Gas of Rotators (11/04)

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1 Solution

First of all, we examine the problem of rigid rotators in three dimensions from a statistical point. The Hamiltonian is $H = p^2/2m + p_\theta^2/2I$, where p is the CM (center mass) momentum, p_θ is the angular momentum and I the moment of inertia. The partition function is then:

$$Z = \frac{1}{N!}\left\{\frac{(mI)^{3/2}V}{\hbar^2}\left(\frac{2\pi}{\beta}\right)^3\right\}^N, \tag{1}$$

where $\beta = 1/kT$. The pressure is then $P = (1/\beta)\left(\frac{\partial \ln Z}{\partial V}\right)_\beta$, which is obviously independent of the nature of the molecules (rod-like or point-like). We shall now try to explain this formal result microscopically.

For this purpose, we examine a collision of such a rigid rod against a wall. We imagine a rod of length L and mass m moving freely towards the wall with CM velocity v_0 and angular velocity ω_0 (for simplification we consider the CM motion normal to the wall, the rest is unaffected). The motion of the CM and the relative rotation with respect to the CM are independent and therefore analyzed separately. For the CM the whole event is but a linear collision against a wall with "infinite" mass. The initial momentum $+mv_0$ simply changes sign to $-mv_0$ and thus, the linear momentum transfer is $\Delta p = 2mv_0$. The total momentum transfer (from all such collisions) per unit time contributes to the pressure. We now examine the contribution of the relative part of the motion. We imagine the rod exactly at the time its "endpoint" hits the wall. The collision is assumed to take place so fast, that the total angular momentum of the system (rod+wall) with respect to the CM is conserved. This assumption is analogous to the assumption of zero impulse (leading to the conservation of linear momentum) in a collision between point particles. Therefore we expect the total angular momentum to be conserved. Before the collision the rod had angular momentum $+I\omega_0$ and afterwards it must have $-\omega_0$, in complete analogy to a linear collision against a wall (in this case the wall is assumed to have "infinite" moment of inertia). This argument can be proved easily by moving to a "Center-of-Angular-Momentum" (CAM) reference frame rotating with angular frequency $I\omega_0/(I+I_{wall}) \approx 0$ with respect to the center of mass between the rod and the wall. In such a reference frame the angular momenta of the wall and the rod are the same before and after the collision but have reversed directions. They are $L_{CAM} = I_{wall}I\omega_0/(I+I_{wall})$ and $L_{wall,CAM} = -I_{wall}I\omega_0/(I+I_{wall})$ before the collision. After the collision, we will have $L'_{CAM} = -L_{CAM}$ and $L'_{wall,CAM} = -L_{wall,CAM}$. It is evident that back to the original (non-rotating) reference frame, the angular momenta will be $L' = -L$ and $L'_{wall} = 2L$. We, thus, find that the angular momentum transferred to the wall is $\Delta L = 2I\omega_0$. Nevertheless, the net angular momentum transferred to the wall from all the possible collisions is zero since the symmetric nature of the collisions eliminates the tendency of the wall to rotate (the sum is zero, since for every possible collision contributing $+2I\omega_0$ there is another equally possible collision contributing $-2I\omega_0$). The situation is different to that of linear momentum because there are two possible directions of rotation, whereas only one possible direction of motion of the wall in the linear case. Therefore, the rotational part does not contribute to the pressure.