example, let X be a countably infinite set, which we enumerate as $\{x, y, x^1, x^2, \ldots\}$, and $\Omega_X := \{S \in 2^X \backslash \{\emptyset\}\colon |S| = \infty \text{ only if } x \in S\}$. Clearly, (X, Ω_X) is a choice space. Now consider the choice correspondence $c := \max(\cdot, \succsim)$ on Ω_X, where $\succsim$ is the preference relation defined by $\cdots \succ x^2 \succ x^1 \succ y$ and $\sim := \{(x, x)\colon x \in X\}$. (Notice that x is not comparable to any other alternative in X.) Then $c(X) = \{x\}$ while $y \in c\{x, y\}$, so c fails to satisfy WARNI.

Corollary 1 shows that this sort of a difficulty cannot arise in finite choice spaces. We shall show below that this is also the case in the context of compact choice spaces.

Since the basic structures of the standard choice theory and a choice theory that would be based on WARNI are quite similar, one might expect that the latter would still have considerable descriptive restrictions. In closing, we note that there are indeed interesting choice correspondences that would *not* be captured by such a theory.

Example 3 *(Reason-Based Choice).* Let $X := \{x, y, z\}$, and consider the decision maker of Example 1. If the choice behavior of this individual is dominance-based, then we would have $c\{x, y, z\} = \{x, y\}$. However, there is some experimental evidence that suggest that in this case many people would in fact choose y uniquely from $\{x, y, z\}$. Psychologists have offered the following explanation: An individual may try to settle her choice problem by looking for "reasons" to choose between the alternatives that she finds difficult to compare, and the fact that y dominates z, and x does not, may well provide a "reason" for the individual to choose y, but not x, from the set $\{x, y, z\}$.[21] It is clear that such a choice behavior (i.e. $c\{x, y\} = \{x, y\}$, $c\{x, z\} = \{x, z\}$, $c\{y, z\} = \{y\}$, and $c\{x, y, z\} = \{y\}$) would violate WARNI (but not the (α)-axiom).

Examples 1 and 3 show that there is a wedge between dominance-based and reason-based decision making. Broadly speaking, the former belongs to the "maximizing paradigm" (The-

[20] This is, in fact, the only result that Bandyopadhay and Sengupta (1993) derives from WARNI, who work within the context of *finite* choice spaces. In the same context, Schwartz (1976) provides an alternative characterization of those choice correspondences that are rationalized by binary relations whose strict parts are transitive.

[21] This phenomenon, also known as the "attraction effect," was first identified by Huber et al. (1982). Among other experimental studies that provide further support for it are Simonson (1989), Tversky and Shafir (1992), Shafir et al. (1993), and Redelmeier and Shafir (1995).

orem 2), while it seems like the latter requires one to build a theory of "procedural" choice. (See Ok, 2005.)

2.6. Indifference vs. incomparability

Theorem 2 indicates that much of our standard maximization paradigm would be retained in the presence of WARNI, provided that we allow for the incompleteness of the individual preference relations. Nevertheless, this result does not readily provide information about when one can determine if an agent is indifferent or indecisive between two or more alternatives, *on the basis of her choice behavior*. In this sense, it does not draw an exact parallel to Theorem 1. After all, Theorem 1 tells us exactly when an agent strictly prefers an alternative x over y (the case where x is chosen uniquely from a set that contains both x and y), when she is indifferent between x and y (the case where both x and y are chosen from a feasible set), and when she is indecisive between x and y (the case that never occurs). The task that we wish to undertake here is harder in that we also wish to understand if a given decision maker who is observed to have deemed both x and y "choosable" in a choice situation, feels indifferent between x and y or is indecisive between these outcomes.[22]

It is not difficult to find a sufficient condition—in terms of "choices"—for determining when an agent, whose choice correspondence c is induced by maximizing a preference relation, is indecisive between two alternatives x and y. It turns out that all we need to check is if (x, y) is c-incomparable.

Proposition 1. *Let (X, Ω_X) be a choice space. If c is a choice correspondence on Ω_X such that $c = \max(\cdot, \succsim)$ for some preference relation $\succsim$ on X, then*

$$\mathcal{I}(c) \subseteq \bowtie .$$

Proof. Take any $x, y \in X$ such that $x \bowtie y$ is false. If either $x \succ y$ or $y \succ x$ holds, then $(x, y) \notin \mathcal{P}(c)$, so $(x, y) \notin \mathcal{I}(c)$. Suppose, then, $x \sim y$, and take any finite $S \in \Omega_X$ with $x \in S$ and $y \notin S$. Notice that if $z \in (c(S)\backslash\{x\})\backslash c(S_{y,-x})$, then $c = \max(\cdot, \succsim)$ implies $y \succ z$. Since $x \sim y$, therefore, we find $x \succ z$, which contradicts $z \in c(S) = \max(S, \succsim)$. The analogous argument shows that $(c(S_{y,-x})\backslash\{y\})\backslash c(S) = \emptyset$ as well. That is, $c(S)\backslash\{x\} = c(S_{y,-x})\backslash\{y\}$. Moreover, since $x \sim y$ and $c = \max(\cdot, \succsim)$, we obviously have $x \in c(S)$ iff $y \in c(S_{y,-x})$. It follows that $(x, y) \notin \mathcal{I}(c)$, and hence the claim. □

This is a useful observation, especially with regards to experimental studies on individual choice. It tells us that, provided that the choice correspondence c of a subject is found to satisfy WARNI (and hence to be rationalized by a possibly incomplete preference relation $\succsim$), we may identify the alternatives that the agent cannot compare decisively (in terms of $\succsim$) simply by examining the contents of $\mathcal{I}(c)$. Put differently, Proposition 1 tells us when one can draw a distinction between indifference and indecisiveness of a decision maker upon *observing* her choice behavior.

Proposition 1 gives only a sufficient condition for identifying pairs of alternatives that the intrinsic preferences of an individual fail to rank. However, as we show next, if the preference re-

[22] All existing studies on revealed preferences that have worked with variations of WARP avoid this issue at the outset by demanding, per force, that choices be rationalized by a complete relation (which need not be transitive).

lation of the agent is regular—a weak requirement, to be sure—this condition becomes necessary as well.

Proposition 2. *Let (X, Ω_X) be a choice space. If c is a choice correspondence on Ω_X such that $c = \max(\cdot, \succsim)$ for some regular preference relation $\succsim$ on X, then*

$$\mathcal{I}(c) = \bowtie.$$

Proof. We only need to prove that $\bowtie \subseteq \mathcal{I}(c)$. To this end, take any $x, y \in X$ with $x \bowtie y$. Then $c = \max(\cdot, \succsim)$ implies $(x, y) \in \mathcal{P}(c)$. Since $\succsim$ is regular, there exists a $z \in X$ such that

$$\left|\bowtie \cap \{(x,z),(y,z)\}\right| = 1 = \left|\left(\succ \cup \succ^{-1}\right) \cap \{(x,z),(y,z)\}\right|.$$

Suppose, without loss of generality, $x \bowtie z$, so we have either $y \succ z$ or $z \succ y$. Since $c = \max(\cdot, \succsim)$, we have $c\{y, z\} = \{y\}$ in the former case, and $c\{y, z\} = \{z\}$ in the latter case, whereas $c\{x, z\} = \{x, z\}$ in either contingencies. Then, for $S := \{x, z\}$, we have $c(S)\backslash\{x\} \neq c(S_{y,-x})\backslash\{y\}$ in the former case, while $x \in c(S)$ but $y \notin c(S_{y,-x})$ in the latter case. Therefore, $(x, y) \in \mathcal{I}(c)$. □

One operational way in which one may detect indifference of a "rational" decision maker between two alternatives x and y with $\{x, y\} = c\{x, y\}$ is to check if from any choice problem T that contains x and y, the agent would choose y whenever she chooses x and vice versa. Thus, if this is not the case, that is, if we observe instead that $x \in c(T)$ while the individual declares that she would never choose y from T, then it would seem more natural to conclude that the agent in fact found x and y incomparable, and not indifferent, in the case of the problem $\{x, y\}$. The next result shows that this is entirely consistent with the notion of c-incomparability introduced above.

