Question 12/02

CHARGED DROP

The shape of a freely suspended liquid drop is kept spherical (with radius *R*) by the surface tension *g*. [For simplicity we assume weightlessness.] Assume that the liquid is conducting and it is being gradually charged. What will happen as the charge *Q* increases?

Extended Solution

According to the comments of Y. Kantor related to my answer to this question (see the web site http://star.tau.ac.il/QUIZ/02/D12.02.html), and the discussions with professor Muñoz of the University of Valladolid, we have studied in a more detailed way the problem and here is our analysis.

As it was well stated in our previous answer, by **simple energy considerations** (electrostatic and surface contributions) we concluded that if the drop charge is

$$Q_e \geq \pi\sqrt{32\varepsilon_0\frac{\left(2^{1/3}-1\right)}{\left(1-2^{-2/3}\right)}}\sqrt{gR^3} = 4.74\pi\sqrt{\varepsilon_0 gR^3} \qquad (1)$$

(g being the surface tension and R the drop radius), the drop could break into two equal spherical drops thus reducing its total energy, but obviously this breaking requires to overcome an energy barrier.

Therefore, we must perform another complementary analysis in terms of the **electrostatic and surface pressures**. When the drop is charged it appears a surface charge density, σ, and consequently an electrostatic pressure, $p = \sigma^2/2\varepsilon_0$, which tends to increase the drop radius. When this pressure is greater than the surface pressure, the Coulomb repulsion makes that some pieces leave the drop, giving place to the drop explosion. Consequently, this pressure must be balanced with the surface pressure, which we can evaluate as follows:

The drop surface is:

$$S = 4\pi R^2 \qquad (2)$$

and the small variations of its radius and surface are related as:

$$dS = 8\pi R dR. \qquad (3)$$

Therefore, the variation in the surface energy is:

$$dW = gdS = 8\pi RgdR, \qquad (4)$$

which can be also written in terms of the surface pressure p as:

$$dW = FdR = p4\pi R^2 dR. \qquad (5)$$

By comparing equations (4) and (5), we find the expression for the surface pressure:

$$p = 2g/R, \qquad (6)$$

showing that R becomes smaller, the pressure increases.

The balance of electrostatic and surface pressures is:

$$\frac{\sigma^2}{2\varepsilon_0} = \frac{2g}{R}, \tag{7}$$

and consequently:

$$\frac{1}{2\varepsilon_0}\frac{Q_p^2}{\left(4\pi R^2\right)^2} = \frac{2g}{R}. \tag{8}$$

Therefore, the new limit for the electric charge in the drop will be:

$$Q_p = 8\pi\sqrt{\varepsilon_0 g R^3} \ . \tag{9}$$

Note that this result is greater than we found by energy considerations (equation (1)):

$$Q_e \geq 4.74\pi\sqrt{\varepsilon_0 g R^3} \, . \tag{10}$$

Therefore, we can conclude that when the electric charge in the drop is

$$Q \leq 4.74\pi\sqrt{\varepsilon_0 g R^3} \, , \tag{11}$$

the drop will be stable.

In the case

$$Q \geq 8\pi\sqrt{\varepsilon_0 g R^3} \ , \tag{12}$$

the drop spontaneously will explode (the energy barrier disappears).

For drop charges ranging between both values, we can expect the occurrence of local minimum of energy (two, three, four… drops) separated by energy barriers.

Note: In the whole analysis we have assumed that the electric field at the drop surface is smaller than the electric field necessary to produce spark in air (dielectric breakdown), which is $E_b \approx 3\times10^6$ V/m. Therefore the electric charge in the drop must be always smaller than:

$$4\pi\varepsilon_0 R^2 E_b \, . \tag{13}$$

In other case, the air behaves as conductive and the drop losses electric charge until the above quoted limit.

Corresponding authors:

J.I.I. de la Torre (e-mail *nacho@usal.es*)

and

J.M. Muñoz Muñoz (e-mail *jmmm@ee.uva.es*)