Floating Bodies of Equilibrium

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Abstract A long cylindrical body of circular cross-section and homogeneous density may float in all orientations around the cylinder axis. It is shown that there are also bodies of non-circular cross-sections which may float in any direction. Apart from those found by Auerbach for $\rho = 1/2$ there are one-parameter families of cross-sections for $\rho \neq 1/2$ which have a p-fold rotation axis. For given p they exist for $p-2$ densities ρ. There are strong indications, that for all $p-2$ densities one has the same family of cross-sections.

1 Introduction

A long standing problem asked by Stanislaw Ulam in the Scottish Book [1] (problem 19) is, whether a sphere is the only solid of uniform density which will float in water in any position. A simpler, two-dimensional, question is: Consider a long log of circular cross-section; it will, obviously, float in any position without tending to rotate. (Of course, the axis of the log is assumed to be parallel to the water surface.) Are there any additional cross-section shapes such that the log will float in any position? This question was answered in 1938 by Auerbach [2] for the special density $\rho = 1/2$ (the density of water is normalized to unity). He showed, that for this special density there is a large variety of cross-sections.

Here we consider the case $\rho \neq 1/2$. We observe that for special densities one may deform the circular cross-section into one with p-fold symmetry axis and mirror symmetry. There are $p-2$ densities, for which this is possible. For given p the densities appear pairwise as ρ and $1-\rho$. For odd p there is also a solution for $\rho = 1/2$. These solutions can be expanded in powers of the deformation ϵ. In polar coordinates (r, ψ) they are written

$$r(\psi) = \bar{r}(1 + 2\epsilon\cos(p\psi) + 2\sum_{n=2}^{\infty} c_n \cos(pn\psi)), \qquad (1)$$

where the coefficients c_n are functions of ϵ, with $c_n = O(\epsilon^n)$. The associated densities depend on ϵ.

I have carried through the calculations of the coefficients up to order ϵ^7 and surprisingly I obtain the same expansion for all $p-2$ solutions, although I did expect agreement only for pairs ρ and $1-\rho$. Thus I conjecture that there is a one-parameter family (parametrized by ϵ) of bodies for given p which floats indifferently at $p-2$ different densities.

The outline of the paper is as follows: In the next section I derive some general properties of these cross-sections. In the following section I consider the special case $\rho = 1/2$. The case of $\rho \neq 1/2$ is treated in first order of the distortion in section 4 and in higher orders in section 5. Section 6 contains a short conclusion.

2 General Considerations

I start with some considerations for general densities ρ. I denote the cross-section area of the log by A, the cross-section above the water be A_1, that below the water A_2, then

$$A_1 = (1-\rho)A, \quad A_2 = \rho A \tag{2}$$

according to Archimedes. Moreover I denote by m the total mass of the log, by $m_{1,2}$ the masses above/below the water-line, by $C_{1,2}$ the center of mass above/below the water-line, by $h_{1,2}$ the distance of $C_{1,2}$ from the water-line, and by $\hat{L}$ the length of the log.

Then the potential energy of the system is

$$\begin{aligned} V &= m_1 g h_1 + (m - m_2) g h_2 = \rho A_1 \hat{L} h_1 g + (A_2 \hat{L} - A_2 \rho \hat{L}) g h_2 \\ &= \rho(1-\rho) A \hat{L} g (h_1 + h_2). \end{aligned} \tag{3}$$

Thus the difference in height between the two centers of mass, $h_1 + h_2$, has to be independent of the orientation of the log. This does not imply, that h_1 and h_2 are separately constant. Moreover the line C_1C_2 connecting the two centers of mass has to be perpendicular to the water-level. In the following I will consider convex simply connected cross-sections. This condition may be relaxed, but it is necessary, that the water-level crosses the circumference of the log at exactly two points denoted by L and R. The distance between these two points be $2l$.

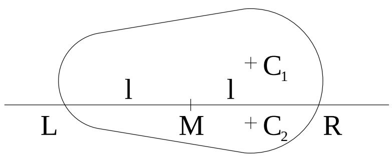

Figure 1

Let us consider now what happens if we rotate the log by an infinitesimal angle $\delta\phi$. Since the area above and below the water has to be conserved the rotation is around the midpoint M of the line LR. Then on the right the area $l^2\delta\phi/2$ rises above the water and on the left the same area disappears below the water-level. Let us consider the shift of the centers of mass in horizontal direction. There are two changes: Due to the rotation C_1 moves to the left by $\delta\phi h_1$ and C_2 to the right by $\delta\phi h_2$. The appearance of the area $l^2\delta\phi/2$ on the right and the disappearance of the same area on the left moves the center C_1 of mass by $\delta\phi(\int_0^l l^2 dl - \int_{-l}^0 l^2 dl)/A_1 = \frac{2}{3}\delta\phi l^3/A_1$ to the right. Similarly the center of mass below the water level C_2 moves to the left by $\frac{2}{3}\delta\phi l^3/A_2$.

In total the shift to the right of C_1 is $(-h_1 + \frac{2}{3}l^3/A_1)\delta\phi$ and that of C_2 $(h_2 - \frac{2}{3}l^3/A_2)\delta\phi$. Both have to be equal since otherwise a torque would be created. Thus we have

$$\frac{2}{3}l^3(\frac{1}{A_1} + \frac{1}{A_2}) = h_1 + h_2. \tag{4}$$

Since the distance $h_1 + h_2$ has to be constant and also A_1 and A_2 are constant, we obtain the important result, that also l is constant. If (4) is not fulfilled, but the left hand side is larger then the right hand side, then the log is in stable equilibrium, if the left hand side is less then the right hand side then the equilibrium is unstable. But here we are interested in an indifferent equilibrium, where equality (4) holds.

When we continue to rotate the log, then in general the mid-point M between L and R will move. It can move only along the direction of the line LR. We consider now the loci of the points M as a function of ϕ. In order to do this we keep however the orientation of the log fixed and rotate the direction of the gravitational force and the orientation of the watersurface. Since M can move only in direction of the line LR whose slope is now given by $\tan\phi$ we can parametrize the loci of M by $x_M(\phi)$ and $y_M(\phi)$, where $s(\phi)d\phi$ is the shift of M along LR as we rotate by $d\phi$. It is now obvious that the solution of the problem can be parametrized by

$$x_R(\phi) = x_M(\phi) + l\cos\phi, \qquad y_R(\phi) = y_M(\phi) + l\sin\phi, \tag{5}$$

$$x_M(\phi) = x_M(0) + \int_0^\phi s(\phi)\cos\phi\,\mathrm{d}\phi, \qquad y_M(\phi) = y_M(0) + \int_0^\phi s(\phi)\sin\phi\,\mathrm{d}\phi. \tag{6}$$

$x_M(\phi)$ and $y_M(\phi)$ parametrize the envelope of the water-lines LR. Considering the boundary of the log we observe, that the part of the circumference which disappears on one side under rotation by $\delta\phi$ below the water-line and appears on the other side is $\sqrt{s^2 + l^2}\delta\phi$ on both sides, since in horizontal direction the contribution is $s\mathrm{d}\phi$ and in vertical direction $l\mathrm{d}\phi$. Thus the part of the circumference below the waterline is constant independent of the orientation. Obviously the envelope of the water-line must be closed, that is

$$\int_0^{2\pi} s(\phi)\cos\phi\,\mathrm{d}\phi = 0, \qquad \int_0^{2\pi} s(\phi)\sin\phi\,\mathrm{d}\phi = 0 \tag{7}$$

It must be noted however, that the boundary of the log is not only given by eq. (5), but it is also obtained by reversing the sign of l, that is not only by considering the points R but also the points L

$$x_L(\chi) = x_M(\chi) - l\cos\chi, \quad y_L(\chi) = y_M(\chi) - l\sin\chi. \tag{8}$$

The problem is now to find envelopes, parametrized by $s(\phi)$, so that both descriptions coincide. Of course the arguments ϕ and χ are different in both expressions in order to describe the same point on the boundary.