Proposition 3. *Let (X, Ω_X) be a choice space, c a choice correspondence on Ω_X that satisfies WARNI, and $(x, y) \in \mathcal{P}(c)$. If there exists a finite set $T \in \Omega_X$ such that $x, y \in T$ and $|c(T) \cap \{x, y\}| = 1$, then (x, y) must be a c-incomparable pair.*

Proof. Let $T \in \Omega_X$ be a finite set with $x, y \in T$ and $|c(T) \cap \{x, y\}| = 1$, and define $S := T\backslash\{y\}$. Suppose first that $x \in c(T)$. Then, if $y \in c(T\backslash\{x\})$, then $\{x, y\} = c\{x, y\}$ and WARNI entail that $y \in c(T)$, contradicting $|c(T) \cap \{x, y\}| = 1$. Thus $y \notin c(T\backslash\{x\}) = c(S_{y,-x})$, while by the (α)-axiom (Remark 2(1)), $x \in c(S)$, so we may conclude that $(x, y) \in \mathcal{I}(c)$. Suppose next that $x \notin c(T)$. In that case, $y \in c(T)$, and an analogous argument yields $y \in c(S_{y,-x})$ while $x \notin c(S)$, allowing us to conclude again that $(x, y) \in \mathcal{I}(c)$. □

Since c-incomparability is shown above to be closely related to the indecisiveness of a decision maker (whose choices abide by WARNI), Proposition 3 can be thought of as providing an easy-to-check sufficiency condition for determining when an agent makes a decision *indecisively*.

2.7. Choice and continuous (incomplete) preference relations

Corollary 1 shows that, in the context of *finite* choice spaces, WARNI yields a full characterization of those choice correspondences that are rationalized by regular, but not necessarily complete, preference relations. In this subsection we generalize this result to the context of all compact choice spaces, thereby substantially extending the coverage of our findings so far. This generalization will be put to good use in the next section where we work with risky choices.

Throughout this subsection, we assume that (X, Ω_X) is a compact choice space. As is common, we view Ω_X as a metric space on its own right by using the Hausdorff metric.[23] Moreover, we say that a preference relation $\succsim$ on X is *continuous* if $\succsim$ is a closed subset of $X \times X$ with respect to the product metric, that is, if, for any convergent sequences (x_n) and (y_n) in X with $x_n \succsim y_n$ for each $n = 1, 2, \ldots$, we have $\lim x_n \succsim \lim y_n$. For consumer-theoretic applications (especially those that involve risk), it would be useful to derive continuous preference relations from the choice behavior of individuals; indeed, this is one of the common themes of the classical revealed preference theory. When WARP is postulated on choice behavior, this is an easy exercise. For, in this case the rationalizing preference relation is complete (Theorem 1), so assuming that c is upper hemicontinuous on Ω_X, one can prove that this relation is in fact continuous. The present setting, however, allows for incomplete preferences, thereby rendering this approach inapplicable. It turns out that upper hemicontinuity is, in general, too demanding a condition to posit on choice correspondences. For instance, the widely used *Pareto choice correspondence* fails to satisfy this property.[24]

To "derive" a continuous preference relation from a choice correspondence c on Ω_X, then, we need to posit weaker continuity properties on it. We propose to impose the following property for this purpose.

(C)-axiom.
$(C1)$ $\mathcal{I}(c)$ is an open subset of X^2.
$(C2)$ For any convergent sequences $(x_n) \in X^\infty$ and $(S_n) \in \Omega_X^\infty$, if, for each n, $x_n \in c(S_n)$ and $(x_n, y_n) \notin \mathcal{I}(c)$ for all $y_n \in S_n$, then $\lim x_n \in c(\lim S_n)$.

The (C)-axiom consists of two parts. While its second part corresponds to a straightforward relaxation of upper hemicontinuity, its first part says simply that if an individual cannot compare two alternatives x and y, and x' and y' are arbitrarily close to these alternatives respectively, then she also cannot compare x' and y'. The Pareto choice correspondence on a Euclidean space, for instance, satisfies both of these properties; we shall point to some other economic examples in the next section.

Combining WARNI with the (C)-axiom yields the following characterization result.

Theorem 3. *Let (X, Ω_X) be a compact choice space and $c : \Omega_X \rightrightarrows X$. Then, c is a choice correspondence on Ω_X that satisfies WARNI and the (C)-axiom if, and only if, there exists a (unique but not necessarily complete) continuous regular preference relation $\succsim$ on X such that $c = \max(\cdot, \succsim)$.*

---

23 Let d denote the metric of X, and let $d(b, A)$ stand for $\inf_{a \in A} d(b, a)$ for any nonempty subset of X. The *Hausdorff metric* is the function $d^H : \Omega_X^2 \to \mathbb{R}_+$ defined by $d^H(S, T) := \max\{\sup_{x \in S} d(x, T), \sup_{y \in T} d(y, S)\}$.

24 *Example.* Let $X := [0, 1]^2$ and define the (Pareto) choice correspondence c on Ω_X by $c(S) := \{x \in S\colon\ y > x$ for no $y \in S\}$. If $S_n := \{(a, b) \in X\colon\ b \leqslant n(1 - a)\}$, $n = 1, 2, \ldots$, then $\lim S_n = X$ and $(1, 0) \in c(S_n)$ for each n, but $(1, 0) \notin c(X)$, so c is not upper hemicontinuous.

While Corollary 1, especially when combined with Propositions 1–3, is suitable for experimental work on individual choice, Theorem 3 extends the coverage of our theory to the context of most economic applications.

3. Choice and expected multi-utility representation

3.1. The main representation theorem

The choice behavior that is envisaged by Theorem 3 suggests modeling "rational" decision-makers as aiming to maximize a regular preference relation on the relevant outcome space. This sort of a model is, however, rather inconvenient to work with in many economic situations that are traditionally investigated by the aid of utility functions. Moreover, the present model is not suitable for choice problems that involve risk, which is obviously a significant shortcoming. The main objective of this section is thus to extend the theory to the case where the items of choice are lotteries, and provide a pragmatic way of introducing incomplete preferences into applied decision theory by using (cardinal) utility functions.[25]

As usual, we designate an arbitrary compact metric space Z as the set of all (riskless) prizes, and let $\mathbf{C}(Z)$ denote the Banach space of all continuous real maps on Z (under the sup-norm). By a *lottery*, we mean a Borel probability measure on Z, and denote the set of all lotteries by $\mathbb{P}(Z)$. For any $z \in Z$, we denote by δ_z the lottery that is degenerate on z, that is, the probability measure that puts full mass on the singleton set $\{z\}$. By δ_Z, we mean the set of all degenerate lotteries on Z, that is, $\delta_Z := \{\delta_z\colon z \in Z\}$.

