Floating Bodies of Equilibrium II

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Abstract In a previous paper (physics/0203061) "Floating bodies of equilibrium" we have shown that there exist two-dimensional non-circular cross-sections of bodies of homogeous density $\rho \neq 1/2$ which can float in any orientation in water, which have a p-fold rotation axis. For given p they exist for $p - 2$ different densities. However, this was found only in a Taylor-expansion in a parameter which described the distortion from the circular shape up to seventh order. Here a differential equation for the boundary curve is given and the non-circular boundary curve can be expressed in terms of an elliptic integral.

1 Introduction

Stanislaw Ulam asks in the Scottish Book [1] (problem 19), whether a sphere is the only solid of uniform density which will float in water in any position. In a recent paper[2] I considered the two-dimensional problem for $\rho \neq 1/2$. (The case $\rho = 1/2$ has been solved by Auerbach [3]). I was able to obtain non-circular two-dimensional cross-sections of bodies which can flow in any orientation in water by a perturbative expansion around the circular solution. These cross-sections have a p-fold rotation symmetry. In polar coordinates r and ψ the boundary-curve could be expanded in powers of a deformation-parameter ϵ

$$r(\psi) = \bar{r}(1 + 2\epsilon \cos(p\psi) + 2 \sum_{n=2}^{\infty} c_n(\epsilon) \cos(np\psi)). \tag{1}$$

The coefficients $c_n = O(\epsilon^n)$ were determined up to order ϵ^7. Although one has solutions for $p - 2$ different densities ρ, it turned out that $r(\psi)$ was the same for all these densities.

Here a non-perturbative solution is given. It is shown that the boundary-curve obeys the differential equation

$$\frac{1}{\sqrt{r^2 + r'^2}} = ar^2 + b + cr^{-2}, \tag{2}$$

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with $r' = \mathrm{d}r/\mathrm{d}\psi$. This equation can be integrated

$$r'^2 = \frac{r^4}{(ar^4 + br^2 + c)^2} - r^2, \tag{3}$$

$$\psi = \pm \int \frac{\mathrm{d}r(ar^4 + br^2 + c)}{r\sqrt{r^2 - (ar^4 + br^2 + c)^2}}$$

$$= \pm \int^{r^2} \frac{\mathrm{d}q(aq^2 + bq + c)}{2q\sqrt{q - (aq^2 + bq + c)^2}} \tag{4}$$

and is thus given by an elliptic integral. With increasing ψ the radius r oscillates periodically between the largest and the smallest radii $r_>$ and $r_<$, resp. The three constants a, b, and c are determined by these extreme radii, and by the periodicity of the boundary. Since r' vanishes for the extrema of r, one has

$$r_{>,<} = ar_{>,<}^4 + br_{>,<}^2 + c. \tag{5}$$

The periodicity is given by

$$\int_{r_<}^{r_>} \frac{\mathrm{d}r(ar^4 + br^2 + c)}{r\sqrt{r^2 - (ar^4 + br^2 + c)^2}} = \frac{\pi}{p}. \tag{6}$$

The solution of the differential equation (2) expanded in powers of ϵ agrees completely with the expansion obtained in [2]. (Actually a factor 3 is missing in the $\epsilon^5 \cos(3\psi)$ term in eq. 86).

I explain now, how I arrived at this differential equation. Denote the two intersections of the water-line with the cross-section by L and R. From the general arguments given in [2] one knows, that the length $2l$ of the water-line has to be independent of the orientation. (Here actually we will fix the orientation of the cross-section and rotate the water-line including the direction of the gravitational force.) The midpoint M of the line moves in the direction of the chord LR as the water-line is rotated. Suppose the coordinates of the intersections are given by $L = (x_M - l, y_M)$ and $R = (x_M + l, y_M)$. If the line moves by an infinitesimal amount $L' = (x_M - l + \mathrm{d}x_L, y_M + \mathrm{d}y_L)$, $R' = (x_M + l + \mathrm{d}x_R, y_M + \mathrm{d}y_R)$, then the condition, that the distance $L'R' = 2l$ is fixed yields $\mathrm{d}x_L = \mathrm{d}x_R$ and the condition, that M moves in the direction of LR yields $\mathrm{d}y_L + \mathrm{d}y_R = 0$. From both conditions we find, that under this change the same length of the perimeter $\mathrm{d}u = \sqrt{\mathrm{d}x_L^2 + \mathrm{d}y_L^2} = \sqrt{\mathrm{d}x_R^2 + \mathrm{d}y_R^2}$ disappears on one side below the water-line and appears on the other side above the water-line. Then $\mathrm{d}x_L = \mathrm{d}x_R = \cos\beta\mathrm{d}u$, $\mathrm{d}y_L = -\mathrm{d}y_R = \sin\beta\mathrm{d}u$. Thus the angles between the tangents on the boundary and the water-line are the same on both ends.
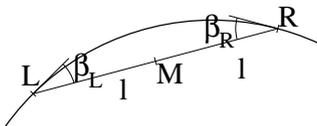

Figure 1: Boundary curve, water-line LR, tangents, and the angles β.