3 The case $\rho = \frac{1}{2}$

In the special case $\rho = \frac{1}{2}$ one has $A_1 = A_2 = A/2$, with the implication that for ϕ and $\phi + \pi$ one has the same line LR separating the part of the log above and that below the water-level, only L and R are exchanged, but M is the same for ϕ and $\chi = \phi + \pi$

$$x_M(\phi) = x_M(\phi + \pi), \quad y_M(\phi) = y_M(\phi + \pi). \tag{9}$$

If we differentiate these equations with respect to ϕ using the representation (6) we obtain the condition

$$s(\phi + \pi) = -s(\phi) \tag{10}$$

and taking it for $\phi = 0$ yields

$$\int_0^\pi s(\phi) \cos\phi \, d\phi = 0, \quad \int_0^\pi s(\phi) \sin\phi \, d\phi = 0 \tag{11}$$

The conditions (10) and (11) and a sufficient 'convexity' is sufficient to obtain a body which flows in each direction. Curves of this type were already discussed by Zindler in section 6 of [3] but apparently without reference to this physical problem.

As an example we consider

$$s(\phi) = a\cos(3\phi) \tag{12}$$

which yields

$$x_M(\phi) = \frac{a}{4}\sin(2\phi) + \frac{a}{8}\sin(4\phi), \quad y_M(\phi) = \frac{a}{4}\cos(2\phi) - \frac{a}{8}\cos(4\phi) - \frac{a}{8}, \tag{13}$$

a curve which has kinks at $(\pm 3\sqrt{3}a/16,\ a/16)$, $(0,\ -a/2)$. For $l > 3a$ the circumference is convex, but it might be that it is sufficient to have $l > a/2$, so that the circumference lies completely outside the loci of M.

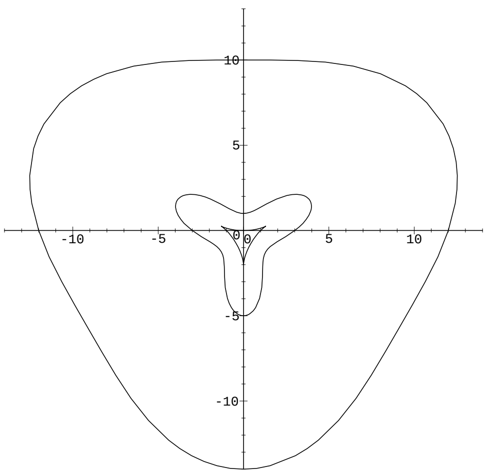

Figure 2. This figure shows the loci of M for $a = 4$ (innermost curve) and the boundaries for $l = 3$ and $l = 12$ (outermost curve). A log with the cross-section of the outermost curve and $\rho = 1/2$ will flow in water in any position without tendency to rotate. The waterline will be tangent to the innermost curve.

One may however, choose a cross-section with very elementary pieces of the boundary. We may e.g. choose a triangle as envelope of the waterlines. Then we obtain a cross-section bounded by six straight lines and six arcs as shown in the next figure. This corresponds to an $s(\phi)$ consisting of a sum of three δ-distributions.

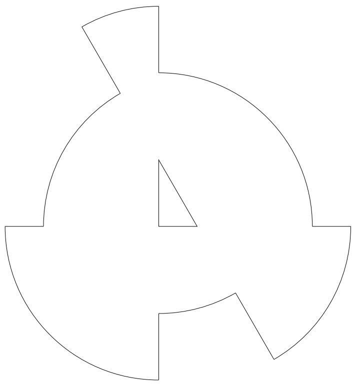

Figure 3. Another example for $\rho = 1/2$.

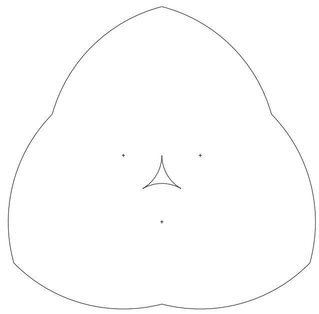

Figure 4. A further example for $\rho = 1/2$.

A further possibility is an envelope of a water-line consisting of three arcs which yields a boundary consisting of six arcs as shown in figure 4. In this case $s(\phi)$ is piecewise constant with alternating signs.

4 The case $\rho \neq \frac{1}{2}$. Linear Theory

We continue our consideration on floating rods for densities $\rho \neq 1/2$. We describe the boundary in terms of polar coordinates $r(\psi)$. Each point of the boundary can be at the water-level in two orientations given by ϕ and χ (figure 5a), thus obeying

$$x(\psi) = r(\psi)\cos\psi = x_M(\phi) + l\cos\phi = x_M(\chi) - l\cos\chi, \quad (14)$$
$$y(\psi) = r(\psi)\sin\psi = y_M(\phi) + l\sin\phi = y_M(\chi) - l\sin\chi. \quad (15)$$

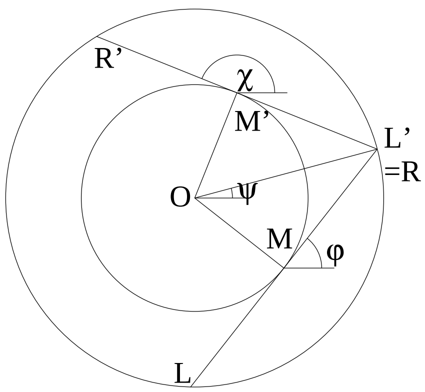

Figure 5a. The two waterlines meeting at R=L’.

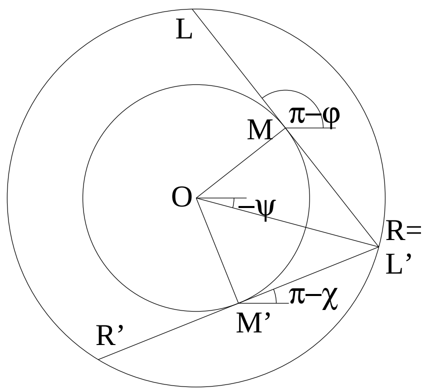

Figure 5b. The picture reflected at the x-axis.