We endow $\mathbb{P}(Z)$ with some metric that induces the topology of weak convergence.[26] It is well-known that this renders $\mathbb{P}(Z)$ a compact metric space. Adopting the notation of the previous section, we let $\Omega_{\mathbb{P}(Z)}$ denote the set of all nonempty closed subsets of $\mathbb{P}(Z)$. (Thus: $(\mathbb{P}(Z), \Omega_{\mathbb{P}(Z)})$ is a compact choice space.) Also, for any $\lambda \in [0,1]$, we define the binary operation $\oplus_\lambda$ on $\mathbb{P}(Z)$ by

$$p \oplus_\lambda q := \lambda p + (1-\lambda)q.$$

As usual, $p \oplus_\lambda q$ corresponds to the compound lottery in which p is played with probability λ and q is played with probability $1-\lambda$. Moreover, for any $(r, S) \in \mathbb{P}(Z) \times \Omega_{\mathbb{P}(Z)}$, we define

$$S \oplus_\lambda r := \{p \oplus_\lambda r\colon\ p \in S\}.$$

A good portion of risk theory is developed under the assumption that there is a best and worst prize in Z in some objective sense. While it simplifies the theoretical analysis considerably, this requirement is met in most economic contexts. Here we adopt an even weaker assumption.

(*B*)-axiom. There is a $z^* \in Z$ such that $\{\delta_{z^*}\} = c(\delta_Z)$.

---

25 The problem of obtaining a (continuous) utility representation for an incomplete preference relation is a highly technical matter, which was first investigated by Aumann (1962) and Peleg (1970). Unfortunately, in the ordinal framework, this exercise is hopelessly difficult (cf. Sondermann, 1980 and Ok, 2002). As shown by Dubra et al. (2004), however, the situation is much more promising in the case of the von Neumann–Morgenstern setup. Our main result, Theorem 4, is in fact built on the findings of that paper.

26 That is, for any sequence (p_n) in $\mathbb{P}(Z)$, $p_n \to p$ means that $\int_Z f\,\mathrm{d}p_n \to \int_Z f\,\mathrm{d}p$ for all $f \in \mathbf{C}(Z)$.

The interpretation of this property is straightforward: there is a riskless outcome $z^* \in Z$ which is the "best" outcome in Z. For instance if Z models a monetary prize space or a product commodity space, then the (B)-axiom would be an unexceptionable assumption.[27]

The following postulate is crucial for the main result of this section.

(I)-axiom. For any $p, q, r \in \mathbb{P}(Z)$, $S \in \Omega_{\mathbb{P}(Z)}$ and any $\lambda \in (0, 1)$:
$(I1)$ If $p \in c(S)$, then $p \oplus_\lambda r \in c(S \oplus_\lambda r)$;
$(I2)$ If $(p \oplus_\lambda r, q \oplus_\lambda r) \in \mathcal{I}(c)$, then $(p, q) \in \mathcal{I}(c)$.

The first part of this property is a straightforward reflection of the standard independence axiom of expected utility theory. It says that if a lottery p is not revealed inferior to any alternative in a choice problem S, then any given mixing of p with another fixed lottery r should not be revealed inferior to the same mixing of any member of S with r. $(I2)$, on the other hand, says that if a particular mixing of a lottery p with another lottery r is revealed to be incomparable to the same mixing of a lottery q with r, then this incomparability must be a consequence of only the *variable* part of these compound lotteries, that is, it must be the case that p and q are incomparable. As in standard theory, the (I)-axiom is quite appealing from the normative perspective. Just like the standard independence axiom, however, it is likely to be found demanding from the descriptive angle.

We need a final bit of notation for the statement of the main result of this section. For any nonempty $\mathcal{U} \subseteq \mathbf{C}(Z)$ and $p \in \mathbb{P}(Z)$, we write $\boldsymbol{E}_{p,\mathcal{U}}$ to mean the function that maps $\mathcal{U}$ into $\mathbb{R}$ as

$$\boldsymbol{E}_{p,\mathcal{U}}(u) := \int_Z u \, \mathrm{d}p.$$

Therefore, for any $p, q \in \mathbb{P}(Z)$, the expression $\boldsymbol{E}_{p,\mathcal{U}} \geqslant \boldsymbol{E}_{q,\mathcal{U}}$ is understood pointwise, that is,

$$\boldsymbol{E}_{p,\mathcal{U}} \geqslant \boldsymbol{E}_{q,\mathcal{U}} \quad \text{if and only if} \quad \int_Z u \, \mathrm{d}p \geqslant \int_Z u \, \mathrm{d}q \in \text{ for all } u \in \mathcal{U}.$$

We write $\boldsymbol{E}_{p,\mathcal{U}} > \boldsymbol{E}_{q,\mathcal{U}}$ to mean $\boldsymbol{E}_{p,\mathcal{U}} \geqslant \boldsymbol{E}_{q,\mathcal{U}}$ but not $\boldsymbol{E}_{q,\mathcal{U}} \geqslant \boldsymbol{E}_{p,\mathcal{U}}$. The following result is the most structured revealed preference theorem that we are able to report in this paper.

Theorem 4. *Let Z be a compact metric space. Then, c is a choice correspondence on $\Omega_{\mathbb{P}(Z)}$ that satisfies WARNI and the (C), (B) and (I)-axioms if, and only if, there exists a nonempty closed and convex set $\mathcal{U}$ in $\mathbf{C}(Z)$ such that*

$$\bigcap_{u \in \mathcal{U}} \arg\max_{z \in Z} u(z) \quad \textit{is a singleton set},$$

and

$$c(S) = \{p \in S\colon\ \boldsymbol{E}_{q,\mathcal{U}} > \boldsymbol{E}_{p,\mathcal{U}} \textit{ for no } q \in S\}, \qquad S \in \Omega_{\mathbb{P}(Z)}.$$

This result shows that the WARNI and the (C), (B) and (I)-axioms yield a choice model with quite a bit of structure. This model envisages that a decision maker is endowed with a *set*

[27] If Z lies in a Euclidean space (or any partially ordered linear space), then the (B)-axiom can be replaced in what follows by the standard monotonicity, or local non-satiation, properties.

of utility functions, where we may interpret each utility function as corresponding to a complete ranking of the alternatives in terms of some attribute.[28] If there are one or more alternatives in a given choice set that the agent finds superior to all other feasible alternatives, then the agent chooses these particular alternatives. On the other hand, the agent becomes indecisive when she needs to choose between two alternatives one of which is superior in one attribute, and the other is superior in another attribute. In this case, the choice model does not specify how the final decision is rendered (which, in our interpretation (Section 2.1), is settled upon (subjective) randomization).[29]

3.2. An application: the preference reversal phenomenon

Among the many experimental observations that refute the basic premises of expected utility theory, a particularly striking one is the so-called *preference reversal* (*PR*) *phenomenon*. The typical experiment that gives rise to this phenomenon asks the decision maker to choose between two lotteries, one offering a high probability of winning a small amount of money, and the other offering a low probability of winning a large payoff. For clarity, let h stand for the lottery that pays \$10 with probability 8/9 and nothing otherwise (i.e. $h := \delta_{10} \oplus_{8/9} \delta_0$), and let ℓ stand for the lottery that pays \$85 with probability 1/9 and nothing otherwise (i.e. $\ell := \delta_{85} \oplus_{1/9} \delta_0$). When confronted with such a choice problem, most individuals are observed to prefer h over ℓ (at least weakly) in the experiments. The subjects are then asked to state the minimum price at which they would be willing to sell h and ℓ (had they owned these lotteries), and surprisingly, most have found to charge a strictly higher price for ℓ than for h; hence the term *preference reversal* (*PR*) *phenomenon*.[30]

To see what can be said about this phenomenon within the choice theory developed here, let $Z := [0, z]$ for some $z > 85$, and assume that the choice correspondence c of a decision maker on $\Omega_{\mathbb{P}(Z)}$ satisfies all hypotheses of Theorem 3. We assume, in addition, that c is *monotonic*, that is, $c\{\delta_a, \delta_b\} = \{\delta_a\}$ for all $a, b \in Z$ with $a > b$. Hence, there exists a continuous preorder $\succsim$ on $\mathbb{P}(Z)$ such that $c = \max(\cdot, \succsim)$, where $\delta_a \succ \delta_b$ for any $a, b \in Z$ with $a > b$. Can such a choice correspondence account for the choices stated above? The answer is easily seen to be no, if $\succsim$ is a *complete* preorder. For, in that case we would have $p \sim \delta_{\eta_c(p)}$ for all $p \in \mathbb{P}(Z)$, where $\eta_c(p)$ is the *minimum selling price* for p induced by c, that is,

$$\eta_c(p) := \inf\bigl\{a \in Z\colon\ c\{p, \delta_a\} = \{\delta_a\}\bigr\}, \quad p \in \mathbb{P}(Z).$$

In turn, this would entail $h \succsim \ell \sim \delta_{\eta_c(\ell)} \succ \delta_{\eta_c(h)} \sim h$, contradicting the transitivity of $\succsim$. It is this observation that makes one view the PR-phenomenon as an apparently serious challenge to the

[28] This set is unique up to taking closed and convex hulls (cf. Dubra et al., 2004). The induced preference relation that rationalizes the choice correspondence c as in Theorem 4 is, in turn, unique. (This is a consequence of the uniqueness part of Theorem 3.)

