Let's consider a thin ruler balanced on a soda can.
Derive an expression for the oscillation period of the ruler. Your expression should include quantities such as the length of the ruler, the mass of the ruler, the radius of the can, etc. Hint: the derivation simplifies if you neglect the thickness of the ruler. If you have the ability, some additional points you might want to consider are, what are the effects of the thickness of the ruler, and/or how does the coefficient of friction affect things?

Solution

Before we start working on the actual solution, I shall make several assumptions that will clarify the problem, and declare the known parameters of the system:

1. The soda can is not allowed to roll along the table and is fixed in one position horizontally. Otherwise, the problem becomes extremely interesting, thus going way beyond the context of this exercise :o)
2. The soda can is an ideal cylinder and both ruler and the can are rigid bodies (no mechanical deformations). The ruler's mass is distributed uniformly along its length and ruler is positioned perpendicularly to the axis of the cylinder.
3. The oscillations are "small" (angular displacement is not greater than 5 deg., or ~0.1 rad, so that sine function can be replaced by the actual radian measure of the angle), i.e. can be described by the theory of small oscillations.
4. The ruler does not slide along the can at any considered angular displacement (except for the very last part of the solution, where effect of friction is examined).

For the ruler of mass m, length L, thickness h and radius R of the soda can, let us now begin the derivation.

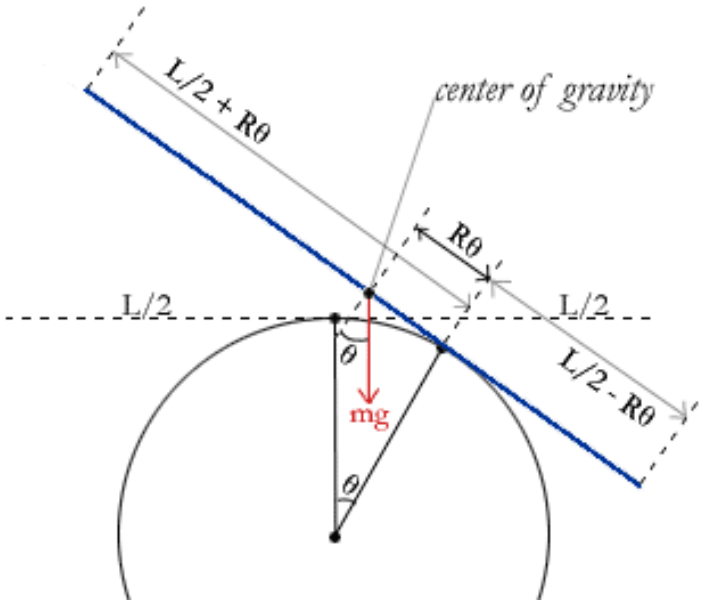

Fig. 1

a) In this part of the solution, h<<R, so the thickness of the ruler may be neglected. From Fig. 1, the torque produced by gravity at the ruler's center of gravity at angular displacement of θ, is equal:

$M = - mg^* (R^*\theta) * \cos(\theta) \approx - mgR\theta$, producing angular acceleration according to $J(d^2\theta/dt^2) = M$, where J is the instantaneous moment of inertia of the ruler. Thus,

$J(d^2\theta/dt^2) + mgR\theta = 0$ (1)

The moment of inertia at any displacement angle θ can be found using parallel-axis theorem, and is equal:

$J = 1/12\ mL^2 + m(R\theta)^2$

As we can see, in general, the moment of inertia is a quadratic function of displacement, which may complicate equation (1) tremendously. Unless, however, R ~ 10L (which is not the case in this particular problem), the θ-dependent term in the expression for J can be neglected. Hence (1), can be presented as

$d^2\theta/dt^2 + (12gR/L^2) *\theta = 0$

So, the period of small oscillations can be represented as

$$\mathbf{T = 2\pi\sqrt{L^2\ /12gR}} \quad \mathbf{(2)}$$

b) Before we consider the effect produced by taking the thickness of the ruler into account, let us examine the mechanics of ruler's oscillations in general. First of all, with R << h, the oscillations will not take place since even at smallest angular displacement; the resulting torque is directed to increase the displacement (Fig. 2 below). In terms of energy, the potential energy of displaced ruler is smaller than for a ruler balanced horizontally–a textbook example of unstable system.