In the following we will use these properties in two forms: (i) as line-condition, requiring, that the envelope of the water-lines is the loci of the midpoints M, so that we obtain L and R by drawing tangents on the envelope and

going a constant distance l along the tangent in both directions to obtain L and R. (ii) as angle-condition, requiring, that moving L and R the same piece u along the perimeter to L' and R', the angles β_L and β_R between the tangents and the chord $L'R'$ always obey $\beta_L = \beta_R$. Both conditions are equivalent. From (i) follows (ii) and from (ii) follows (i).

In a next step consider the limit of large p. Then r will oscillate on a distance of order $\bar{r}/p$, which is small in comparison to $\bar{r}$ itself. I assume, that then one can neglect the overall average curvature $1/\bar{r}$ and one may find a periodic function f of $\xi = \bar{r}\psi$, which obeys the line-condition. Indeed assuming, that this holds for $l \gg \bar{r}/p$, that is on a distance of many periods, one can derive in this linear case a differential equation, which has to be obeyed by the function f. This will be done in subsection 2.1. Expanding the solution in powers of ϵ is in agreement with the terms leading in $1/p$ which were found in eq. 86 of [2].

Unfortunately this procedure is not applicable for finite p. Here I work with a new conjecture: Although for physical reasons the periodicity p has to be an integer, one may try it with non-integer p. If p differs from an integer infinitesimally, then going around by the angle 2π one arrives at the same function shifted however, in ψ-direction by an infinitesimal angle 2α. I conjecture that there is also a line between L and R belonging to angles $\psi_{L,R}$ differing by $\delta\psi \approx 2\pi$ of constant small length $2l$, which obeys the line-condition. This condition yields a differential equation. For the linear case this is done in subsection 2.2. It is in agreement with that derived for large distance l. For the circular case one obtains the differential equation in subsection 3.1, which is identical to eq. (2).

Finally it is shown in subsections 2.3 and 3.2, that the angle-condition holds for the curves described by the so obtained differential equations which are shifted by arbitrary distances in the linear case and by arbitrary angles in the circular case, resp. This last property can now also be used for two points L and R on the same curve, and thus shows that eq. (2) yields the boundaries for cross-sections with $\rho \neq 1/2$.

The derivation for the linear approximation is given, since it is somewhat simpler, but one may skip section 2 and go directly to the derivation in section 3.

2 Linear Approximation

Consider now the above mentioned linear case. Let us start from the loci of the middle points M of the chord LR, described by Cartesian coordinates (g, ξ). Then drawing the tangent on this curve and moving along the tangent by the distance l in both directions one obtains the points R and L, which should lie on the same curve described by $f(\xi)$. Thus the functions g and f should obey

$$g(\xi) \pm \frac{lg'(\xi)}{\sqrt{1 + g'^2(\xi)}} = f(\xi \pm \frac{l}{\sqrt{1 + g'^2(\xi)}}). \tag{7}$$

2.1 Large Distance

Assume now, that the functions are periodic with periodicity 2λ and $f(\xi + \lambda) = -f(\xi) = f(-\xi)$, similarly for g. Then $f(n\lambda) = 0$ for integer n and f has extrema at $\xi = o\lambda/2$ with odd o. If we seek solutions for large l, then we put $l = l_o + \delta$

with $l_o = o\lambda/2$ and small δ. We write

$$\frac{l}{\sqrt{1 + g'^2(\xi)}} = l_o + \delta_1, \quad \delta_1 = \frac{l_o + \delta}{\sqrt{1 + g'^2(\xi)}} - l_o = \delta - l_o g'^2/2 + ... \quad (8)$$

and obtain by expansion in δ_1 and addition and subtraction of the equations

$$g(\xi) = (-)^{(o-1)/2}(\delta_1 f'(\xi + \lambda/2) + \frac{\delta_1^3}{6} f'''(\xi + \lambda/2) + ...) \quad (9)$$

$$(l_o + \delta)\frac{g'(\xi)}{\sqrt{1 + g'^2(\xi)}} = (-)^{(o-1)/2}(f(\xi + \lambda/2) + \frac{\delta_1^2}{2} f''(\xi + \lambda/2) + ...). \quad (10)$$