We consider ϕ and χ to be functions of ψ and differentiate these equations with respect to ψ,

$$
\begin{aligned}
r'\cos\psi - r\sin\psi &= \phi'(s(\phi)\cos\phi - l\sin\phi) = \chi'(s(\chi)\cos\chi + l\sin\chi), &(16)\\
r'\sin\psi + r\cos\psi &= \phi'(s(\phi)\sin\phi + l\cos\phi) = \chi'(s(\chi)\sin\chi - l\cos\chi). &(17)
\end{aligned}
$$

Multiplying these equations by $\cos\psi$ and $\sin\psi$ and adding or subtracting them we obtain

$$
\begin{aligned}
r' &= \phi'(s(\phi)\cos(\psi-\phi) + l\sin(\psi-\phi))\\
&= \chi'(s(\chi)\cos(\chi-\psi) + l\sin(\chi-\psi)). &(18)\\
r &= \phi'(s(\phi)\sin(\phi-\psi) + l\cos(\phi-\psi))\\
&= \chi'(s(\chi)\sin(\chi-\psi) - l\cos(\chi-\psi)). &(19)
\end{aligned}
$$

We first consider the trivial solution of a circular cross-section, where r and s are constant. For these we obtain

$$\phi = \psi + \delta_0, \qquad \chi = \psi + \pi - \delta_0, \tag{20}$$

$$s = s_0 = r_0\sin\delta_0, \qquad l = r_0\cos\delta_0, \quad r = r_0 = \sqrt{s_0^2 + l^2}. \tag{21}$$

For the circular case δ_0 is the angle between the water-line LR and the radius OR in figure 5a.

In the next step we allow a variation of our four functions r, s, ϕ and χ in order to admit non-circular solutions. For this purpose we write

$$r = r_0 + \hat{r}(\psi), \quad s(\phi) = s_0 + \hat{s}(\phi), \quad \phi = \psi + \delta_0 + \hat{\phi}(\psi), \quad \chi = \psi + \pi - \delta_0 + \hat{\chi}(\psi). \tag{22}$$

Then the equations (18) to (19) read

$$
\begin{aligned}
\hat{r}' &= (1+\hat{\phi}')(\hat{s}(\psi+\delta_0+\hat{\phi})\cos(\delta_0+\hat{\phi}) - r_0\sin\hat{\phi}), &(23)\\
&= (1+\hat{\chi}')(-\hat{s}(\psi+\pi-\delta_0+\hat{\chi})\cos(\delta_0-\hat{\chi}) - r_0\sin\hat{\chi}), &(24)\\
r_0 + \hat{r} &= (1+\hat{\phi}')(\hat{s}(\psi+\delta_0+\hat{\phi})\sin(\delta_0+\hat{\phi}) + r_0\cos\hat{\phi}), &(25)\\
&= (1+\hat{\chi}')(\hat{s}(\psi+\pi-\delta_0+\hat{\chi})\sin(\delta_0-\hat{\chi}) + r_0\cos\hat{\chi}). &(26)
\end{aligned}
$$

4.1 Linearization

We assume, that the quantities $\hat{r}$, $\hat{s}$, $\hat{\phi}$, $\hat{\chi}$ are small quantities, which can be expanded in powers of an expansion parameter ϵ. Suppose we have solved the equations (23) to (26) up to some order in ϵ and wish to calculate the next order. Then by expanding these equations in powers of ϵ we may extract the contributions in this new order. The linear terms in this new order are unknown, the non-linear ones are already known. We bring the linear terms on the left-hand side of the equations (23) to (26) and the other terms on the right-hand side. This yields the equations

$$
\begin{aligned}
\hat{r}' + r_0\hat{\phi} - \cos\delta_0\hat{s}(\psi+\delta_0) &= I_1(\psi), &(27)\\
\hat{r}' + r_0\hat{\chi} + \cos\delta_0\hat{s}(\psi+\pi-\delta_0) &= I_2(\psi), &(28)\\
\hat{r} - r_0\hat{\phi}' - \sin\delta_0\hat{s}(\psi+\delta_0) &= I_3(\psi), &(29)\\
\hat{r} - r_0\hat{\chi}' - \sin\delta_0\hat{s}(\psi+\pi-\delta_0) &= I_4(\psi). &(30)
\end{aligned}
$$

where the non-linear terms are in the $I(\psi)$. Let us expand now our unknowns in Fourier series,

$$\hat{r} = \sum_k \hat{r}_k \mathrm{e}^{\mathrm{i}k\psi}, \quad \hat{s}(\phi) = \sum_k \hat{s}_k \mathrm{e}^{\mathrm{i}k\phi}, \quad \hat{\phi} = \sum_k \hat{\phi}_k \mathrm{e}^{\mathrm{i}k\psi}, \tag{31}$$

$$\hat{\chi} = \sum_k \hat{\chi}_k \mathrm{e}^{\mathrm{i}k\psi}, \quad I_n = \sum_k I_{n,k} \mathrm{e}^{\mathrm{i}k\psi}. \tag{32}$$

Then our system of equations reads

$$\mathrm{i}k\hat{r}_k + r_0\hat{\phi}_k + C_{1,k}\hat{s}_k = I_{1,k}, \qquad C_{1,k} = -\cos\delta_0 \mathrm{e}^{\mathrm{i}k\delta_0} \tag{33}$$

$$\mathrm{i}k\hat{r}_k + r_0\hat{\chi}_k + C_{2,k}\hat{s}_k = I_{2,k}, \qquad C_{2,k} = +\cos\delta_0 \mathrm{e}^{\mathrm{i}k(\pi-\delta_0)} \tag{34}$$

$$\hat{r}_k - \mathrm{i}kr_0\hat{\phi}_k + C_{3,k}\hat{s}_k = I_{3,k}, \qquad C_{3,k} = -\sin\delta_0 \mathrm{e}^{\mathrm{i}k\delta_0} \tag{35}$$

$$\hat{r}_k - \mathrm{i}kr_0\hat{\chi}_k + C_{4,k}\hat{s}_k = I_{4,k}, \qquad C_{4,k} = -\sin\delta_0 \mathrm{e}^{\mathrm{i}k(\pi-\delta_0)}. \tag{36}$$

The determinant of this linear system of equations is $(1-k^2)\hat{C}_k$ with

$$\begin{aligned} \hat{C}_k &= \mathrm{i}k(C_{2,k} - C_{1,k}) + C_{4,k} - C_{3,k} \\ &= \begin{cases} 2\mathrm{i}(k\cos\delta_0\cos(k\delta_0) + \sin\delta_0\sin(k\delta_0)) & k \text{ even} \\ -2(k\cos\delta_0\sin(k\delta_0) - \sin\delta_0\cos(k\delta_0)) & k \text{ odd.} \end{cases} \end{aligned} \tag{37}$$