[29] Alternatively, we can think of each $u \in \mathcal{U}$ as the utility function of a "future self" of the agent. With this interpretation, Theorem 4 can be thought of as providing a purely choice-theoretic foundation for the recently popular *multi-objective* (or *multi-self*) decision making paradigm.

[30] This phenomenon was first demonstrated by Slovic and Lichtenstein (1968), and then explored carefully by Grether and Plott (1979). While later studies by Holt (1986), Karni and Safra (1987), Segal (1988), and Safra et al. (1990) have shown that the problem may in fact be the violation of a certain aspect of the independence axiom, the experimental studies of Tversky et al. (1990), and Loomes et al. (1991) have shown that the PR-phenomenon indeed points to something deeper than this.

positive strength of the expected utility paradigm; but this, notably, only when $\succsim$ is *assumed* to be complete.

Another way of saying this is that if c is a monotonic choice correspondence on $\Omega_{\mathbb{P}(Z)}$ that satisfies WARP and the (C)-axiom, then it simply cannot choose h over ℓ while $\eta_c(\ell) > \eta_c(h)$.[31] However, as we demonstrate next, these properties can well be satisfied by a choice correspondence that satisfies the postulates of Theorem 4.

Example 4. Let $\mathcal{U}$ be the set of all continuous and strictly increasing real functions on $Z = [0, z]$, and define the choice correspondence c on $\Omega_{\mathbb{P}(Z)}$ by

$$c(S) = \{p \in S\colon\ \boldsymbol{E}_{q,\mathcal{U}} > \boldsymbol{E}_{p,\mathcal{U}} \text{ for no } q \in S\}.$$

Clearly, $c(S)$ consists of all the first order stochastically undominated elements of S, that is, $c = \max(\cdot, \succcurlyeq_{\mathrm{FSD}})$, where $\succcurlyeq_{\mathrm{FSD}}$ stands for the first order stochastic dominance partial order on $\Omega_{\mathbb{P}(Z)}$. By Theorem 4, c satisfies WARNI and the (C), (B) and (I)-axioms. Moreover, it is not difficult to verify that

$$\eta_c(p) \equiv \sup\{\mathrm{ce}_u(p)\colon\ u \in \mathcal{U}\} = \max\bigl(\mathrm{supp}(p)\bigr) \quad \text{for all } p \in \mathbb{P}(Z),$$

where $\mathrm{ce}_u(p)$ is the certainty equivalent of the lottery p relative to the von Neumann–Morgenstern utility function u. (Here $\mathrm{supp}(p)$ stands for the smallest closed subset A of Z with $p(A) = 1$.) Thus $\eta_c(h) = 10$ and $\eta_c(\ell) = 85$, so we see that $\{\delta_{\eta_c(\ell)}\} = c\{\delta_{\eta_c(h)}, \delta_{\eta_c(\ell)}\}$. Yet h and ℓ are $\succcurlyeq_{\mathrm{FSD}}$-incomparable, so we have $h \in c\{h, \ell\}$.[32]

This example shows that insofar as one considers WARNI and the (C), (B) and (I)-axioms as basic tenets of rationality, one need not consider the PR-phenomenon as "irrational." Perhaps more importantly, it demonstrates that an individual whose choice behavior is modeled as in Theorem 4 is likely to *overprice* the lottery ℓ (that offers a low probability of winning a large payoff) in the sense that she would strictly prefer receiving the minimum selling price for ℓ to participating in this lottery. Curiously, this is precisely the phenomenon observed by Tversky et al. (1990) who found that over 65% of the subjects overpriced ℓ in this sense.

We conclude with a caveat. The theory presented here cannot account for the following *strong* version of the PR-phenomenon:

$$\{h\} = c\{h, \ell\} \quad \text{and} \quad \{\delta_{\eta_c(\ell)}\} = c\{\delta_{\eta_c(h)}, \delta_{\eta_c(\ell)}\}. \tag{2}$$

Indeed, a quick application of Theorem 3 shows that no monotonic choice correspondence c that satisfies the WARNI and (C)-axiom can satisfy these two equations. Surprisingly, none of the

[31] For, WARP and the (C)-axiom together imply that $c\{h, \delta_{\eta_c(h)}\} = \{h, \delta_{\eta_c(h)}\}$ and $c\{\ell, \delta_{\eta_c(\ell)}\} = \{\ell, \delta_{\eta_c(\ell)}\}$, and this, in turn, entails that we cannot have $h \in c\{h, \ell\}$ and $\{\delta_{\eta_c(\ell)}\} = c\{\delta_{\eta_c(h)}, \delta_{\eta_c(\ell)}\}$ simultaneously. (The proof of the first claim is straightforward. To see the second one, suppose all four of these statements hold, and let $S := \{h, \delta_{\eta_c(h)}, \ell, \delta_{\eta_c(\ell)}\}$. Then, if $h \in c(S)$, then $\delta_{\eta_c(h)} \in c(S)$ by the (β)-axiom, but this contradicts $\{\delta_{\eta_c(\ell)}\} = c\{\delta_{\eta_c(h)}, \delta_{\eta_c(\ell)}\}$ in view of the (α)-axiom. If, on the other hand, $h \notin c(S)$, then $\eta_c(h), \ell \notin c(S)$ as well, by the (α) and (β)-axioms. Thus in this case $\{\delta_{\eta_c(\ell)}\} = c(S)$, but this contradicts $\{\ell, \delta_{\eta_c(\ell)}\} = c\{\ell, \delta_{\eta_c(\ell)}\}$ by the (β)-axiom.)

[32] These conclusions would also be obtained if we used here the second order stochastic dominance partial order $\succcurlyeq_{\mathrm{SSD}}$ instead of $\succcurlyeq_{\mathrm{FSD}}$. (For, if $c = \max(\cdot, \succcurlyeq_{\mathrm{SSD}})$, then $\eta_c(p)$ equals the expected value of p for any $p \in \mathbb{P}(Z)$.) Nevertheless, we do not, of course, claim here that subjects should be modeled as first or second order stochastic dominance maximizers in these experiments; we only wish to show in Example 4 the *possibility* of exhibiting the PR phenomenon upon maximizing an incomplete preference relation.

previous experiments on the PR-phenomenon distinguish between the strong form (2) and the following *weak* form of this phenomenon:

$$\{h,\ell\} = c\{h,\ell\} \quad \text{and} \quad \{\delta_{\eta_c(\ell)}\} = c\{\delta_{\eta_c(h)}, \delta_{\eta_c(\ell)}\}. \tag{3}$$

By contrast, this distinction is quite important for the present study, for (3) is consistent with maximizing an incomplete preference relation and (2) is not.[33]

4. Conclusion

The standard theory of (static) individual choice is based on the premise of maximizing a utility function. This theory embodies strong predictive power, notably because it is a universal theory that applies to any decision theoretic situation. Unfortunately, there is substantial evidence that suggests that the explanatory power of this theory is not satisfactory. It is therefore of interest if one can modify the standard theory in a way that increases its explanatory power, without losing its universal applicability to choice problems.