Note that $f(\xi + l_o + \delta_1) = (-)^{(o-1)/2} f(\xi + \frac{\lambda}{2} + \delta_1)$ and $f(\xi - l_o - \delta_1) = (-)^{(o+1)/2} f(\xi + \frac{\lambda}{2} - \delta_1)$. If f is of order l_o^0, then we realize from the second equation, that g' and thus g is of order $1/l_o$. The first equation yields, that δ_1 is of order $1/l_o$. We put $\delta = c/l_o$. Furthermore we write $(-)^{(o-1)/2} l_o g = \hat{g}$ and collect the leading terms of these two equations

$$\hat{g} = (c - \hat{g}'^2/2)f', \quad \hat{g}' = f, \quad (11)$$

where the argument of g is ξ and of f is $\xi + \lambda/2$. From these two equations we obtain differential equations for f and $\hat{g}$. Taking the derivative of the second equation and substituting into the first one yields

$$\hat{g} = (c - \hat{g}'^2/2)\hat{g}''. \quad (12)$$

If we multiply this equation by $\hat{g}'$, we can integrate it and obtain

$$\hat{g}^2 = (c - \hat{g}'^2/4)\hat{g}'^2 + c_2. \quad (13)$$

In order to obtain a differential equation for f we solve this equation for $\hat{g}$ and differentiate it. This yields

$$\hat{g}' = \frac{\hat{g}''(c\hat{g}' - \hat{g}'^3/2)}{\sqrt{c_2 + c\hat{g}'^2 - \hat{g}'^4/4}} \quad (14)$$

and substituting $\hat{g}' = f$ the equation

$$\sqrt{c_2 + cf^2 - f^4/4} = (c - f^2/2)f' \quad (15)$$

or

$$(c - f^2/2)^2(1 + f'^2) = c_2 + c^2, \quad (16)$$

which we will use in the form

$$\frac{1}{\sqrt{1 + f'^2}} = af^2 + b \quad (17)$$

with new integration constants a and b.

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2.2 Infinitesimal distance

We perform now a similar calculation, but assume, that we can represent $f(\xi + \alpha)$ and $f(\xi - \alpha)$ similarly by g, where now α and l are infinitesimally small. Thus we assume

$$g(\xi) \pm \frac{lg'(\xi)}{\sqrt{1 + g'^2(\xi)}} = f(\xi \mp \alpha \pm \frac{l}{\sqrt{1 + g'^2(\xi)}}). \tag{18}$$

It will turn out, that we need the expansion in α to first order and in l in third order. We obtain

$$g = f + \frac{l^2}{2(1 + g'^2)} f'', \tag{19}$$

$$g' = f' - \frac{\alpha}{l} f' \sqrt{1 + g'^2} + \frac{l^2}{6(1 + g'^2)} f'''. \tag{20}$$

Differentiating the first equation yields

$$g' = f' + \frac{l^2 f'''}{2(1 + g'^2)} - \frac{l^2 g' g'' f''}{(1 + g'^2)^2}. \tag{21}$$

Equating both expressions and choosing $\alpha = cl^3$ yields

$$cf' \sqrt{1 + g'^2} + \frac{f'''}{3(1 + g'^2)} - \frac{g' g'' f''}{(1 + g'^2)^2} = 0. \tag{22}$$

In the limit $\alpha \to 0$ one has $g = f$. With this substitution and multiplication by $1/\sqrt{1 + f'^2}$ we may integrate this equation

$$cf + \frac{f''}{3(1 + f'^2)^{3/2}} + c_1 = 0. \tag{23}$$

Multiplication by f' allows a second integration which yields

$$\frac{1}{\sqrt{1 + f'^2}} = \frac{3c}{2} f^2 + 3c_1 f + 3c_2. \tag{24}$$

Comparison with the equation obtained for large distances (17) shows agreement, if we change the notation for the constants. There is no linear term in eq.(17) like the term $3c_1 f$ in eq.(24). The reason is, that in eq. (17) we required f to vanish on the average. But of course a constant can be added to f and g, yielding such a linear term. Thus both procedures yield the same differential equation.

