Solution to the "Reference Frames" problem from the Physics Questions/Problems at TAU page.

This is actually one of the "paradoxes", which might have facilitated the introduction of the Special Theory of Relativity.

To summarize the resolution of the apparent paradox: The argument, presented in the problem is "classical" (non-relativistic). This approach to the problem is valid when $V_0 << c$, or as the relativistic treatment of the problem will show, we rather need $V_0E << c$. In this case, the magnetic field in the particle reference frame ($V_0E/c << 1$) is negligible (or zero to the same order of approximation, that the whole approach is valid). Therefore, we get no magnetic field in the reference frame of the particle and consequently no acceleration in the horizontal direction. The "paradox" is resolved. All this is valid unless either V_1 or V_0 aren't small compared to c, and then the full relativistic treatment is needed.

The full relativistic treatment of the problem will show us that there indeed exists acceleration in the horizontal direction and the particle doesn't really draw a parabolic trajectory, instead it's movement is more complicated.

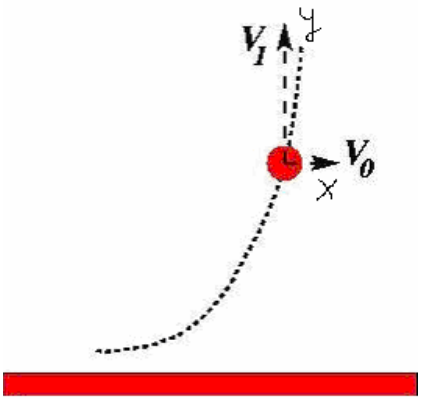
Here is the solution of the problem:

The Lorentz Force, acting on the particle in the EM field:

$$\frac{d\vec{p}}{dt}(\equiv \dot{\vec{p}}) = e \cdot \vec{E} + \frac{e}{c} \cdot (\vec{V} \times \vec{H})$$

Due to symmetry, the particle movement is effectively confined to the plane.

Denote the horizontal axis by x, vertical by y (rightwards and upwards):



Since in all inertial reference frames we get the same physics, let's look at the plate reference frame, which is convenient since $\vec{H} = 0$ there.

We have:

$$\dot{\vec{p}} = e \cdot \vec{E} \qquad \vec{E} = E_x \cdot \hat{x} + E_y \cdot \hat{y} = 0 \cdot \hat{x} + E_y \cdot \hat{y}$$

Therefore:

$$\dot{p}_x = e \cdot E_x = 0 \Rightarrow p_x(t) = 0 + p_x(0) = p_x(0) = \frac{mV_0}{\sqrt{1 - V_0/c}}$$

$$\dot{p}_y = e \cdot E_y \Rightarrow p_y(t) = e \cdot E \cdot t + p_y(0) = e \cdot E \cdot t$$

Moreover:

$$\vec{p} = \frac{K}{c^2} \cdot \vec{v} \Rightarrow \vec{v} = \frac{c^2}{K} \cdot \vec{p}$$

$$v_x = \frac{c^2}{K} \cdot p_x \qquad v_y = \frac{c^2}{K} \cdot p_y$$

Where

$$K = \sqrt{mc^4 + p^2c^2} = \sqrt{m \cdot c^4 + (p_x(0) \cdot c)^2 + (e \cdot E \cdot t \cdot c)^2} = \sqrt{m \cdot c^4 + c \cdot (\frac{mV_0}{\sqrt{c - V_0}} \cdot c)^2 + (e \cdot E \cdot t \cdot c)^2}$$

Summarizing, we get:

$$v_x(t) = \frac{c^2}{K} \cdot p_x(t) = \frac{c^2}{K} \cdot \frac{mV_0}{\sqrt{1 - V_0/c}} = \frac{c^2}{\sqrt{m \cdot c^4 + c \cdot (\frac{mV_0}{\sqrt{c - V_0}} \cdot c)^2 + (e \cdot E \cdot t \cdot c)^2}} \cdot \frac{mV_0}{\sqrt{1 - V_0/c}}$$

$$v_y(t) = \frac{c^2}{K} \cdot p_y(t) = \frac{c^2}{K} \cdot e \cdot E \cdot t = \frac{c^2 \cdot e \cdot E \cdot t}{\sqrt{m \cdot c^4 + c \cdot (\frac{mV_0}{\sqrt{c - V_0}} \cdot c)^2 + (e \cdot E \cdot t \cdot c)^2}}$$

We see that both $v_x(t)$ and $v_y(t)$ depend on time, and the particle trajectory is more complicated than simple classic parabola. There is a horizontal acceleration after all! (The horizontal velocity is actually decreasing). I must admit, not a very intuitive result.

Alex Shpunt
Tel-Aviv, Israel
08-March-2003