Generally one has

$$\hat{s}_k = \frac{\hat{I}_k}{\hat{C}_k}, \quad \hat{I}_k = \mathrm{i}k(I_{2,k} - I_{1,k}) + I_{4,k} - I_{3,k}. \tag{38}$$

If $\hat{C}_k = 0$, (which is always the case, if $k = 0, \pm 1$) then a solution exists only, if $\hat{I}_k = 0$. With the exception of $k = \pm 1$ the choice of $\hat{s}_k$ is arbitrary, and one obtains

$$\begin{aligned} (1-k^2)\hat{r}_k &= I_{3,k} + \mathrm{i}kI_{1,k} - (\mathrm{i}kC_{1,k} + C_{3,k})\hat{s}_k \\ &= I_{4,k} + \mathrm{i}kI_{3,k} - (\mathrm{i}kC_{2,k} + C_{4,k})\hat{s}_k \end{aligned} \tag{39}$$

$$(1-k^2)r_0\hat{\phi}_k = I_{1,k} - \mathrm{i}kI_{3,k} - (C_{1,k} - \mathrm{i}kC_{3,k})\hat{s}_k \tag{40}$$

$$(1-k^2)r_0\hat{\chi}_k = I_{2,k} - \mathrm{i}kI_{4,k} - (C_{2,k} - \mathrm{i}kC_{4,k})\hat{s}_k \tag{41}$$

We consider now the cases $k = \pm 1$, $k = 0$, and the special case, where $\hat{C}_k$ vanishes for some other integer $k = p$.

4.2 Translational Invariance and the case $k = \pm 1$

In this subsection we show that the vanishing of the determinant for $k = \pm 1$ is related to the translational invariance of the problem. For $k = \pm 1$ we have

$$\hat{s}_{\pm 1} = \int_0^{2\pi} \mathrm{d}\phi s(\phi)\mathrm{e}^{\pm\mathrm{i}\phi} = x_M(2\pi) - x_M(0) \pm \mathrm{i}(y_M(2\pi) - y_M(0)) = 0. \tag{42}$$

Thus we obtain only a solution, if

$$I_{3,\pm 1} = \mp\mathrm{i}I_{1,\pm 1}, \quad I_{4,\pm 1} = \mp\mathrm{i}I_{2,\pm 1}. \tag{43}$$

Then $\hat{\phi}$ and $\hat{\chi}$ can be expressed by

$$r_0\hat{\phi}_{\pm 1} = \mp \mathrm{i}\hat{r}_{\pm 1} + I_{1,\pm 1}, \quad r_0\hat{\chi}_{\pm 1} = \mp \mathrm{i}\hat{r}_{\pm 1} + I_{2,\pm 1} \tag{44}$$

with arbitrary $\hat{r}$.

This arbitrary choice of $\hat{r}_{\pm 1}$ is related to the translational invariance of the problem. We are free to move the system by $(\delta x, \delta y)$. Then also the angle ψ and the distance r from the origin will change,

$$\bar{x}(\psi + \delta\psi) = \bar{r}(\psi + \delta\psi)\cos(\psi + \delta\psi) = x(\psi) + \delta x = r(\psi)\cos\psi + \delta x, \tag{45}$$

$$\bar{y}(\psi + \delta\psi) = \bar{r}(\psi + \delta\psi)\sin(\psi + \delta\psi) = y(\psi) + \delta y = r(\psi)\sin\psi + \delta y. \tag{46}$$

With $\bar{r}(\psi) = r(\psi) + \delta r(\psi)$ one obtains

$$(r(\psi + \delta\psi)) + \delta r)\cos(\psi + \delta\psi) = r(\psi)\cos(\psi) + \delta x, \tag{47}$$

$$(r(\psi + \delta\psi)) + \delta r)\sin(\psi + \delta\psi) = r(\psi)\sin(\psi) + \delta y. \tag{48}$$

and thus

$$\delta x = (r'\delta\psi + \delta r)\cos\psi - r\sin\psi\delta\psi, \tag{49}$$

$$\delta y = (r'\delta\psi + \delta r)\sin\psi + r\cos\psi\delta\psi, \tag{50}$$

which yields

$$r\delta\psi = \delta y\cos\psi - \delta x\sin\psi, \tag{51}$$

$$\delta r = \delta x\cos\psi + \delta y\sin\psi - \frac{r'}{r}(\delta y\cos\psi - \delta x\sin\psi). \tag{52}$$

Since we presently consider the circular solution, $r' = 0$ holds, and

$$\delta r = \mathrm{e}^{\mathrm{i}\psi}\hat{r}_1 + \mathrm{e}^{-\mathrm{i}\psi}\hat{r}_{-1} \tag{53}$$

with $\hat{r}_{\pm 1} = \frac{\delta x \pm \mathrm{i}\delta y}{2}$. Since we have shifted the system parallel, $s(\phi)$ and $s(\chi)$ did not change, $s_{\pm 1} = 0$, but $\bar{\phi}(\psi + \delta\psi) = \phi(\psi)$, which yields

$$\delta\phi = -\phi'\delta\psi = -\frac{\delta y}{r}\cos\psi + \frac{\delta x}{r}\sin\psi = -\frac{\mathrm{i}\hat{r}_1}{r}\mathrm{e}^{\mathrm{i}\psi} + \frac{\mathrm{i}\hat{r}_{-1}}{r}\mathrm{e}^{-\mathrm{i}\psi}. \tag{54}$$

One obtains the same result for $\delta\chi$. This is in agreement with the homogeneous equations (33) to (36) for $k = \pm 1$.

4.3 Variation of the density and the case $k = 0$

A small but constant variation of $\delta_0 \to \delta_0 + \alpha$ yields for constant $l = r_0\cos\delta_0$

$$\hat{r}_0\cos\delta_0 - r_0\alpha\sin\delta_0 = 0. \tag{55}$$

This variation of δ_0 corresponds to a variation of the density ρ. For s, ϕ and χ we obtain

$$\hat{s}_0 = \hat{r}_0\sin\delta_0 + r_0\alpha\cos\delta_0, \quad \hat{\phi}_0 = \alpha, \quad \hat{\chi}_0 = -\alpha. \tag{56}$$

Elimination of α yields

$$\hat{r}_0 = \sin\delta_0\hat{s}_0, \quad r_0\hat{\phi}_0 = \cos\delta_0\hat{s}_0, \quad r_0\hat{\chi}_0 = -\cos\delta_0\hat{s}_0 \tag{57}$$

in agreement with the homogeneous solution of (33) to (36) for $k = 0$.

