Question 08/04 *FALLING MAGNET*
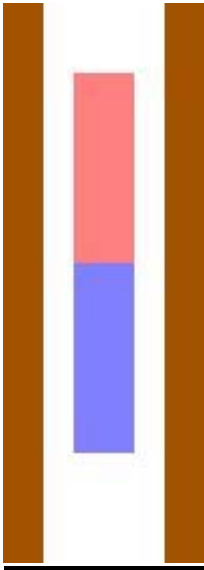

Bar magnet falling in a vertical copper pipe quickly reaches "terminal" velocity and stops accelerating. Find that velocity as a function of the features of the magnet and the pipe.

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APPROXIMATE SOLUTION:

(for a more detailed study and precise results, see **European Journal of Physics, volume 25, issue 5, pages 593 - 604**)

As is well known, the equation of motion for a damped fall is:

$$M \frac{dv}{dt} = Mg - kv, \tag{1}$$

where k (disregarding the aerodynamic friction due to extremely low speed, as we will see later) represents the damping coefficient associated with the induced currents and M the mass of the magnet. If we integrate Eq. (1), we obtain:

$$v(t) = \frac{Mg}{k}\left(1 - e^{-\frac{k}{M}t}\right). \tag{2}$$

It is observed experimentally that the asymptotic speed (Mg/k) is reached almost instantaneously ($\tau = M/k \ll 1$ s). For this reason we concluded that, for a tube length l of several decimeters, the speed can be considered practically uniform. For example, for a copper tube with dimensions $l = 1$ m, inner radius $b = 6.5$ mm, outer radius $c = 7.5$ mm (see Fig. 1) and a SmCo-magnet of mass $M = 9.3$ g, height $h = 10$ mm and radius $a = 6$ mm, a falling time of 12.3 s is obtained. We thus find:

$$v = \frac{l}{t} = \frac{Mg}{k} \approx 81 \, mm/s \quad k \approx 1.12 \, kg/s \quad \tau = \frac{M}{k} \approx 8 \, ms. \tag{3}$$

It is difficult to relate the damping coefficient k to the geometrical and electromagnetic parameters of the experiment, but we shall do so approximately by using energy conservation. It suffices to say that the variation in the potential energy of the magnet is equivalent to the energy dissipated in the tube through the Joule effect.

a) The Dipolar Contribution

a) **The Dipolar Contribution**

According to Fig. 2, the magnitude of the induced azimuthal electric field, E, in a filamentary loop of radius ρ ($b \le \rho \le c$) at a distance z from the magnet (represented by its magnetic moment $\boldsymbol{m}$) in terms of the magnetic vector potential, A, is:

$$E(\rho, z) = \left| \frac{\partial A(\rho, z, t)}{\partial t} \right| = v \left| \frac{\partial A}{\partial z} \right| = v \left| \frac{\partial}{\partial z} \left(\frac{\mu_0}{4\pi} \frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3} \right) \right| =$$

$$= v \frac{\partial}{\partial z} \left[\frac{\mu_0 m \rho}{4\pi \left(\rho^2 + z^2 \right)^{3/2}} \right] = v \frac{3\mu_0 m}{4\pi} \frac{\rho z}{\left(\rho^2 + z^2 \right)^{5/2}}. \tag{4}$$

The power dissipated in the tube due to the dipole contribution will be:

$$P_{dip} = \int_{tube} \sigma E^2 dv = \sigma v^2 \frac{9 \mu_0^{\,2} m^2}{16\pi^2} \int_{-q}^{p} \int_{b}^{c} \frac{\rho^2 z^2}{\left(\rho^2 + z^2 \right)^5} 2\pi \rho \, d\rho \, dz. \tag{5}$$

We can do the integral in Eq. (5) using that b and c are much smaller than p and q, and obtain:

$$P_{dip} \approx \sigma v^2 \frac{15 \mu_0^{\,2} m^2}{1024} \left(\frac{1}{b^3} - \frac{1}{c^3} \right). \tag{6}$$