2.3 Arbitrary Distance

We show now the following **theorem**. Consider two curves C_L and C_R which are governed by eq. (17) with the same constants a and b, and point $L \in C_L$ with Cartesian coordinates ξ_L, f_L and $R \in C_R$ with coordinates ξ_R, f_R. If now the angle β_L between the tangent in L and $\overline{LR}$ equals the angle β_R between the tangent in R and $\overline{RL}$, and $f_L \neq f_R$, then this property will hold between any two points $L' \in C_L$ and $R' \in C_R$, which are obtained by moving from L and R by the same arc u in the same direction along the curves. Simultaneously the distance $\overline{L'R'} = \overline{LR}$ is constant.

In order to show this we first express the angles β_L and β_R. Let us denote the angles of the curves against the ξ-axis by $\phi_{L,R}$,

$$\tan\phi_{L,R} = f'_{L,R}, \quad \cos\phi_{L,R} = \frac{1}{\sqrt{1+f'^2_{L,R}}} = af^2_{L,R} + b, \qquad (25)$$

$$\sin\phi_{L,R} = \frac{f'_{L,R}}{\sqrt{1+f'^2_{L,R}}} = \sqrt{1 - (af^2_{L,R}+b)^2} =: W_{L,R} \qquad (26)$$

and the angle of LR against the ξ-axis by γ,

$$\tan\gamma = \frac{f_R - f_L}{\xi_R - \xi_L}, \quad \sin\gamma = \frac{f_R - f_L}{2l}, \quad \cos\gamma = \sqrt{1 - \frac{(f_R - f_L)^2}{(2l)^2}} =: W_\gamma, \quad (27)$$

where $2l$ is the distance LR.

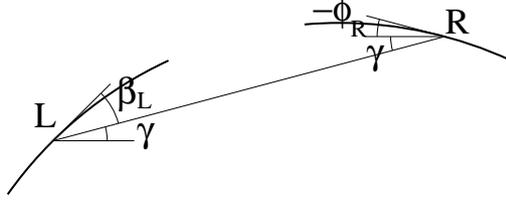
Figure 2: In this figure ϕ_L is given by $\beta_L + \gamma$. The angle ϕ_R is negative in this figure, and $\beta_R = \gamma - \phi_R$.

Then we have $\beta_L = \phi_L - \gamma$, $\beta_R = \gamma - \phi_R$. The condition $\beta_L = \beta_R$ can now be expressed in various ways. From $\cos(\phi_L - \gamma) = \cos(\gamma - \phi_R)$ one obtains

$$(af_L^2 + b)W_\gamma + \frac{f_R - f_L}{2l}W_L = (af_R^2 + b)W_\gamma + \frac{f_R - f_L}{2l}W_R, \qquad (28)$$

which for $f_R \neq f_L$ reduces to

$$a(f_R + f_L)W_\gamma = \frac{W_L - W_R}{2l}. \qquad (29)$$

We show now, that

$$n := \beta_L - \beta_R = \phi_L + \phi_R - 2\gamma \qquad (30)$$

is independent of u, if initially it vanishes, $n = 0$, and $f_R \neq f_L$ holds. For this purpose we calculate the derivative $\frac{dn}{du}$ and show, that it vanishes, and thus, that n remains 0. For this purpose we calculate the derivatives needed and indicate the derivative with respect to u by a dot,

$$\dot{f}_{L,R} = \sin(\phi_{L,R}) = W_{L,R}, \qquad (31)$$

$$\dot{W}_{L,R} = \frac{dW_{L,R}}{df_{L,R}}\dot{f}_{L,R} = -2af_{L,R}(af^2_{L,R} + b), \qquad (32)$$

$$\dot{W}_\gamma = -\frac{(f_L - f_R)(W_L - W_R)}{4l^2 W_\gamma}. \qquad (33)$$

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We use generally for the derivatives of the angles

$$\dot{\phi} = \cos\phi(\dot{\sin\phi}) - \sin\phi(\dot{\cos\phi}) \tag{34}$$

and obtain

$$\dot{\phi}_{L,R} = -2af_{L,R} \tag{35}$$

and

$$\dot{\gamma} = W_\gamma \frac{W_R - W_L}{2l} - \frac{(f_R - f_L)^2(W_L - W_R)}{8l^3 W_\gamma}. \tag{36}$$

We make now use of (29) and obtain

$$\dot{\gamma} = -a(f_L + f_R), \tag{37}$$

from which we conclude $\dot{n} = 0$, provided $f_L \neq f_R$ ∎

For $f_L = f_R$ one has $W_L = \pm W_R$. If the minus-sign applies, then n vanishes for all f_L and l. In this case, however, $W_\gamma = 1$, $\dot{W}_\gamma = 0$ and we obtain $\dot{n} = -4af + 2W_L/l$. One can easily check, that for $f_R = f_L + \epsilon$ one obtains $n = \epsilon(2af/W_L - 1/l) + O(\epsilon^2)$, so that $\dot{n} = 0$ holds also as we approach $f_L = f_R$. Thus the theorem is proven.

