Solution to Problem "Delicate Balance" 09/03

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Abstract

Calculation of maximal time of observed "equilibrium" of a pencil, taking Quantum and Thermal effects into account.

1 Quantum Effects

We first consider the case of a particle, described by a Hamiltonian

$$H = \frac{p_\theta^2}{2m} + mgl\cos\theta, \tag{1.1}$$

where $p_\theta = ml^2\dot{\theta}$ is the particle's angular momentum. The pencil's mass m is assumed to be concentrated at the end of a rod with length l. The initial conditions are $\theta(0) \equiv \theta_0$ and $\dot{\theta}(0) \equiv \dot{\theta}_0$. We assume the initial conditions to be restricted by Heisenberg's Uncertainty Principle:

$$(\Delta p_\theta)_0(\Delta\theta)_0 \geq \hbar/2, \tag{1.2}$$

which in our case, yields:

$$\theta_0\dot{\theta}_0 \geq \hbar/(2ml^2). \tag{1.3}$$

We assume those initial conditions allowed by the Uncertainty Principle, which are closest to the desired classical equilibrium conditions $\theta_0 = \dot{\theta}_0 = 0$, and we will move on to calculate how much time it will take until the pencil assumes an angle which -being observed by the naked eye- will be considered as the end of the equilibrium. We expect this final angle to be very small, of the order $\theta_1 \sim 0.1$ radians.

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Therefore, for small angles, the equation of motion becomes:

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = mgl\sin\theta \approx mgl\theta = ml^2\ddot{\theta}. \tag{1.4}$$

Rearranging terms, gives:

$$\ddot{\theta} - \omega^2\theta = 0, \tag{1.5}$$

where $\omega^2 = g/l$. Solving for the angle is trivial, and yields:

$$\theta(t) = c_1 e^{\omega t} + c_2 e^{-\omega t}, \tag{1.6}$$

where:

$$\left.\begin{array}{l} c_1 = \frac{1}{2}(\theta_0 + \dot{\theta}_0/\omega) \\ c_1 = \frac{1}{2}(\theta_0 + \dot{\theta}_0/\omega) \end{array}\right\} \tag{1.7}$$

Solving eq. 1.6) for the elapsed time τ until θ takes the final value θ_1 gives:

$$e^{\omega\tau} = \frac{1}{2c_1}\left[\theta_1 + \sqrt{\theta_1^2 - (\theta_0^2 - \dot{\theta}_0^2/\omega^2)}\right]. \tag{1.8}$$

We notice that the maximal time τ corresponds to the minimum values of θ_0 and $\dot{\theta}_0$ allowed by the Uncertainty Principle. We thus expect them to be of the order given by the (=) in the Uncertainty Relation eq. (1.3) and they can, thus, be neglected compared to θ_1. Therefore, eq. (1.8) becomes

$$e^{\omega\tau} \approx \frac{1}{2c_1}\left[\theta_1 + \sqrt{\theta_1^2)}\right] = \frac{\theta_1}{c_1}. \tag{1.9}$$

Evidently, for τ to take its maximum value, c_1 should be assume its minimum value. By assuming that the equality in the Uncertainty Relation holds $\dot{\theta}_0 = \hbar/2ml^2\theta_0$, it is easy to minimize c_1 as a function of θ_0:

$$2c_1(\theta_0) = \theta_0 + \frac{\hbar}{2ml^2\omega\theta_0}, \tag{1.10}$$

and obtain:

$$\theta_0 = \sqrt{\frac{\hbar}{2ml^2\omega}}. \tag{1.11}$$

Using this value for θ_0 in order to calculate c_1 and inserting the latter to eq. (1.9), we get:

$$\tau = \sqrt{\frac{l}{g}}\ln\left\{\theta_1\sqrt{\frac{2mgl^2\omega}{\hbar}}\right\}. \tag{1.12}$$

Typical values for the pencil are ~ 10 cm length and ~ 10 gr mass. If we consider θ_1 to be ~ 0.1 radians, we have $\tau_{\text{quantum}} \approx 3.47$ sec.

2 Thermal Effects

Again, we initiate from the Hamiltonian of the problem, eq. (1.1). From the Classical Theorem for the Equipartition of Energy to the degrees of freedom, we get

$$\langle p_\theta^2 \rangle = mkT, \tag{2.1}$$

which gives:

$$\left\langle \dot{\theta}^2 \right\rangle = \frac{kT}{ml^2}. \tag{2.2}$$

Thus, the initial conditions can be assumed to be:

$$\theta_0 = 0 \tag{2.3}$$

$$\dot{\theta}_0 = \sqrt{\frac{kT}{ml^2}}, \tag{2.4}$$

in which case, the constants of integration become:

$$c_1 = -c_2 = \frac{\dot{\theta}_0}{2\omega}. \tag{2.5}$$

Solving for the elapsed time, gives:

$$e^{\omega\tau} = \frac{\theta_1}{2c_1}\left[1 + \sqrt{1 + 4\left(\frac{c_1}{\theta_1}\right)^2}\right]. \tag{2.6}$$

Since $c_1/\theta_1 \ll 1$, we have:

$$\tau \approx \frac{1}{\omega}\ln\left(\frac{\theta_1}{c_1}\right). \tag{2.7}$$

Using the value obtained for $\dot{\theta}_1$, we have:

$$\tau = \sqrt{\frac{l}{g}}\ln\left\{2\theta_1\sqrt{\frac{mgl}{kT}}\right\}. \tag{2.8}$$

Using the same typical values as before and $kT \sim 4 \cdot 10^{-21}$ Joules, we find $\tau_{\text{thermal}} \approx 1.95$ sec.

The results can be generalized to the form:

$$\tau \approx \sqrt{\frac{l}{g}}\ln\left\{\theta_1\sqrt{\frac{2mgl}{E_0}}\right\}, \tag{2.9}$$

where E_0 is the mean energy corresponding to 1 degree of freedom (in the Classical case) or to 1 quantum of energy (in the Quantum case).

From this point of view, for the Quantum Restrictions treated in Section (1), 1 quantum of energy corresponds to $E_0 = \hbar\omega$, because the "frequency" ω of the "oscillations" is the same in Quantum physics, from Bohr's Correspondence Principle (i.e. unaffected by large quantum numbers), thus:

$$\tau_{\text{quantum}} \approx \sqrt{\frac{l}{g}} \ln\left\{\theta_1 \sqrt{\frac{2mgl}{\hbar\omega}}\right\}, \tag{2.10}$$

which is identical to eq. (1.12). Through this generalization, the case of thermal radiation could also be examined. In this case, $E_0 = hc/\lambda$, where λ is the wavelength corresponding to a maximum in Planck's distribution of radiation. Through Wien's Law, this wavelength can be substituted for the temperature of the radiating source. Other approximations are also possible, mainly in the form of perturbations, but they will not be discussed here.