4.4 The case $\hat{C}_p = 0$, $p \neq 0, \pm 1$

If $\hat{C}_p(\delta_0)$ equals 0 for another p, then again we have a homogeneous solution. This allows us to find non-circular solutions. Let us first consider the zeroes of $\hat{C}_k(\delta_0)$. We find, the $\hat{C}$ vanishes for $\cos\delta_0 = 0$. It turns out, that this is not a single zero, but a three-fold zero. Since moreover $\hat{C}_k(-\delta_0) = (-)^k \hat{C}_k(\delta_0)$, one finds, that $\hat{C}_k$ is $\cos^3\delta_0$ times a polynomial of order $k-2$ in $\cos\delta_0$ for even k, and $\sin\delta_0 \cos^3\delta_0$ times a polynomial of order $k-3$ in $\cos\delta_0$ for odd k. Moreover one has $C_k(\pi - \delta_0) = (-)^{k-1} C_k(\delta_0)$. Thus the polynomials are even in $\cos\delta_0$ or simply polynomials in $\cos(2\delta_0)$. With the abbreviations

$$c_{\delta_0} = \cos\delta_0, \quad s_{\delta_0} = \sin\delta_0, \quad c_{2\delta_0} = \cos(2\delta_0) \tag{58}$$

we write

$$\hat{C}_{2k+2} = 4\mathrm{i} c_{\delta_0}^3 P_k(c_{2\delta_0}), \tag{59}$$

$$\hat{C}_{2k+3} = -16 s_{\delta_0} c_{\delta_0}^3 Q_k(c_{2\delta_0}) \tag{60}$$

and obtain

$$P_0 = 1, \qquad Q_0 = 1, \tag{61}$$

$$P_1 = 6c_{2\delta_0} - 4, \qquad Q_1 = 4c_{2\delta_0} - 1, \tag{62}$$

$$P_2 = 20c_{2\delta_0}^2 - 16c_{2\delta_0} + 1, \qquad Q_2 = 12c_{2\delta_0}^2 - 4c_{2\delta_0} - 2, \tag{63}$$

$$P_3 = 56c_{2\delta_0}^3 - 48c_{2\delta_0}^2 - 12c_{2\delta_0} + 8, \qquad Q_3 = 32c_{2\delta_0}^3 - 12c_{2\delta_0}^2 - 12c_{2\delta_0} + 2 \tag{64}$$

Limits for the zeroes of $\hat{C}_p$ can be easily given, since the sign of $\hat{C}_p$ can be easily determined, if δ_0 assumes integer multiples of $\pi/(2p)$. Thus for even p one has zeroes $\delta_{0,l}$ obeying $\frac{\pi}{p}(l - \frac{1}{2}) < \delta_{0,l} < \frac{\pi}{p} l$ for $l = 1, 2, ... \frac{p}{2} - 1$. For odd p one has zeroes $\frac{\pi}{p} l < \delta_{0,l} < \frac{\pi}{p}(l + \frac{1}{2})$ for $l = 1, 2, ... \frac{p-3}{2}$.

The density corresponding to the angle δ_0 is

$$\rho_0 = \frac{1}{2} - \frac{\delta_0}{\pi} - \frac{\sin(2\delta_0)}{2\pi}. \tag{65}$$

Thus in first order in $\hat{s}$ we may distort the body from its circular shape by adding a distortion described by

$$\hat{s}(\phi) = \hat{s}_p \mathrm{e}^{\mathrm{i}p\phi} + \hat{s}_{-p} \mathrm{e}^{-\mathrm{i}p\phi}. \tag{66}$$

and obtain a non-circular body which floats in any orientation. In lowest order of the distortion it has density ρ_0. The angles and densities for $p = 4...9$ are listed below.

p	l	$\delta_0[^0]$	ρ_0
4	1	24.095	0.24751
5	1	37.761	0.13611
6	1	15.439	0.33255
6	2	46.670	0.08184
7	1	26.291	0.22753
7	2	52.959	0.05273
8	1	11.431	0.37466
8	2	34.361	0.16080
8	3	57.645	0.03585
9	1	20.261	0.28403
9	2	40.605	0.11713
9	3	61.273	0.02543

Only the solutions with positive δ_0 are listed. Besides them the angles $-\delta_0$ and the densities $1-\rho_0$ are also solutions. For odd p one has in addition the solution $\delta_0 = 0$ and $\rho_0 = 1/2$. In total we have $p-2$ solutions δ_0 for given p without the unphysical solutions $\cos\delta_0 = 0$ (see the discussion after eq. 90). Thus with increasing p the densities become arbitrarily dense.

We may also add some $\hat{s}_0$, which however will not change the circular shape but the density in first order in $\hat{s}_0$.

5 Higher Orders in the Distortion

Our strategy will now be to calculate the contributions to $\hat{I}_k$ in higher orders in the distortion. As long as $k \neq p$, this will yield contributions to $\hat{s}_k$ which depend in higher orders on $\hat{s}_0$ and $\hat{s}_{\pm p}$, but we have also to make sure, that $\hat{I}_0$ and $\hat{I}_{\pm p}$ vanish in all orders in the distortion.

5.1 Rotational and Reflection Invariance

The non-circular solution we have found in first order in the distortion has a p-fold rotation axis and mirror symmetry. In higher orders in the distortion we will obtain higher harmonics and quite generally we can expand $\psi - \delta_0$, $\chi - \pi + \delta_0$, s and r in a Fourier-series which contains only terms proportional to $\mathrm{e}^{\mathrm{i}np\psi}$ with integer n. The reason is, that we wish to keep the p-fold rotation axis. Then only components $\mathrm{e}^{\mathrm{i}m\psi}$ are allowed, which for $\psi \to \psi + 2\pi/p$ reproduce themselves, thus $\mathrm{e}^{\mathrm{i}m2\pi/p} = 1$ is required, which implies, that m is an integer multiple of p. For our expansion we use now $s_p = u$ and $s_{-p} = v$ as expansion parameters

$$\frac{\hat{s}_k}{r_0} = \begin{cases} u & k = p \\ v & k = -p \\ \sum_{n1,n2} c_{\mathrm{s}n_1,n_2} u^{n_1} v^{n_2} \delta_{n,n_1-n_2} & k = np \\ 0 & \text{otherwise} \end{cases} \tag{67}$$

$$\frac{\hat{r}_k}{r_0} = \begin{cases} \sum_{n_1,n_2} c_{\mathrm{r}n_1,n_2} u^{n_1} v^{n_2} \delta_{n,n_1-n_2} & k = np \\ 0 & \text{otherwise} \end{cases} \tag{68}$$

and for $\hat{\phi}$ and $\hat{\chi}$ with coefficients c_ϕ and c_χ similarly as for $\hat{r}$ with c_r. Thus we have

$$r(\psi) = r_0 \sum_{m,n} c_{\mathrm{r}m,n} u^m v^n \mathrm{e}^{\mathrm{i}p(m-n)\psi}, \tag{69}$$

$$s(\phi) = r_0 \sum_{m,n} c_{\mathrm{s}m,n} u^m v^n \mathrm{e}^{\mathrm{i}p(m-n)\phi}, \tag{70}$$

similarly for $\phi(\psi)$ and $\chi(\psi)$.