In this paper we have focused on the need of weakening the main hypothesis of rational choice theory, namely the weak axiom of revealed preference in a way to allow a "rational" individual to settle her choices by maximizing a possibly incomplete preference relation. The theory developed here provides a choice-theoretic foundation for incomplete preferences, the importance of which was forcefully defended by many authors in the earlier literature (e.g., Aumann, 1962 and Bewley, 1986). Moreover, this theory allows us to distinguish between an agent being indifferent vs. indecisive on the basis of her observed choice behavior. Finally, in the presence of standard axioms of static decision theory, this model takes the form of maximizing a *vector*-valued utility function, which is a time-honored formulation of decision making with multiple objectives.

Acknowledgments

We thank Jean Pierre Benoît, Juan Dubra, Özgür Kıbrıs, Yusufcan Masatlioglu, Debraj Ray, Larry Samuelson, Uzi Segal, Rani Spiegler, and Andy Schotter for helpful comments. We also thank the Associate Editor and an anonymous referee whose comments have led to a significant improvement in the paper. The generous support of the Center for Experimental Social Science and the C.V. Starr Center for Applied Economics at New York University are gratefully acknowledged.

[33] The previous experimental literature on the PR-phenomenon does not distinguish between (2) and (3), because in all of these studies (with the exception of Tversky et al. (1990), subjects were never given the opportunity to choose more than one alternative. In the experiment of Tversky et al. (1990), on the other hand, subjects were given the chance to choose more than one alternative, but the payment scheme was such that the experimenter would pick randomly among the uneliminated lotteries to finalize the payment. Unfortunately, this procedure is rather problematic, due to the potential uncertainty aversion of the subjects. Thus, the fact that no subject has chosen multiple alternatives in that experiment cannot be interpreted as the chosen alternatives were in fact revealed superior to the unchosen ones. All in all, we need new experiments to get an idea about the relative frequencies of (2) and (3) type phenomena, and to determine to what extent the present choice model would be able to cope with these.

Appendix A. Proofs of theorems

We begin with two simple observations.

Lemma 1. *Let (X, Ω_X) be a choice space, and c any choice correspondence on Ω_X. If c satisfies WARNI, then it satisfies the (α)-axiom. Moreover, if $\{x\} = c(S)$ for some $(x, S) \in X \times \Omega_X$, then $\{x\} = c(T)$ for any $T \in \Omega_X$ with $x \in T \subseteq S$.*

Proof. Take any $S, T \in \Omega_X$ with $T \subseteq S$. Let $x \in T$ and $x \in c(S)$. Now observe that, trivially, $x \in c(S)$ and $y \in S$ holds for *all* $y \in c(T)$. Thus, if c satisfies WARNI, we must have $x \in c(T)$. To prove the second assertion, notice that $\{x\} = c(S)$ and $x \in T \subseteq S$ imply $x \in c(T)$ by the (α)-axiom. Take any $y \in T\backslash\{x\}$. Since $y \notin c(S)$, WARNI implies that, for all $T' \in \Omega_X$ with $x \in T'$, we must have $y \notin c(T')$. Setting $T' = T$ yields $y \notin c(T)$. □

The following lemma settles the uniqueness assertion of Theorem 2.

Lemma 2. *If $\succsim$ and $\succsim'$ are two regular preference relations on a nonempty set X such that*

$$\max(S, \succsim) = \max(S, \succsim')$$

for every $S \subseteq X$ with $|S| = 2$, then $\succsim = \succsim'$.

Proof. Take any $x, y \in X$. Observe that

$$x \succ y \quad \text{iff} \quad \{x\} = \max\bigl(\{x, y\}, \succsim\bigr) \quad \text{iff} \quad \{x\} = \max\bigl(\{x, y\}, \succsim'\bigr) \quad \text{iff} \quad x \succ' y$$

so we have $\succ = \succ'$. Now let $x \sim' y$, but assume that $x \sim y$ does not hold. Then, since $\succ = \succ'$, we have $x \bowtie y$, so by regularity of $\succsim$, there exists a $z \in X$ such that

$$\bigl|\bowtie \cap \bigl\{(x, z), (y, z)\bigr\}\bigr| = 1 = \bigl|\bigl(\succ \cup \succ^{-1}\bigr) \cap \bigl\{(x, z), (y, z)\bigr\}\bigr|.$$

Suppose, without loss of generality, $x \bowtie z$, so we have either $y \succ z$ or $z \succ y$. Then, since $\succ = \succ'$, we have $x \sim' y \succ' z$ in the former case, and $z \succ' y \sim' x$ in the latter case. In either contingencies, then, $\max(\{x, z\}, \succsim) \neq \max(\{x, z\}, \succsim')$, a contradiction. Thus $\sim' \subseteq \sim$, and the converse containment is obtained by interchanging the roles of $\sim$ and $\sim'$ in this argument. □

Proof of Theorem 2. Consider first the binary relation $\sim$ defined on X by

$$x \sim y \quad \text{if and only if} \quad x = y \text{ or } (x, y) \in \mathcal{P}(c)\backslash\mathcal{I}(c).$$

Obviously, $\sim$ is reflexive. It is also symmetric. To see this, take any $(x, y) \in \mathcal{P}(c)\backslash\mathcal{I}(c)$, and notice that $(y, x) \in \mathcal{P}(c)$ holds obviously. We wish to show that $(y, x) \notin \mathcal{I}(c)$. To this end, take any finite set $T \in \Omega_X$ with $y \in T$ and $x \notin T$. Let $S := T_{x,-y}$. Since $(x, y) \notin \mathcal{I}(c)$, we have $x \in c(S)$ iff $y \in c(S_{y,-x})$, that is, $y \in c(T)$ iff $x \in c(T_{x,-y})$. Moreover,

$$c(T_{x,-y})\backslash\{x\} = c(S)\backslash\{x\} = c(S_{y,-x})\backslash\{y\} = c(T)\backslash\{y\}.$$

Thus $(y, x) \notin \mathcal{I}(c)$, and we may conclude that $\sim$ is symmetric.

Now define the binary relation $\succ$ on X by

$$x \succ y \quad \text{if and only if} \quad \{x\} = c\{x, y\} \text{ and } x \neq y. \tag{A.1}$$

It is obvious that $\succ$ is asymmetric and disjoint from $\sim$. Finally, we define $\succsim := \succ \cup \sim$. It is readily verified that $\succsim$ is a reflexive binary relation on X the symmetric and asymmetric parts of which equal $\sim$ and $\succ$, respectively.

Suppose that $x \in c(S)$ for some $S \in \Omega_X$, but that $y \succ x$ for some $y \in S$. By definition of $\succsim$, the latter statement implies $\{y\} = c\{x, y\}$ which contradicts the (α)-axiom. Thus $c \subseteq \max(\cdot, \succsim)$. To establish the converse containment, take any $S \in \Omega_X$, let $x \in \max(S, \succsim)$ and assume that $x \notin c(S)$. By WARNI, there exists a $z \in c(S)$ such that $x \notin c(T)$ for any $T \in \Omega_X$ with $z \in T$. But then $\{z\} = c\{x, z\}$, that is, $z \succ x$, contradicting that $x \in \max(S, \succsim)$. We thus conclude that $c = \max(\cdot, \succsim)$ which is a fact we use frequently below.