Obviously instead of considering two curves, we may consider two points L and R on the same curve, which implies, that the midpoints of these chords are the envelope of these lines as we increase u.

3 Circular Case

The point R with polar coordinates $(r, \psi - \delta\psi)$ lie on the tangent of the envelope through M with polar coordinates (μ, ψ) a distance l apart. Then we have

$$r^2 = \mu^2 + \frac{2\mu\mu' l}{\sqrt{\mu^2 + \mu'^2}} + l^2 \tag{38}$$

Denote the angle OMR by γ. Then γ is related to the slope of the tangent by $-\cos\gamma = \mu'/\sqrt{\mu^2 + \mu'^2}$ and $\sin\gamma = \mu/\sqrt{\mu^2 + \mu'^2}$. For $\delta\psi$ we obtain $\tan\delta\psi = l\sin\gamma/(\mu - l\cos\gamma) = l\mu/(\mu\sqrt{\mu^2 + \mu'^2} + l\mu')$. Thus we obtain

$$\sqrt{\mu^2 \pm \frac{2\mu\mu' l}{\sqrt{\mu^2 + \mu'^2}} + l^2} = r(\psi \mp \alpha \pm \arctan\frac{l\mu}{\mu\sqrt{\mu^2 + \mu'^2} \pm l\mu'}), \tag{39}$$

where the upper sign yields r_R on the curve rotated by α and the lower sign r_L on the curve rotated by $-\alpha$.

3.1 Infinitesimal angle

We expand both sides of eq. (39) up to order l^3 and α and add and subtract the equations for both signs. Then we obtain

$$\mu + \frac{l^2\mu}{2(\mu^2 + \mu'^2)} = r + l^2\left(-\frac{r'\mu'}{\mu(\mu^2 + \mu'^2)} - \frac{r''}{2(\mu^2 + \mu'^2)}\right), \tag{40}$$

$$\mu' - \frac{l^2\mu'}{2(\mu^2 + \mu'^2)} = r' - \frac{\alpha}{l}r'\sqrt{\mu^2 + \mu'^2} + l^2\left(\frac{r'(-\mu^2 + 3\mu'^2)}{3\mu^2(\mu^2 + \mu'^2)}\right.$$
$$\left. - \frac{r''\mu'}{\mu(\mu^2 + \mu'^2)} + \frac{r'''}{6(\mu^2 + \mu'^2)}\right). \tag{41}$$
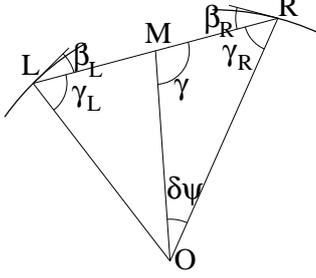
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Figure 3: The angles in the circular case. The angles ϕ determined by r' are related to β and γ by $\phi_L = \gamma_L + \beta_L - \pi/2$, $-\phi_R = \gamma_R + \beta_R - \pi/2$.

Now we differentiate eq. (40)

$$
\begin{aligned}
&\mu' + l^2\left(\frac{\mu'(-\mu^2 + \mu'^2)}{2(\mu^2 + \mu'^2)^2} - \frac{\mu\mu'\mu''}{(\mu^2 + \mu'^2)^2}\right) \\
&= r' + l^2\left(\frac{r'\mu'^2(3\mu^2 + \mu'^2)}{\mu^2(\mu^2 + \mu'^2)^2} - \frac{r''\mu'(2\mu^2 + \mu'^2)}{\mu(\mu^2 + \mu'^2)^2}\right. \\
&\left. + \frac{\mu''r'(-\mu^2 + \mu'^2)}{\mu(\mu^2 + \mu'^2)^2} - \frac{r''\mu''\mu'}{(\mu^2 + \mu'^2)^2} + \frac{r'''}{2(\mu^2 + \mu'^2)}\right)
\end{aligned}
\tag{42}
$$

and subtract eq. (41) from this derivative. The resulting equation contains terms proportional to α/l and to l^2. Thus we set α proportional l^3. In the limit $l \to 0$ the function μ approaches r and similarly for the derivatives. Thus we have

$$
\begin{aligned}
f : &= -\frac{\alpha}{l^3}r'\sqrt{r^2 + r'^2} \\
&- \frac{r'(r^2 + 4r'^2)}{3(r^2 + r'^2)^2} + \frac{r'r''(r^2 - r'^2)}{r(r^2 + r'^2)^2} + \frac{r'r''^2}{(r^2 + r'^2)^2} - \frac{r'''}{3(r^2 + r'^2)} = 0.
\end{aligned}
\tag{43}
$$