Secondly we require for our solution the mirror symmetry, which holds in first order in the distortion. If the x-axis is the axis for the mirror symmetry of the cross-section, then (fig. 5b)

$$r(\psi) = r(-\psi), \quad s(\phi) = s(\pi - \phi), \quad \phi(\psi) = \pi - \chi(-\psi). \tag{71}$$

In first order in u and v we have

$$\hat{s}(\phi) = u\mathrm{e}^{\mathrm{i}p\phi} + v\mathrm{e}^{-\mathrm{i}p\phi} = \hat{s}(\pi - \phi) = u\mathrm{e}^{\mathrm{i}p(\pi-\phi)} + v\mathrm{e}^{-\mathrm{i}p(\pi-\phi)}. \tag{72}$$

Thus u and v are connected by $v = (-)^p u$. Moreover we require $\hat{s}$ to be real, which implies $v = u^*$. Thus for even p we have real u and v, whereas for odd p they are imaginary. We now obtain

$$\begin{aligned} r(\psi) &= r_0 \sum_{m,n} c_{\mathrm{r}m,n} u^{m+n} (-)^{np} \mathrm{e}^{\mathrm{i}p(m-n)\psi} \\ = r(-\psi) &= r_0 \sum_{m,n} c_{\mathrm{r}m,n} u^{m+n} (-)^{np} \mathrm{e}^{-\mathrm{i}(m-n)\psi} \\ &= r_0 \sum_{n,m} c_{\mathrm{r}n,m} u^{m+n} (-)^{mp} \mathrm{e}^{\mathrm{i}p(m-n)\psi}. \end{aligned} \tag{73}$$

Comparison of the coefficients for the same powers of u and of $\mathrm{e}^{\mathrm{i}p\psi}$ yields

$$c_{\mathrm{r}m,n} = (-)^{(m-n)p} c_{\mathrm{r}n,m}. \tag{74}$$

Similarly one obtains

$$c_{\mathrm{s}m,n} = c_{\mathrm{s}n,m}, \quad c_{\chi m,n} = (-)^{(m-n)p+1} c_{\phi n,m}. \tag{75}$$

We also see that under this transformation eq.(27) becomes the negative of (28) and eq.(29) becomes (30). In both cases ψ translates into $-\psi$. Thus we have

$$I_{1,m,n} = (-)^{(m-n)p+1} I_{2,n,m}, \quad I_{3,m,n} = (-)^{(m-n)p} I_{4,n,m}. \tag{76}$$

Moreover one has

$$C_{1,(m-n)p} = -C_{2,(n-m)p}, \quad C_{3,(m-n)p} = C_{4,(n-m)p}. \tag{77}$$

5.2 The Expansion

In determining the expansion coefficients c_r, c_s, c_ϕ, and c_χ we proceed as follows. First we expand the equations (27) to (30) in powers of $u^m v^n$. Then we solve the coefficient equations with increasing $n+m$. From the four equations with coefficients $u^m v^n$ we use the first three ones in order to express $c_{\mathrm{r}m,n}$, $c_{\phi m,n}$, and $c_{\chi m,n}$ by the coefficients with either smaller m or n, which appear in the expressions $I_{1..4,m,n}$ and by the coefficient $c_{\mathrm{s}m,n}$. If $|m-n| > 1$, then we may use the fourth equation in order to determine $c_{\mathrm{s}m,n}$. If $m = n$, then due to the reflection symmetry $I_{3,n,n} = I_{4,n,n}$ and thus $\hat{I} = 0$. Thus the four equations are linearly dependent and $c_{\mathrm{s}n,n}$ is undetermined at this stage. When we come to $(m,n) = (n+1,n)$, then in general $\hat{I}_{n+1,n}$ will not vanish. However it depends on $c_{\mathrm{s}n,n}$, which is not yet determined. $c_{\mathrm{s}n,n}$ enters in the form

$$\hat{I}_{n+1,n} = ... + 2\mathrm{i}\frac{c_{\delta_0}^3 p(p^2-1)}{W} c_{\mathrm{s}n,n} \tag{78}$$

for even p. For odd p the factor i is absent. Thus $c_{\mathrm{s}n,n}$ is uniquely determined from $\hat{I}_{n+1,n} = 0$. $c_{\mathrm{s}n+1,n}$ is already fixed to be zero with the exception of $n = 0$, where we have normalized it to $c_{\mathrm{s}1,0} = 1$. Due to the reflection symmetry the equations for $(n, n+1)$ are simultaneously solved. A problem could occur, if not only $\hat{C}_{\pm p} = \hat{C}_0 = 0$, but also for some m one would have $\hat{C}_{mp} = 0$. In our calculations which we performed up to $m+n=7$, we did not observe any dangerous denominators in the coefficients. The denominators of $c_{\rm r}$, c_ϕ, and c_χ contained only powers of p^2, p^2-1, and W (see below), and the denominators of $c_{\rm s}$ only powers of p^2, despite of the more complicated expressions $\hat{C}_{np}$, which one might have expected in the denominators.

The expansion was performed by use of Maple. In order to implement the condition $\hat{C}_p = 0$, eq.(37) I used the representation

$$\left.\begin{array}{r} \cos(p\delta_0) = -s_{\delta_0}/W, \\ \sin(p\delta_0) = pc_{\delta_0}/W \end{array}\right\} \quad \text{for even } p \tag{79}$$

$$\left.\begin{array}{r} \cos(p\delta_0) = pc_{\delta_0}/W, \\ \sin(p\delta_0) = s_{\delta_0}/W \end{array}\right\} \quad \text{for odd } p \tag{80}$$

$$\text{with } W = \pm\sqrt{p^2 c_{\delta_0}^2 + s_{\delta_0}^2}, \tag{81}$$

where I used the explicit representation of W only at a very late stage.

Here I give the results for r, s, ϕ, and χ up to order $u^m v^n$ with $m+n \le 2$. For even p I obtain

$$\begin{aligned} \frac{r}{r_0} &= 1 + \frac{W}{p^2-1}(u\mathrm{e}^{\mathrm{i}p\psi} + v\mathrm{e}^{-\mathrm{i}p\psi}) + \frac{(2p^2-1)W^2}{2(p^2-1)^2p^2}(u^2\mathrm{e}^{2\mathrm{i}p\psi} + v^2\mathrm{e}^{-2\mathrm{i}p\psi}) \\ &- \frac{p^4(5+3c_{2\delta_0}) - 4p^2c_{2\delta_0} + c_{2\delta_0} - 1}{4(p^2-1)^2p^2}uv + \ldots \end{aligned} \tag{82}$$

$$\begin{aligned} \frac{s}{r_0} &= c_{\delta_0} + u\mathrm{e}^{\mathrm{i}p\psi} + v\mathrm{e}^{-\mathrm{i}p\psi} + \frac{(4p^2-1)s_{\delta_0}}{2p^2}(u^2\mathrm{e}^{2\mathrm{i}p\psi} + v^2\mathrm{e}^{-2\mathrm{i}p\psi}) \\ &+ \frac{s_{\delta_0}}{2p^2}uv + \ldots \end{aligned} \tag{83}$$

