Dipole in Shell

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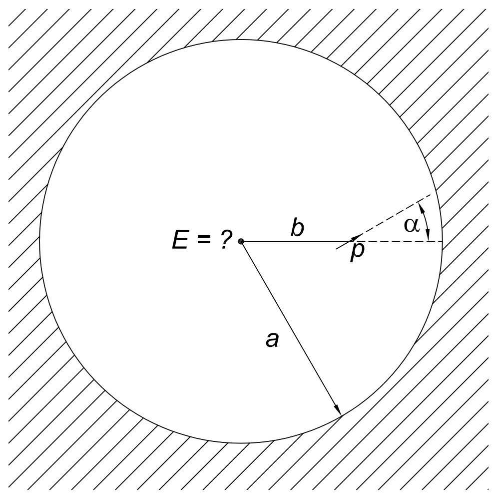
1 Problem

What is the (vector) electric field strength at the center of a spherical cavity of radius a in a grounded conductor if a point dipole $\mathbf{p}$ is placed at distance b ($0 < b < a$) from the center so as to make angle α to a radius vector?



2 Solution

This problem is from Yakov Kantor's Physics Quiz site, http://star.tau.ac.il/QUIZ/

We solve via the image method for spherical geometry. Recall that a charge q at radius $b < a$ results in an image charge q' at a^2/b, such that the potential at $r = a$ vanishes:

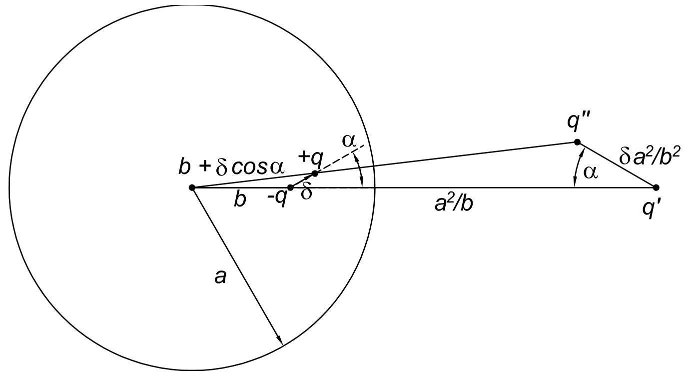
$$\phi(r = a) = 0 = \frac{q}{a - b} + \frac{q'}{a^2/b - a} = \frac{q}{a - b} + \frac{q'b/a}{a - b}\,, \tag{1}$$

and hence $q' = -qa/b$.

For the present case of a point dipole $\mathbf{p}$, we can think of this as due to charges $\pm q$ separated by distance $\delta = q/p$, with the charge $-q$ at radius b and the charge $+q$ at radius $b + \delta \cos\alpha$ as shown below.

The image transformation $r \to a^2/r$ is conformal (angle preserving), so the triangle formed by the center of the sphere and charges $\pm q$ is similar to that formed by the center and the image charges q' and q''. Image charge $q' = +qa/b$ is at distance a^2/b from the center. Hence, the distance between the two image charges is

$$\delta' = \delta\frac{a^2}{b^2}\,. \tag{2}$$



The image charge q'' has value

$$q'' = -q\frac{a}{b+\delta\cos\alpha} \approx -q\frac{a}{b} + q\delta\cos\alpha\frac{a}{b^2}\,. \tag{3}$$

Thus, taking the limit $\delta \to 0$, $q \to \infty$ such that $q\delta = p$, the two image charges can be thought of as a dipole of strength

$$p' = -q\frac{a}{b}\delta\frac{a^2}{b^2} = -p\frac{a^3}{b^3}\,, \tag{4}$$

at angle α to the radius vector as shown above, **plus** an additional charge of magnitude

$$q''' = p\frac{a}{b^2}\cos\alpha, \tag{5}$$

located at distance a^2/b from the center. If $\alpha = \pm\pi/2$, the image of the dipole $\mathbf{p}$ is a pure dipole.

To calculate the electric field at the center, we recall that the field of a point dipole $\mathbf{p}$ is

$$\mathbf{E} = \frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3}\,, \tag{6}$$

where $\mathbf{r}$ points from the dipole to the observer. In the present problem, we define $\hat{\mathbf{z}}$ to point from the center to the dipoles, so $\hat{\mathbf{r}} = -\hat{\mathbf{z}}$. Dipole $\mathbf{p}$ is taken to lie in the x-z plane, so that

$$\mathbf{p} = p\hat{\mathbf{z}}\cos\alpha + p\hat{\mathbf{x}}\sin\alpha, \tag{7}$$

and

$$\mathbf{p}' = p\frac{a^3}{b^3}\hat{\mathbf{z}}\cos\alpha - p\frac{a^3}{b^3}\hat{\mathbf{x}}\sin\alpha. \tag{8}$$

Then, the field at the center of the cavity due to the dipole, and its image dipole + extra charge q''' is

$$\begin{aligned}
\mathbf{E}(0) &= \frac{3(\mathbf{p}\cdot\hat{\mathbf{z}})\hat{\mathbf{z}} - \mathbf{p}}{b^3} + \frac{3(\mathbf{p}'\cdot\hat{\mathbf{z}})\hat{\mathbf{z}} - \mathbf{p}'}{(a^2/b)^3} - \frac{q'''\hat{\mathbf{z}}}{(a^2/b)^2} \\
&= \frac{2p\hat{\mathbf{z}}\cos\alpha - p\hat{\mathbf{x}}\sin\alpha}{b^3} + \frac{2p\hat{\mathbf{z}}\cos\alpha + p\hat{\mathbf{x}}\sin\alpha}{a^3} - \frac{p\hat{\mathbf{z}}\cos\alpha}{a^3} \\
&= p\hat{\mathbf{z}}\cos\alpha\left(\frac{2}{b^3} + \frac{1}{a^3}\right) - p\hat{\mathbf{x}}\sin\alpha\left(\frac{1}{b^3} - \frac{1}{a^3}\right)\,.
\end{aligned} \tag{9}$$