To show that $\succsim$ is transitive, take any distinct $x, y, z \in X$ with $x \succsim y \succsim z$. If $x \succ y \succ z$, then, $\{x\} = c\{x, y\}$ and $\{y\} = c\{y, z\}$, so by applying the (α)-axiom twice, we get $\{x\} = c\{x, y, z\}$, and it follows from Lemma 1 that $\{x\} = c\{x, z\}$, that is, $x \succ z$. If $x \succ y \sim z$, then let $S := \{x, z\}$, and notice that if $z \in c(S)$ was the case, then $y \sim z$ would imply that $y \in c(S_{z,-y}) = c\{x, y\}$, contradicting $x \succ y$. If, on the other hand, $x \sim y \succ z$, then let $S := \{y, z\}$, and note that $x \sim y$ implies $\emptyset = c(S)\backslash\{y\} = c(S_{x,-y})\backslash\{x\}$, so $c\{x, z\} = c(S_{x,-y}) = \{x\}$, that is, $x \succ z$.

Finally, consider the case $x \sim y \sim z$. We may assume that x, y and z are distinct, otherwise the claim is trivial. By the findings of the previous paragraph, it is evident that $x \succ z$ or $z \succ x$ cannot hold, so we must have $(x, z) \in \mathcal{P}(c)$. We wish to show that $(x, z) \notin \mathcal{I}(c)$. Take any finite $S \in \Omega_X$ with $x \in S$ and $z \notin S$. If $x \in c(S)$ but $z \notin c(S_{z,-x})$, then since $c = \max(\cdot, \succsim)$, we must have $w \succ z$ for some $w \in S\backslash\{x\}$. Since $z \sim y \sim x$, then, we also have $w \succ y \sim x$, and hence $w \succ x$, contradicting $x \in \max(S, \succsim)$. Conversely, if $x \notin c(S)$ but $z \in c(S_{z,-x})$, then $w \succ x$, and hence $w \succ y$, for some $w \in S$. Now let $T := S_{z,-x}\backslash\{y\}$. By the ($\alpha$)-axiom $z \in c(T)$, and yet, since $w \in T_{y,-z}$, we have $y \notin c(T_{y,-z})$, contradicting $y \sim z$. We thus conclude that

$$x \in c(S) \quad \text{if and only if} \quad z \in c(S_{z,-x}).$$

On the other hand, if $c(S)\backslash\{x\} \neq c(S_{z,-x})\backslash\{z\}$, say if there exists a $w \in c(S)\backslash\{x\}$ which does not belong to $c(S_{z,-x})$, then $x \sim y \sim z \succ w$, so we have $x \succ w$, contradicting $w \in c(S)$. The analogous argument yields also that there is no $w \in c(S_{z,-x})\backslash\{z\}$ that does not belong to $c(S)$, and hence we may conclude that

$$c(S)\backslash\{x\} = c(S_{z,-x})\backslash\{z\}.$$

Thus $(x, z) \in \mathcal{P}(c)\backslash\mathcal{I}(c)$, that is, $x \sim z$, as we sought.

It remains to establish that $\succsim$ is regular. To this end, take any $x, y \in S$ such that $x \bowtie y$. By definition of $\succsim$, then, there exists a finite $S \in \Omega_X$ with $x \in S$ and $y \notin S$ such that at least one of the following cases is true.

(a) $x \in c(S)$ and $y \notin c(S_{y,-x})$;
(b) $x \notin c(S)$ and $y \in c(S_{y,-x})$;
(c) there is a $z \in (c(S)\backslash\{x\})\backslash c(S_{y,-x})$;
(d) there is a $z \in (c(S_{y,-x})\backslash\{y\})\backslash c(S)$.

If case (a) is true, then there exists a $z \in S_{y,-x}$ such that $z \succ y$. Since $x \in c(S)$ we cannot have $z \succ x$. We cannot have $x \succsim z$ either, for otherwise, by transitivity of $\succsim$ we obtain the contradiction $x \succ y$. Thus: $x \bowtie z \succ y$. Moreover, analogous arguments yield a $z \in X$ with $y \bowtie z \succ x$ in case (b), and show that we must have $y \succ z \bowtie x$ and $x \succ z \bowtie y$ in cases (c) and (d), respectively.

To prove the second assertion of Theorem 2, assume that Ω_X includes all countable subsets of X, and let $\succsim$ be a preference relation on X, and suppose $c = \max(\cdot, \succsim)$. We wish to show that

c satisfies WARNI. To this end, take any $S \in \Omega_X$ and $y \in S$, and suppose that for each $x \in c(S)$ we can find a $T \in \Omega_X$ such that $x \in T$ and $y \in c(T)$. It follows that $x \succ y$ cannot hold for any $x \in c(S)$, that is,

$$y \in \max\bigl(c(S) \cup \{y\}, \succsim\bigr). \tag{A.2}$$

To obtain a contradiction, suppose that $y \notin c(S)$, that is, there exists an $x^1 \in S$ such that $x^1 \succ y$. By (A.2), $x^1 \notin c(S)$, so there is an $x^2 \in S$ such that $x^2 \succ x^1$. Applying (A.2) again and proceeding inductively, we obtain a sequence $(x_m) \in S^\infty$ such that $x^{m+1} \succ x^m$ for each $m \in \mathbb{N}$. But then $S := \{x^1, x^2, \ldots\} \in \Omega_X$, but $c(S) = \max(S, \succsim) = \emptyset$, contradicting c being a choice correspondence on Ω_X. □

Lemma 3. *Let X be a metric space and $\succsim$ a continuous preference relation on X. If S is a nonempty compact subset of X, then* $\max(S, \succsim) \neq \emptyset$.

Proof. Define $\trianglerighteq := \succ \cup \{(x,x)\colon\ x \in X\}$, and note that $(S, \trianglerighteq)$ is a partially ordered set. By the Hausdorff Maximal Principle, therefore, there exists a nonempty set $T \subseteq S$ such that $(T, \trianglerighteq)$ is a linearly ordered set, and if $T' \subseteq S$ is such that $(T', \succ)$ is a linearly ordered set, then $T \subset T'$ cannot hold.

Define

$$U(x) := \{y \in S\colon\ y \succsim x\}, \quad x \in S.$$

Since $\succsim$ is continuous, it is clear that $U(x)$ is a closed subset of S for each $x \in S$. Moreover, $\{U(x)\colon\ x \in T\}$ has the finite intersection property. (Indeed, if F is any nonempty finite subset of T, then there exists an $x^* \in F$ with $x^* \succsim x$ for all $x \in F$ since $\succsim$ is a linear order on F. It thus follows that x^* belongs to $\bigcap\{U(x)\colon\ x \in F\}$, so the latter set is nonempty.) Since S is compact, therefore, we have $\bigcap\{U(x)\colon\ x \in T\} \neq \emptyset$. Thus there exists an $x_* \in S$ with $x_* \in U(x)$ for all $x \in T$. But then $x_* \in \max(S, \succsim)$, because if $y \succ x_*$ for some $y \in S$, then $(T \cup \{y\}, \succ)$ is a linearly ordered set, contradicting the maximality of T. □

Proof of Theorem 3. Let $\succsim$ be a continuous regular preference relation on X and define $c : \Omega_X \rightrightarrows X$ by $c(S) := \max(S, \succsim)$. By Lemma 3, c is a choice correspondence on Ω_X. To show that c satisfies WARNI, take any $S \in \Omega_X$ and $y \in S$, and suppose that $y \notin c(S)$. We wish to prove that there exists an $x \in c(S)$ such that $x \succ y$. Define

$$U(y) := \{z \in S\colon\ z \succsim y\}.$$

Since $\succsim$ is continuous and S is compact, $U(y)$ is a nonempty compact subset of S. By Lemma 3, therefore, there is an $x \in \max(U(y), \succsim)$. Notice that if $x \notin \max(S, \succsim)$, then $x' \succ x$ for some $x' \in U(y)$, which is impossible. Thus $x \in \max(S, \succsim)$. Moreover, since $y \notin \max(S, \succsim)$, there is a $z \in U(y)$ with $z \succ y$, and it follows that $x \succ y$, as we sought.