This differential equation can be integrated. One finds, that

$$
f = \frac{\sqrt{r^2 + r'^2}}{r}\frac{\mathrm{d}f_1}{\mathrm{d}\psi} = \frac{\sqrt{r^2 + r'^2}}{r^3}\frac{\mathrm{d}f_3}{\mathrm{d}\psi}
\tag{44}
$$

with

$$
f_1 = \frac{-r''r + r^2 + 2r'^2}{3(r^2 + r'^2)^{3/2}} - \frac{\alpha r^2}{2l^3},
\tag{45}
$$

$$
f_3 = -\frac{r^3(r + r'')}{3(r^2 + r'^2)^{3/2}} - \frac{\alpha r^4}{4l^3},
\tag{46}
$$

Thus f_1 and f_3 are constants. Elimination of r'' yields the differential equation of the bounding curve quoted in the introduction, eq. (2).

$$
\frac{1}{\sqrt{r^2 + r'^2}} = ar^2 + b + cr^{-2}
\tag{47}
$$

with constants $a = \lim 3\alpha/l^3$, $b = 3f_1/2$, and $c = -3f_2/2$.

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3.2 Arbitrary angle

Now we consider the two curves rotated against each other by an arbitrary angle 2α and perform similar calculations to those for the linear case.

We first express the angles

$$\beta_L = \phi_L - \gamma_L + \frac{\pi}{2}, \quad \beta_R = \frac{\pi}{2} - \phi_R - \gamma_R. \tag{48}$$

and show the following **theorem**: If the condition $\beta_L = \beta_R$, that is

$$n := \phi_L - \gamma_L + \phi_R + \gamma_R = 0 \tag{49}$$

is fulfilled and $r_L \neq r_R$ and the points L and R move along their curves by the same arc u to points L' and R', then the condition $n = 0$ remains fulfilled, that is for all these chords $\beta_L = \beta_R$ holds. Simultaneously the distance $\overline{LR} = \overline{L'R'}$ remains constant.

In order to show this we calculate $\mathrm{d}n/\mathrm{d}u$. For this purpose we first list the sines and cosines of the angles

$$\cos\phi_{L,R} = \frac{r_{L,R}}{\sqrt{r_{L,R}^2 + r_{L,R}'^2}} = r_{L,R}(ar_{L,R}^2 + b + \frac{c}{r_{L,R}^2}) \tag{50}$$

$$\sin\phi_{L,R} = \frac{r_{L,R}'}{\sqrt{r_{L,R}^2 + r_{L,R}'^2}} =: W_{L,R} \tag{51}$$

$$\cos\gamma_{L,R} = \frac{4l^2 + r_{L,R}^2 - r_{R,L}^2}{4lr_{L,R}} \tag{52}$$

$$\sin\gamma_{L,R} = \frac{W_\gamma}{4lr_{L,R}} \tag{53}$$

$$W_\gamma = \sqrt{-16l^4 + 8l^2(r_L^2 + r_R^2) - (r_L^2 - r_R^2)^2} \tag{54}$$

Their derivatives with respect to u indicated by a dot are

$$\dot{r}_{L,R} = \sin\phi_{L,R} = W_{L,R} \tag{55}$$

$$\dot{W}_{L,R} = -\cos\phi_{L,R}\frac{\mathrm{d}\cos\phi_{L,R}}{\mathrm{d}r_{L,R}} \tag{56}$$

Performing the derivatives for r_L and r_R and using (55) one obtains

$$\dot{W}_\gamma = \frac{8r_Lr_Rl}{W_\gamma}(W_R\cos\gamma_L + W_L\cos\gamma_R) \tag{57}$$