$$\begin{aligned} \phi &= \psi + \delta_0 - \frac{\mathrm{i}(pc_{\delta_0} + \mathrm{i}s_{\delta_0})(c_{\delta_0} - \mathrm{i}ps_{\delta_0})}{W(p^2-1)}u\mathrm{e}^{\mathrm{i}p\psi} \\ &+ \frac{\mathrm{i}(pc_{\delta_0} - \mathrm{i}s_{\delta_0})(c_{\delta_0} + \mathrm{i}ps_{\delta_0})}{W(p^2-1)}v\mathrm{e}^{-\mathrm{i}p\psi} \\ &- \frac{(p^2c_{\delta_0} - 2\mathrm{i}ps_{\delta_0} + c_{\delta_0})s_{\delta_0}(pc_{\delta_0} + \mathrm{i}s_{\delta_0})^2}{2W^2(p^2-1)p^2}u^2\mathrm{e}^{2\mathrm{i}p\psi} \\ &- \frac{(p^2c_{\delta_0} + 2\mathrm{i}ps_{\delta_0} + c_{\delta_0})s_{\delta_0}(pc_{\delta_0} - \mathrm{i}s_{\delta_0})^2}{2W^2(p^2-1)p^2}v^2\mathrm{e}^{-2\mathrm{i}p\psi} \\ &+ \frac{s_{\delta_0}c_{\delta_0}(5p^2-1)}{2(p^2-1)p^2}uv + \ldots \end{aligned} \tag{84}$$

$$\begin{aligned} \chi &= \psi + \pi - \delta_0 + \frac{\mathrm{i}(pc_{\delta_0} + \mathrm{i}s_{\delta_0})(c_{\delta_0} - \mathrm{i}ps_{\delta_0})}{W(p^2-1)}v\mathrm{e}^{-\mathrm{i}p\psi} \\ &- \frac{\mathrm{i}(pc_{\delta_0} - \mathrm{i}s_{\delta_0})(c_{\delta_0} + \mathrm{i}ps_{\delta_0})}{W(p^2-1)}u\mathrm{e}^{\mathrm{i}p\psi} \\ &+ \frac{(p^2c_{\delta_0} - 2\mathrm{i}ps_{\delta_0} + c_{\delta_0})s_{\delta_0}(pc_{\delta_0} + \mathrm{i}s_{\delta_0})^2}{2W^2(p^2-1)p^2}v^2\mathrm{e}^{-2\mathrm{i}p\psi} \\ &+ \frac{(p^2c_{\delta_0} + 2\mathrm{i}ps_{\delta_0} + c_{\delta_0})s_{\delta_0}(pc_{\delta_0} - \mathrm{i}s_{\delta_0})^2}{2W^2(p^2-1)p^2}u^2\mathrm{e}^{2\mathrm{i}p\psi} \end{aligned}$$

$$- \frac{s_{\delta_0} c_{\delta_0} (5p^2-1)}{2(p^2-1)p^2} uv + ... \qquad (85)$$

For odd p one has to replace u by $-\mathrm{i}u$ and v by $\mathrm{i}v$ in r, ϕ and χ, but one has to leave the factors u and v unchanged in s.

In a next step I have fixed the radius averaged over all angles to $\bar{r}$ and further denoted the relative amplitude of the first harmonic by 2ϵ, that is $r(\psi) = \bar{r}(1+2\epsilon\cos(p\psi))$+higher harmonics. Then to my surprise I obtained a solution depending only on ϵ and p, but not explicitely on δ_0 up to seventh order in ϵ. Here I report the result up to fifth order

$$\begin{aligned}
\frac{r(\psi)}{\bar{r}} &= 1 + 2\epsilon\cos(p\psi) \\
&+ 2\left(\frac{2p^2-1}{2p^2}\epsilon^2 + \frac{(p^2-1)(30p^6-47p^4+12p^2-3)}{48p^6}\epsilon^4\right)\cos(2p\psi) \\
&+ 2\left(-\frac{(p^2-1)(3p^4-14p^2+3)}{16p^4}\epsilon^3\right. \\
&+ \left.\frac{(p^2-1)^2(9p^8+160p^6-94p^4+24p^2-3)}{256p^8}\epsilon^5\right)\cos(3p\psi) \\
&- 2\frac{30p^8-77p^6+83p^4-33p^2+3}{48p^6}\epsilon^4\cos(4p\psi) \\
&+ 2\frac{(p^2-1)(75p^{10}-1011p^8+1414p^6-974p^4+255p^2-15)}{768p^8}\epsilon^5\cos(5p\psi) \\
&+ O(\epsilon^6). \qquad (86)
\end{aligned}$$

If this expansion continues in higher orders without dependence on δ_0, which seems quite likely, then we have for fixed p and ϵ a non-circular shape which can float in any direction for $p-2$ different densities. This is an intriguing result.

The densities are functions of ϵ,

$$\begin{aligned}
\pi\rho &= \pi\rho_0 - \frac{(p^2-1)^2 s_{\delta_0} c_{\delta_0}^3}{p^2 W^2}\epsilon^2 - \frac{(p^2-1)^2 s_{\delta_0} c_{\delta_0}^3 P_\rho}{32p^6 W^4}\epsilon^4 + O(\epsilon^6), \qquad (87) \\
P_\rho &= 3p^8 - 26p^6 - 20p^4 - 22p^2 + 1 + c_{2\delta_0}(3p^8 - 24p^6 + 6p^4 + 16p^2 - 1). \qquad (88)
\end{aligned}$$

The ratio of the part of the circumference below the water-level a to the total circumference $a_{\rm tot}$ also depends on ϵ

$$\begin{aligned}
\frac{2\pi a}{a_{\rm tot}} &= \pi - 2\delta_0 - \frac{s_{\delta_0} c_{\delta_0}(p^2-1)^2}{p^2 W^2}\epsilon^2 + \frac{s_{\delta_0} c_{\delta_0}(p^2-1)^2 P_{\rm a}}{64p^6 W^4}\epsilon^4 + O(\epsilon^6), \qquad (89) \\
P_{\rm a} &= 13p^8 + 50p^6 - 12p^4 + 14p^2 - 1 + c_{2\delta_0}(13p^8 - 6p^4 - 8p^2 + 1). \qquad (90)
\end{aligned}$$

For $\rho = 1/2$ all determinants $(k^2-1)\hat{C}_k$ vanish for odd k. This is the source for the large variety of cross-sections at $\rho = 1/2$. We may, however, use the above equations, which were initially solved for general c_{δ_0} and s_{δ_0} and finally insert $s_{\delta_0} = 0$, $c_{\delta_0} = 1$, and again we obtain the same boundary (to seventh order in ϵ).