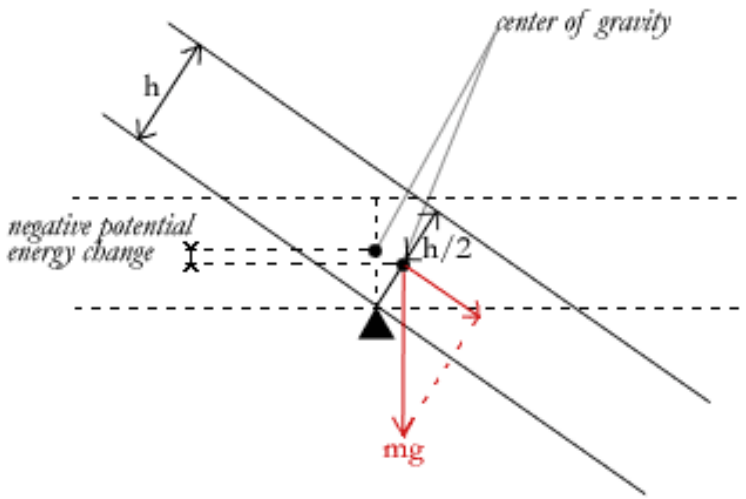

Fig. 2

So, although expression (2) produces infinity in case of an infinitely thin ruler angularly displaced around a fixed point (a case of indifferent equilibrium), the reality cannot be effectively described by (2). The necessity of introducing the value of thickness of the ruler is now clear, not only in finding a more general expression for the period of oscillations, but, most importantly, useful to analyze stability conditions for our system.

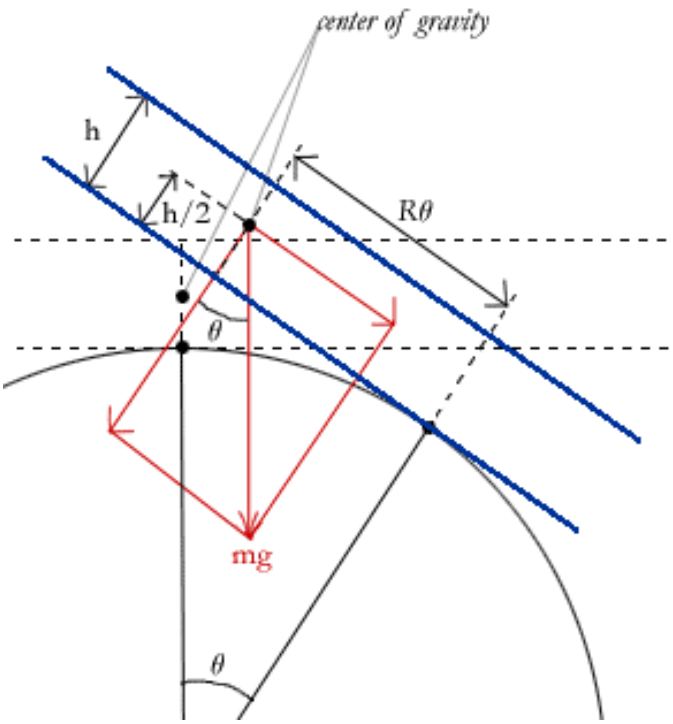

Fig. 3

From the Fig, 3, the torque produced by gravitation force at the ruler's center of gravity relative to the instant axis of rotation is equal

$$M = -(mg*(R\theta)*\cos(\theta) - mg*\, h/2 * \sin(\theta)) \approx -mg(R - h/2)*\theta;$$

The moment of inertia J (again, neglecting angular dependence), can be represented by
$J = 1/12\ mL^2$ (in general, it is a bit more complicated expression, but we are still considering a thin ruler with h << L, though the reader is free to find and use the complete expression).

Thus, the following equation for the ruler's oscillations can be written as:

$$d^2\theta/dt^2 + [12g(R- h/2)/L^2] *\theta = 0 \quad (3)$$

and $$\mathbf{T = 2\pi\sqrt{L^2/12g(R - h/2)}} \quad (4)$$

As we can see, harmonic solution to (3) can only exist (making oscillations possible) when $R > h/2$; otherwise, the system becomes unstable and the whole idea of oscillations becomes irrelevant (the value of T becomes a complex value at $R < h/2$). At $h \ll R$, expression (4) becomes identical to (2) – a natural consequence.

c) In all previous steps of our solution, it was assumed that the ruler does not slide along the can surface. It is important to notice, though, that the friction between the smooth surfaces of a soda can and a wooden/plastic ruler is not particularly large. It can easily be derived that a body starts sliding off an inclined surface if the inclination angle θ is related to the friction coefficient k the following way:

$\tan(\theta) \geq k$, or, for small angles, $\theta \geq k$.

Hence, it is of utmost importance that the amplitude of oscillations does not exceed the value of k (in radians). Otherwise, the ruler will simply slide off the can. This makes the assumption of "small" oscillations rather preeminent to this problem in order for it to be well-defined. The actual period of oscillations is not directly affected by the value of friction coefficient since, if any sliding between the surfaces is assumed, the analysis (and, possibly, any meaningful analysis) presented in parts a) and b) becomes obsolete.