Therefore, the balance of energy, $P_{dip} = Mgv$, allows us to write:

$$\sigma = \frac{1024}{15} \frac{Mg}{\mu_0^{\,2} m^2} \frac{c^3 b^3}{c^3 - b^3} \frac{1}{v}. \tag{7}$$

To evaluate the accuracy that is obtained by including only the magnetic dipole contribution, we determined the magnetic moment of the SmCo-magnet experimentally. To do so, we suspended it from a very thin cotton thread and measured its oscillating period in the presence of the Earth's magnetic field. The value of its horizontal component in our laboratory, measured with a tangent compass, is $B_h = 25.2$ μT, yielding a period $T = 0.447$ s. From the expression:

$$T = 2\pi \sqrt{\frac{J}{mB_h}},$$

<div align="right">(8)</div>

where J represents the inertial moment of the magnet ($J = Ma^2/4 + Mh^2/12$), the magnetic moment was found to be 1.26 Am^2. If we substitute this result in Eq. (7), the conductivity is found to be $24\times10^6\,\Omega^{-1}\mathrm{m}^{-1}$, which corresponds to approximately 40% of the published value for copper alloys, thus illustrating the need to include higher-order multipole contributions.

b). Other Approximations

Instead of proceed to carry out a standard multipole expansion, we would prefer previously try some other options in order to reach better approximation.

The most convenient corresponds to the considering of a collection of n equal point dipoles, of magnetic moment $\boldsymbol{m}/n$, uniformly spaced at distances h/n in the symmetry axis of the tube representing the falling magnet. In this way the calculation of the induced azimuthal electric field, $\boldsymbol{E}$, corresponds to a simple superposition of n contributions like Eq. (4). But according to Eq. (5), the dissipated power would be a complicated polynomial expression with $(n^2+n)/2$ contributions. Except for small values of n it constitutes a tedious problem without major interest.

Another possibility would be the substitution of the magnet by two opposite surface densities of magnetic pole strength at a distance h apart. The calculation is very difficult because this model involves sources out of the symmetry axis, and it does not contribute anything to the best understanding of the problem.
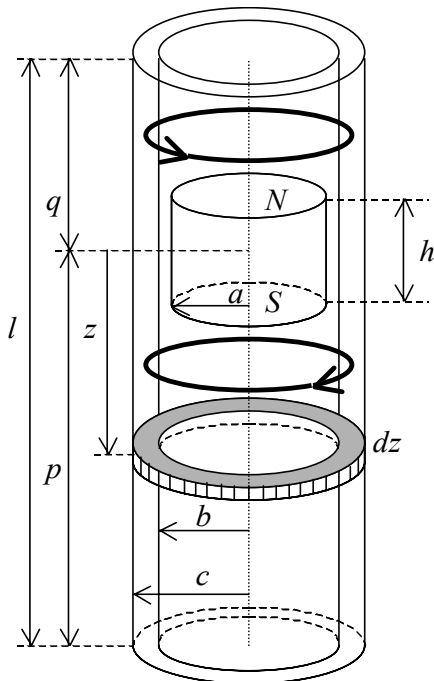

Figure 1.- Description and dimensions of the metallic tube. The currents induced above and below the magnet are shown.
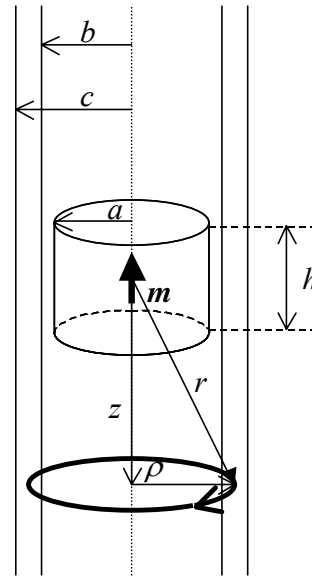

Figure 2.- Notation used in the dipolar contribution calculation.