Finally we may reconsider the "solution" $c_{\delta_0} = 0$, $s_{\delta_0} = \pm 1$. In this special case formally the boundary and the envelope of water-lines coincide. However, the parametrization is different, since r is given as a function of the polar angle

ψ, whereas x_M, y_M are given as function of the direction of the tangent on the curve. Despite the fact that there is no solution for $\rho = 0$ and $\rho = 1$ except the circle (in this case the height of the center of gravity above or below the water-line has to be constant, which according to Montejano [4] is only fulfilled by a circle), our equations are fulfilled for arbitrary closed boundaries, since in this limit $l = 0$ and thus always $\phi = \chi$. That $c_{\delta_0} = 0$ appears as a multiple solution is probably a consequence of the fact, that the fictitious water-line is parallel to the boundary and thus the coincidence of R and L' is fulfilled in higher orders. We note that formal substitution of $c_{\delta_0} = 0$ and $s_{\delta_0} = \pm 1$ again yields the same boundary.

There remains the question of the convergency of the ϵ-expansion. I do not have a definite answer to this. However, if one describes the boundary by x_M, y_M for $c_{\delta_0} = 0$, then it is apparent, that there is a unique solution only as long as the cross-section is convex, since otherwise x_M, y_M are no longer unique functions of ϕ. Thus the radius of convergency cannot be larger then the maximal ϵ for a convex boundary, which can be estimated $\epsilon \approx 1/(2p^2)$. From the numerical calculations it seems likely, that this is the radius of convergency for $c_{\delta_0} = 0$. For larger c_{δ_0}, that is for ρ closer to $1/2$ the radius of convergency seems to be larger.

We show now several examples. For given p and ϵ we have plotted points of the envelope of the water-line for different ρ in intervals of $6^0 = \pi/30$ and the corresponding points on the boundary for various cases. For odd p I include the result for $\rho = 1/2$. As long as $\epsilon \leq 1/(2p^2)$, I also include the points for $c_{\delta_0} = 0$, which lie on the boundary. If one would include the points calculated for $c_{\delta_0} = 0$ for $\epsilon = 1/p^2$, then one would see that they scatter around and do not fit on a smooth boundary, indicating the non-convergency of the ϵ-expansion in this case. All points have been calculated from the expansion of x_M, y_M to order ϵ^7.

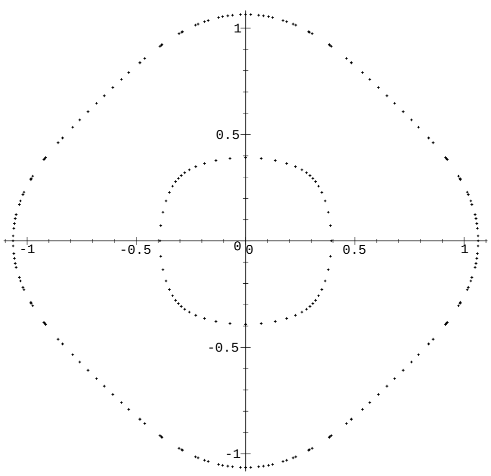

Figure 6. $p = 4$, $\epsilon = 1/(2p^2)$.

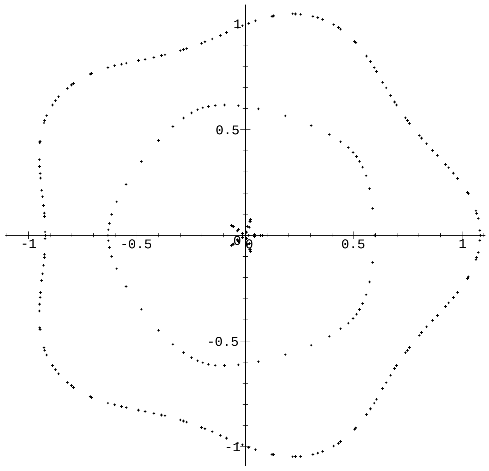

Figure 7. $p = 5$, $\epsilon = 1/p^2$.

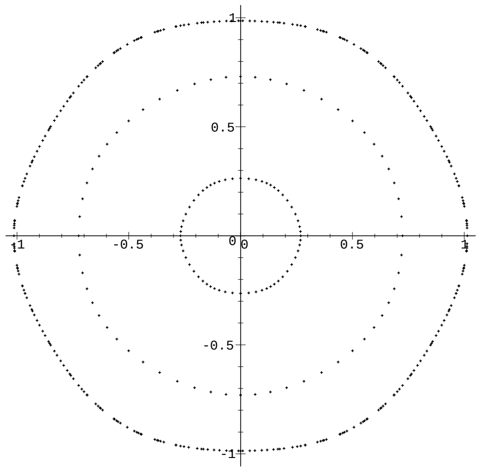

Figure 8. $p = 6$, $\epsilon = 1/(4p^2)$.

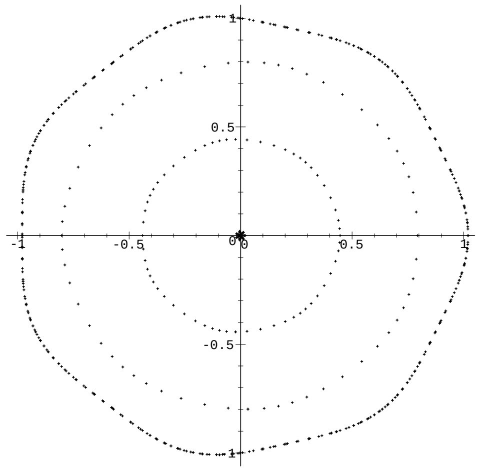

Figure 9. $p = 7$, $\epsilon = 1/(2p^2)$.

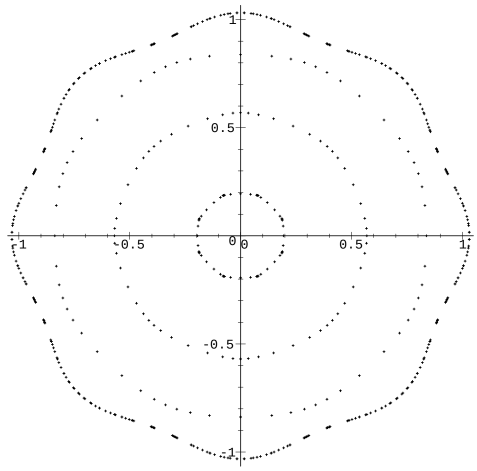

Figure 10. $p = 8$, $\epsilon = 1/p^2$.

6 Conclusion

Here we have shown, that for densities different from 1/2 it is possible to have logs with non-circular cross-sections, which float indifferently in all orientations around the axis. The cross-sections which are presented have a p-fold rotation axis and p mirror lines. For each p we have a one-parameter family of solutions at $p-2$ densities. From the $\rho \to 1-\rho$ symmetry it is expected, that the solutions are pairwise equal. However, the expansion in the deformation parameter ϵ yields $p-2$ equal solutions up to order ϵ^7, from which I conjecture, that one and the same manifold of cross-sections floats indifferently for all $p-2$ densities. It would be very useful to obtain the curves in a non-perturbative way, in order to verify or falsify the conjecture.

We have obtained these solutions by deforming the circular solution. One may also try to deform the solutions for $\rho = 1/2$ or the solutions here obtained by looking, if for some ϵ again a non-trivial deformation is possible which might yield a solution of even lower symmetry.

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