It remains to prove that c satisfies the (C)-axiom. We begin with verifying the $(C1)$-axiom. Take any $(x, y) \in \mathcal{I}(c)$, and note that to establish $(x, y) \in \operatorname{int}(\mathcal{I}(c))$, it is enough to prove that $\mathcal{I}(c)$ contains all but finitely many terms of any sequence that converges to (x, y). Let (x_n, y_n) be an arbitrary sequence in $X \times X$ with $(x_n, y_n) \to (x, y)$ as $n \to \infty$. If this sequence did not stay in $\mathcal{I}(c)$ eventually, there would exist a subsequence (x_{n_k}, y_{n_k}) in $(X \times X)\backslash\mathcal{I}(c)$. To obtain a contradiction from this situation, assume first that $\{x_{n_k}\} = c\{x_{n_k}, y_{n_k}\}$ for infinitely many k so that there exists a subsubsequence $(x_{n_{k_l}}, y_{n_{k_l}})$ such that $\{x_{n_{k_l}}\} = \max(\{x_{n_{k_l}}, y_{n_{k_l}}\}, \succsim)$ for each l. But then $x_{n_{k_l}} \succ y_{n_{k_l}}$ for each l, and we get $x \succsim y$ by continuity of $\succsim$. This is a contradiction, for we have $x \bowtie y$ by Proposition 2. Assume then that there exists an integer K such that $\{x_{n_k}, y_{n_k}\} = c\{x_{n_k}, y_{n_k}\} = \max(\{x_{n_k}, y_{n_k}\}, \succsim)$ for each $k \geqslant K$. Then, since $(x_{n_k}, y_{n_k}) \notin \mathcal{I}(c)$, Proposition 2 implies that $x_{n_k} \sim y_{n_k}$ for each $k \geqslant K$, so letting $k \to \infty$, we find $x \sim y$ by continuity of $\succsim$, contradicting $(x, y) \in \mathcal{I}(c)$. To verify the $(C2)$-axiom, take any convergent sequences (x_n) and (S_n) in X and Ω_X respectively, and assume that $x_n \in c(S_n)$ and $(x_n, y_n) \notin \mathcal{I}(c)$ for all $y_n \in S_n$ and all $n \in \mathbb{N}$. Pick any $y \in \lim S_n$. Then there exists a sequence (y_n) in X with $y_n \in S_n$ and $\lim y_n = y$. Since $x_n \in \max(S_n, \succsim)$, we cannot have $y_n \succ x_n$. If, on the other hand, $y_n \bowtie x_n$ held, then we would obtain the contradiction $(x_n, y_n) \in \mathcal{I}(c)$ by Proposition 2. Thus $x_n \succsim y_n$ for all n, and since $\succsim$ is continuous, we get $x \succsim y$. Since y was arbitrary in S, this means that $x \in c(S)$, as we sought.

Conversely, assume that c is a choice correspondence on Ω_X that satisfies the WARNI and the (C)-axiom, and let $\succsim$ be the preference relation defined on X defined in the proof of Theorem 2. Given what is established in that proof, all we need to verify here is the continuity of $\succsim$. To this end, take any sequence (x_n, y_n) in $\succsim$ such that $(x_n, y_n) \to (x, y)$ as $n \to \infty$ for some $x, y \in X$. If $x = y$ or $y \notin c\{x, y\}$, there is nothing to prove, so we assume in what follows that $x \neq y$ and $y \in c\{x, y\}$. Now suppose $x_n \succ y_n$ holds for infinitely many n so that there exists a subsequence (x_{n_k}, y_{n_k}) in $\succ$. By definition of $\succsim$, we then have $\{x_{n_k}\} = c\{x_{n_k}, y_{n_k}\}$, so $(x_{n_k}, y_{n_k}) \notin \mathcal{I}(c)$ for each k. Since $(x_{n_k}, y_{n_k}) \to (x, y)$ as $k \to \infty$, the $(C1)$-axiom implies $(x, y) \notin \mathcal{I}(c)$ and the $(C2)$-axiom gives $x \in c\{x, y\}$. Thus $x \succsim y$ obtains in this case.

Suppose next that there exists an integer N such that $x_n \sim y_n$ for each $n \geqslant N$. If $x_n = y_n$ for infinitely many n, there is nothing to prove, so it is without loss of generality to let $x_n \neq y_n$ for each $n \geqslant N$. Then $\{x_n, y_n\} = c\{x_n, y_n\}$ and $(x_n, y_n) \notin \mathcal{I}(c)$ for each $n \geqslant N$. So applying again the $(C1)$ and $(C2)$-axioms in conjunction, we find $\{x, y\} = c\{x, y\}$ and $(x, y) \notin \mathcal{I}(c)$, that is, $x \succsim y$. This proves that $\succsim$ is a continuous preference relation. □

The proof of Theorem 4 is based on the following theorem which is recently proved by Dubra et al. (2004).

The expected multi-utility theorem. *Let Z be a compact metric space, and let $\succsim$ be a continuous preference relation on $\mathbb{P}(Z)$ such that $p \succsim q$ implies $p \oplus_\lambda r \succsim q \oplus_\lambda r$ for all $p, q, r \in \mathbb{P}(Z)$ and all $\lambda \in (0, 1)$. Then there exists a nonempty closed and convex $\mathcal{U} \subseteq \mathbf{C}(Z)$ such that $p \succsim q$ if and only if $\boldsymbol{E}_{p,\mathcal{U}} \geqslant \boldsymbol{E}_{q,\mathcal{U}}$.*

Proof of Theorem 4. To prove the "if" part, define the preorder $\trianglerighteq$ on $\mathbb{P}(Z)$ by

$$p \trianglerighteq q \quad \text{if and only if} \quad \boldsymbol{E}_{p,\mathcal{U}} \geqslant \boldsymbol{E}_{q,\mathcal{U}},$$

which is easily checked to be continuous. Notice that p and q are two lotteries on Z that are non-comparable by $\trianglerighteq$ if and only if there exist $u, v \in \mathcal{U}$ such that $\boldsymbol{E}_{p,\mathcal{U}}(u) > \boldsymbol{E}_{q,\mathcal{U}}(u)$ and $\boldsymbol{E}_{q,\mathcal{U}}(v) > \boldsymbol{E}_{p,\mathcal{U}}(v)$. Now let z^* be the only member of $\bigcap_{u \in \mathcal{U}} \arg\max_{z \in Z} u(z)$, and define $r(\lambda) := \delta_{z^*} \oplus_\lambda p$ for any $\lambda \in [0, 1]$. (Clearly, neither p nor q may equal δ_{z^*}.) It is evident that $\boldsymbol{E}_{r(\lambda),\mathcal{U}} > \boldsymbol{E}_{p,\mathcal{U}}$ for all $0 < \lambda < 1$, while for small enough $\lambda > 0$ we have $\boldsymbol{E}_{q,\mathcal{U}}(v) > \boldsymbol{E}_{r(\lambda),\mathcal{U}}(v)$. Thus, for small enough $\lambda > 0$, $r(\lambda) \triangleright p$ while $r(\lambda)$ and q are non-comparable by $\trianglerighteq$. We thus conclude that $\trianglerighteq$ is regular. Thus, by Theorem 3, $c := \max(\cdot, \trianglerighteq)$ is a choice correspondence that satisfies WARNI and the (C)-axiom. But it is readily verified that c also satisfies the (B) and (I)-axioms, and hence follows the "if" part of the theorem.