Thus we obtain using again eq. (34)

$$\dot{\phi}_{L,R} = -\cos^2\phi_{L,R}\frac{\mathrm{d}\cos\phi_{L,R}}{\mathrm{d}r_{L,R}} - W_{L,R}^2\frac{\mathrm{d}\cos\phi_{L,R}}{\mathrm{d}r_{L,R}}$$

$$= -\frac{\mathrm{d}\cos\phi_{L,R}}{\mathrm{d}r_{L,R}} = -3ar_{L,R}^2 - b + \frac{c}{r_{L,R}^2}, \tag{58}$$

$$\dot{\gamma}_{L,R} = \frac{\cos\gamma_{L,R}\dot{W}_\gamma}{4lr_{L,R}} + \frac{W_\gamma(r_{R,L}W_{R,L} - r_{L,R}W_{L,R})}{8l^2r_{L,R}^2}. \tag{59}$$

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Now we use $n = 0$ in order to simplify the derivatives $\dot\gamma$. First we evaluate $\sin(\phi_L + \gamma_R) = \sin(\gamma_L - \phi_R)$ which yields

$$W_R \cos\gamma_L + W_L \cos\gamma_R = \frac{W_\gamma}{4l}\left(\frac{\cos\phi_R}{r_L} - \frac{\cos\phi_L}{r_R}\right) \tag{60}$$

Inserting this expression into eq.(57) yields

$$\dot W_\gamma = 2(r_L \cos\phi_L - r_R \cos\phi_R) \tag{61}$$

Further we determine the products $W_\gamma W_{L,R}$ from $\cos(\phi_L - \gamma_L) = \cos(\phi_R + \gamma_R)$ and $\cos(\phi_L + \gamma_R) = \cos(\phi_R - \gamma_L)$, which yield

$$\cos\phi_L \cos\gamma_L + \frac{W_\gamma W_L}{4lr_L} = \cos\phi_R \cos\gamma_R - \frac{W_\gamma W_R}{4lr_R}, \tag{62}$$

$$\cos\phi_L \cos\gamma_R - \frac{W_\gamma W_L}{4lr_R} = \cos\phi_R \cos\gamma_L + \frac{W_\gamma W_R}{4lr_L}. \tag{63}$$

For $r_L \neq r_R$ one obtains

$$W_\gamma W_L = 8al^2 r_L r_R^2 + (ar_L^3 + br_L)(4l^2 + r_R^2 - r_L^2) - \frac{c(4l^2 + r_L^2 - r_R^2)}{r_L}, \tag{64}$$

$$W_\gamma W_R = -8al^2 r_R r_L^2 - (ar_R^3 + br_R)(4l^2 + r_L^2 - r_R^2) + \frac{c(4l^2 + r_R^2 - r_L^2)}{r_R} \tag{65}$$

and thus

$$\dot\gamma_L = -a(r_L^2 + 2r_R^2) - b + \frac{c}{r_L^2}, \tag{66}$$

$$\dot\gamma_R = a(2r_L^2 + r_R^2) + b - \frac{c}{r_R^2}, \tag{67}$$

from which one concludes $\dot n = \dot\phi_L - \dot\gamma_L + \dot\phi_R + \dot\gamma_R = 0$ provided $r_L \neq r_R$ ∎

For $r_L = r_R$ one has $\gamma_L = \gamma_R$ and $\phi_L = \pm\phi_R$. If the minus sign applies, then $\beta_L = \beta_R$ and $n = 0$. In this case we are not allowed to divide by $r_L - r_R$, which we did in our derivation. It turns out, however, that in this limit $\dot W_\gamma = 0$ and one obtains

$$\dot n = \frac{W_\gamma W_L}{2r_L l^2} + c_0(r_L), \tag{68}$$

$$c_0(r_L) = -6ar_L^2 - 2b + \frac{2c}{r_L^2}$$

$$= \frac{2a}{r_L^2}((r_>^2 + r_L^2)(r_<^2 + r_L^2) - 4r_L^4) + \frac{2(r_> r_< - r_L^2)}{r_L^2(r_> + r_<)}. \tag{69}$$

3.3 Application to the boundary

We may now apply the theorem to our original problem. In this problem the two curves are identical and constitute the boundary. Suppose we have now a curve obeying the differential equation (2) with integer p. Then we have to look for points L, R on this curve which have either the properties $n = 0$ and $r_L \neq r_R$ or otherwise the properties $r_L = r_R$ and $\dot n = 0$, eq. (68). We will now

show, that there are at least $p - 2$ solutions, which obey the second condition. For this purpose we divide the curve into pieces $A_1, \ldots A_p$ in which r decreases from $r_>$ to $r_<$ as ψ increases and into pieces $B_1, \ldots B_p$, where r increases from $r_<$ to $r_>$. We now look for solutions $r_L = r_R$ and $\dot{n} = 0$, where r_R is in A_1 and r_L in B_k. Thus we have to find a solution of (68). We realize, that with the exception of $l = 0$ the first term in (68) vanishes for $r = r_>$ and $r = r_<$, since $W_L = 0$, because $r' = 0$ at these extrema.

