Physics Questions/Problems

Question 02/02

MAXIMAL GRAVITY

Consider a celestial body (planet) of mass M and volume V consisting of material of uniform density M/V. If the body is spherical, then its radius is $R=(3V/4\{pi\})^{1/3}$, and the free-fall acceleration on its surface is $g=GM/R^2=GM(4\{pi\}/3V)^{2/3}$, where G is the gravitational constant. Consider all possible shapes of the body. What is the largest g, that can be achieved at one point on a surface, and what is the shape of the body for which such acceleration is achieved?

ANSWER:

We consider a body, built by the rotation of an area $x = f(y)$ ($y = 0, \ldots, D = \beta D_S$) around the y-axis, with the same uniform density and the same mass as a sphere of the diameter D_S. The free fall acceleration at the point $x = y = 0$ is $g = \alpha g_S$, compared with that for the sphere.

Then the maximal gravity is by a factor $\alpha = 3*5^{-2/3} \approx 1.02599$ larger than for a sphere and the area, building the body by rotation, is given by $x^2 + y^2 = D^{4/3} y^{2/3}$ with $D = \beta D_S$ and $\beta = 5^{1/3} / 2 \approx 0.85499$.

Definitions and analytical calculations:

From general symmetry considerations it is clear, that the body searched for must be a rotationally symmetric one.
If $L(\varphi)$ is the distance of a point at the area above mentioned from the point $x = y = 0$ under an angle φ ($\varphi = 0, \ldots, \Phi \le \pi/2$) relative to the y-axis (see picture), then $\alpha = g / g_S$ is given by

$$\alpha = \frac{3\int_0^{\Phi} \frac{L(\varphi)}{D} \sin\varphi \cos\varphi \, d\varphi}{\left\{4\int_0^{\Phi} \left[\frac{L(\varphi)}{D}\right]^3 \sin\varphi \, d\varphi\right\}^{\frac{1}{3}}}.$$

Calculations for a circle ($L = D \cos\varphi$), for an ellipse (half axes D/2 and $b \cdot D/2$, $L = \frac{D\, b^2 \cos\varphi}{1+\left(b^2-1\right)\cos^2\varphi}$), and for a rectangle (sides D and a D, $L = \begin{cases} \frac{D}{\cos\varphi} & \left(\varphi \le \arctan a\right) \\ \frac{a\,D}{\sin\varphi} & \left(\varphi \ge \arctan a\right) \end{cases}$) give the α-values 1, 1.02204 (for optimal $b = 1.3898$) and 1.00682 (for optimal $a = 0.60961$), respectively ($\Phi = \pi/2$ in all cases). (I.e., even a cylindrical body can have a larger gravity than an equivalent sphere!)

Varying the parameters for a circle by setting $L = D \cos^k \varphi$ and choosing arbitrary $\Phi \le \pi/2$ gives

$$\alpha = \frac{3}{k+2} \underbrace{\left(\frac{3k+1}{4}\right)^{\frac{1}{3}}}_{\max. = 3\cdot 5^{-\frac{2}{3}}} \underbrace{\frac{1-\cos^{k+2}\Phi}{\left(1-\cos^{3k+1}\Phi\right)^{\frac{1}{3}}}}_{\max. = 1},$$

$$\left(k = \frac{1}{2}\right) \qquad \left(\Phi = \frac{\pi}{2}\right)$$

and therefore ***an optimal value $\alpha = 3 \cdot 5^{-2/3} \approx 1.02599$ for $k = ½$ and $\Phi = \pi/2$.*** This corresponds to an area, building the ***rotational body, described by $x^2 + y^2 = D^{4/3} y^{2/3}$ with $\beta = 5^{1/3} / 2 \approx 0.85499$ or by $L = D \cos^{1/2} \varphi$.***

I make the assertion, that this is generally the optimal shape of the body and the largest possible acceleration under the assumptions described at the beginning.

Firstly **some arguments** suggesting this assertion:

1. This is the largest value α which ever has been found (including other forms like a parabola or a curve of the third order, here not discussed).

2. The contribution for the largest distance in any direction of φ is $\cos \varphi / L^2 \equiv 1 / D^2$, i.e. is equal for all directions !!!

To **proof** the assertion made above should require the solution of the equation $d\alpha / dL = 0$. That is, at least for my knowledge, analytically not possible. Therefore, I tried to solve the problem **numerically** using Mathematica 4.1.

φ has been incremented in N_i equidistant steps (N_i up to 10 000). Each value of L(φ) was incremented or decremented by a number of values in a geometric progression (1.8^{-m}, m = 1, ... , 21), the corresponding L(φ) were changed if the value of α was increased.

The results were the following:

3. The curve, assumed as optimal, was changed not once. The difference of the analytically and numerically calculated curves from the 10 000 L-values gave a root of the mean squared error (σ) of $1.2 \cdot 10^{-9}$.

4. Starting from other curves (circle, ellipse, several rectangles), the final curves approximate in each case our curve proposed as optimum. E.g., for the start with the three curves above mentioned, the root of the mean squared deviation from the optimal analytical curve was after 80 000 – 90 000 changes (in (2 ·) 10 000 · 21 steps) between 2 and $3 \cdot 10^{-6}$. The difference of the numerically calculated α value from those for the optimal curve was in all three cases smaller than $6 \cdot 10^{-12}$.

Point 3 could be discussed as a local minimum. However, the additional result 4 with different starting curves (e.g. narrower and broader rectangles) shows us with a probability near to certainty that the result proposed is the correct one.

In the Mathematica 4.1 file GravitPict.nb the meaning of φ and L(φ) and the curves, discussed in the text, are shown: circle (blue), optimal curve (red), ellipse (green), rectangle (violet)