Conversely, assume that c is a choice correspondence on $\Omega_{\mathbb{P}(Z)}$ that satisfies WARNI and the (C), (B) and (I)-axioms, and let $\succsim$ be the preference relation defined on X defined in the proof of Theorem 2. We showed in the proofs of Theorem 3 and Theorem 2 that this relation is continuous and $c = \max\{\cdot, \succsim\}$. Observe next that $p \succsim q$ implies $(p, q) \notin \mathcal{I}(c)$. Fix any $0 < \lambda < 1$. If $q \oplus_\lambda r \notin c\{p \oplus_\lambda r, q \oplus_\lambda r\}$, then $p \oplus_\lambda r \succ q \oplus_\lambda r$ holds obviously, so consider instead the case where $q \oplus_\lambda r \in c\{p \oplus_\lambda r, q \oplus_\lambda r\}$. But $p \succsim q$ implies $p \in c\{p, q\}$, so the $(I1)$-axiom then yields

$$\{p \oplus_\lambda r, q \oplus_\lambda r\} = c\{p \oplus_\lambda r, q \oplus_\lambda r\}.$$

Notice that if $(p \oplus_\lambda r, q \oplus_\lambda r) \in \mathcal{I}(c)$, then the $(I2)$-axiom yields $(p, q) \in \mathcal{I}(c)$, a contradiction. Thus $(p \oplus_\lambda r, q \oplus_\lambda r) \notin \mathcal{I}(c)$, that is, $p \oplus_\lambda r \sim q \oplus_\lambda r$. We have just showed that $p \succsim q$ implies $p \oplus_\lambda r \succsim q \oplus_\lambda r$ for any $p, q, r \in \mathbb{P}(Z)$ and any $\lambda \in (0, 1)$. We may thus apply the expected multi-utility theorem to find a nonempty closed and convex $\mathcal{U} \subseteq \mathbf{C}(Z)$ such that $p \succsim q$ iff $\boldsymbol{E}_{p,\mathcal{U}} \geqslant \boldsymbol{E}_{q,\mathcal{U}}$. Finally, by Lemma 1 and the (B)-axiom, we have $\{\delta_{z^*}\} = \max(\{\delta_z, \delta_{z^*}\}, \succsim)$ for each $z \in Z$. It follows that $\bigcap_{u \in \mathcal{U}} \arg\max_{z \in Z} u(z) = \{z^*\}$, and the proof is complete. □

References

Arrow, K., 1959. Rational choice functions and orderings. Economica 26, 121–127.

Aumann, R., 1962. Utility theory without the completeness axiom. Econometrica 30, 445–462.

Bade, S., 2005. Nash equilibrium in games with incomplete preferences. Econ. Theory 26, 309–332.

Bandyopadhay, T., Sengupta, K., 1993. Characterization of generalized weak orders and revealed preference. Econ. Theory 3, 571–576.

Bewley, T., 1986. Knightian uncertainty theory: Part I. Discussion paper No. 807. Cowles Foundation.

Chernoff, H., 1954. Rational selection of decision functions. Econometrica 22, 422–443.

Danan, E., 2003. A behavioral model of individual welfare. Mimeo. Université de Paris.

Dubra, J., Maccheroni, F., Ok, E.A., 2004. Expected utility theory without the completeness axiom. J. Econ. Theory 115, 118–133.

Grether, D., Plott, C., 1979. Economic theory of choice and the preference reversal phenomenon. Amer. Econ. Rev. 69, 623–638.

Hansson, B., 1968. Choice structures and preference relations. Synthese 18, 443–458.

Holt, C., 1986. Preference reversals and the independence axiom. Amer. Econ. Rev. 76, 508–515.

Huber, J., Payne, J., Puto, C., 1982. Adding asymmetrically dominated alternatives: Violations of regularity and the similarity hypothesis. J. Cons. Res. 9, 90–98.

Kalai, G., Rubinstein, A., Spiegler, R., 2002. Rationalizing choice functions by multiple rationales. Econometrica 70, 2481–2488.

Karni, E., Safra, Z., 1987. 'Preference reversal' and the observability of preferences by experimental methods. Econometrica 55, 675–685.

Kreps, D., 1988. Notes on the Theory of Choice. Westview Press, Boulder, CO.

Levi, I., 1986. Hard Choices: Decision Making under Unresolved Conflict. Cambridge Univ. Press, Cambridge.

Levy, G., 2004. A model of political parties. J. Econ. Theory 115, 250–277.

Loomes, G., Starmer, C., Sugden, R., 1991. Observing violations of transitivity by experimental methods. Econometrica 59, 425–439.

Mandler, M., 2005. Incomplete preferences and rational intransitivity of choice. Games Econ. Behav. 50, 255–277.

Masatlioglu, Y., Ok, E.A., 2005. Rational choice with status-quo bias. J. Econ. Theory 121, 1–29.

Moulin, H., 1985. Choice functions over a finite set: A summary. Soc. Choice Welfare 2, 147–160.

Nehring, K., 1997. Rational choice and revealed preference without binariness. Soc. Choice Welfare 14, 403–425.

Ok, E.A., 2002. Utility representation of an incomplete preference relation. J. Econ. Theory 104, 429–449.

Ok, E.A., 2005. Independence of irrelevant alternatives and rational choice. Mimeo. NYU.

Peleg, B., 1970. Utility functions for partially ordered topological spaces. Econometrica 38, 93–96.

Redelmeier, D., Shafir, E., 1995. Medical decision making in situations that offer multiple alternatives. J. Amer. Med. Assoc. 273, 302–305.

Richter, M., 1966. Revealed preference theory. Econometrica 34, 625–645.

Roemer, J., 1999. The democratic political economy of progressive income taxation. Econometrica 67, 1–19.

Safra, Z., Segal, U., Spivak, A., 1990. Preference reversal and nonexpected utility behavior. Amer. Econ. Rev. 80, 922–930.

Sagi, J., 2005. Anchored preference relations. J. Econ. Theory. In press.
Samuelson, P., 1938. A note on the pure theory of consumers' behavior. Economica 5, 61–71.
Schwartz, T., 1976. Choice functions, 'rationality' conditions, and variations on the weak axiom of revealed preference. J. Econ. Theory 13, 414–427.
Segal, U., 1988. Does the preference reversal phenomenon necessarily contradict the independence axiom?. Amer. Econ. Rev. 78, 233–236.
Seidenfeld, T., Schervish, M., Kadane, J., 1995. A representation of partially ordered preferences. Ann. Statist. 23, 2168–2217.
Sen, A., 1971. Choice functions and revealed preferences. Rev. Econ. Stud. 38, 307–317.
Sen, A., 1993. Internal consistency of choice. Econometrica 61, 495–521.
Sen, A., 1997. Maximization and the act of choice. Econometrica 65, 745–779.
Shafir, E., Simonson, I., Tversky, A., 1993. Reason-based choice. Cognition 49, 11–36.
Simonson, I., 1989. Choice based on reasons: The case of attraction and compromise effects. J. Cons. Res. 16, 158–174.
Slovic, P., Lichtenstein, S., 1968. The relative importance of probabilities and payoffs in risk-taking. J. Exper. Psychol. 78, 1–18.
Sondermann, D., 1980. Utility representation for partial orders. J. Econ. Theory 23, 183–188.
Suzumura, K., 1983. Rational Choice, Collective Decisions, and Social Welfare. Cambridge Univ. Press, Cambridge.
Tversky, A., Shafir, E., 1992. Choice under conflict: The dynamics of the deferred decision. Psychol. Sci. 3, 358–361.
Tversky, A., Slovic, P., Kahneman, D., 1990. The causes of preference reversal. Amer. Econ. Rev. 80, 204–217.