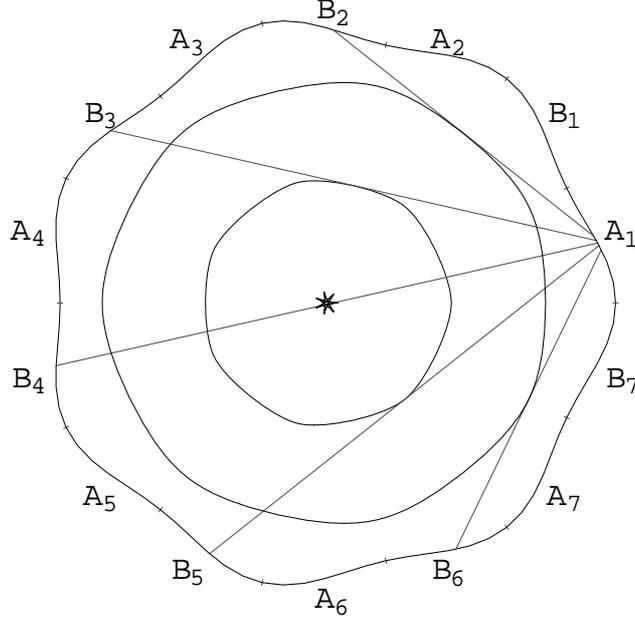

Figure 4: Boundary for $p = 7$ and $\epsilon = 1/49$ (eq. 1) with the pieces A_n and B_n. The outermost curve is the boundary, the inner curves are the envelopes of the water-lines. The innermost one for density $\rho = 1/2$, the two others for densities 0.22748 and 0.77252, and for 0.05269 and 0.94731.

On the other hand we can evaluate c_0 for these extrema

$$c_0(r_>) = -\frac{2(r_> - r_<)(1 + 2ar_>(r_> + r_<)^2)}{r_>(r_> + r_<)} \tag{70}$$

$$c_0(r_<) = \frac{2(r_> - r_<)(1 + 2ar_<(r_> + r_<)^2)}{r_<(r_> + r_<)} \tag{71}$$

The denominator $\sqrt{r^2 - (ar^4 + br^2 + c)^2}$ of the integral (4) vanishes only at the extreme points $r = r_>$ and $r = r_<$. If we express b and c in terms of a, $r_>$, and $r_<$, then

$$r^2 - (ar^4 + br^2 + c)^2 = (r_> - r)(r - r_<) \left(\frac{1}{r_> + r_<} + a(r + r_>)(r + r_<) \right)$$

$$\times \quad (r_> + r)(r + r_<) \left(\frac{1}{r_> + r_<} + a(r - r_>)(r - r_<) \right) \tag{72}$$

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Thus $1+a(r+r_>)(r+r_<)(r_>+r_<) > 0$, and therefor the factors $(1+2ar_>(r_>+r_<)^2)$ and $(1+2ar_<(r_>+r_<)^2)$ in the expressions for c_0 are positive. Therefore one has $c_0(r_>) < 0$ and $c_0(r_<) > 0$. Therefore $\dot{n}$ changes sign between $r = r_>$ and $r = r_<$. Since $\dot{n}$ is a continuous function of r_L, there must be a zero in between. This argument holds for all pieces B_k with the exception of those adjacent to A_1. (For those adjacent l may become 0. In this limit one finds $\dot{n} = 0$.) Therefore we have at least $p-2$ solutions, which is in agreement with those found in [2]. We leave aside the question, whether for sufficiently large ratios $r_>/r_<$ more solutions appear and whether there are other solutions for $\rho \neq 1/2$. We finally mention that in the near-circular case $(r_> - r_<)/(r_> + r_<) \ll 1$ the periodicity p is related to a by $p^2 = 1 + 8ar^3$, where r is some intermediate radius.

4   Conclusion

We have given a closed representation of the boundaries of logs floating in any direction with $\rho \neq 1/2$ in terms of a differential equation which can be solved by an elliptic integral. The differential equation could be obtained, since points of two of these curves rotated against each other can be connected by chords so that by progressing on the two curves by the same length of the perimeter the length of the chord does not change. Using that this holds also for two curves rotated against each other by an infinitesimal angle connected by a chord of infinitesimal length the differential equation could be derived. The differential equation of the linear analogue of this curve has also been given.

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