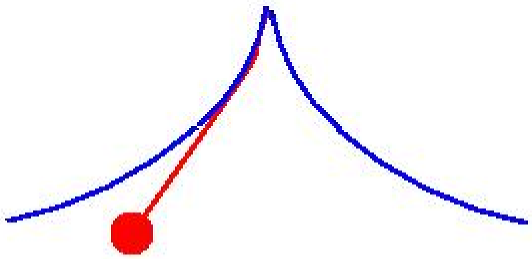
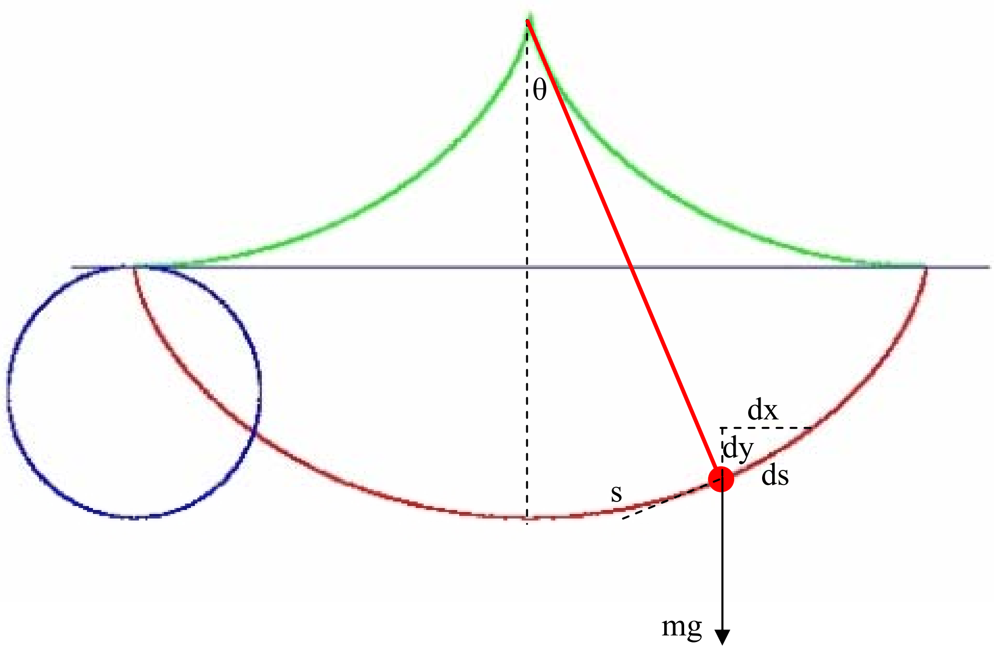


Period of a pendulum depends only on its length only for very small oscillations. For large oscillations the period depends on the amplitude. Such amplitude-dependence can be eliminated by making the string of the pendulum (shown in red) to rap around a limiting curve (shown in blue). What is the shape of this curve?

Solution:

Let us suppose that a particle with mass m is moving without friction on the vertical concave curve in the figure below, in a uniform gravitational field with the intensity g, with the initial conditions: at $t = 0$, $x = 0$, $y = 0$ and $v_y = 0$.



The equation of motion for the particle is:

$$m\ddot{s} = -mg\sin\theta = -mg\frac{dy}{ds} \Rightarrow \ddot{s} + g\frac{dy}{ds} = 0$$

In order to obtain the same equation as in the case of a mathematic pendulum

$$\ddot{s} + \frac{g}{l}s = 0$$

it follows that

$$\sin\theta = \frac{dy}{ds} = \frac{s}{l}, \tag{1}$$

where l is the length of the pendulum. From (1) we get

$$y = \frac{s^2}{2l} = \frac{l}{2}\sin^2\theta\,, \tag{2}$$

where we used the initial conditions.

But $\quad dx = ds\cos\theta \quad (3)$

From (1) and (3) it follows that $\quad dx = l\cos^2\theta d\theta\,,$

Which integrated gives $\quad x = \frac{l}{4}(2\theta + \sin 2\theta) \quad (4)$

With the notations: $\varphi=2\theta$ and $l=4R$, eqs. (2) and (4) will give the parametric equations of the trajectory:

$$\begin{cases} x = R(\varphi + \sin\varphi) \\ y = R(1 - \cos\varphi) \end{cases} \tag{5}$$

Eqs. (5) are the parametric eqs. of a cycloid (red curve). Under these conditions, the period of this cycloidal pendulum is $T = 2\pi\sqrt{\frac{l}{g}} = 4\pi\sqrt{\frac{R}{g}}$ and the shape of the limiting curve (green curve) is that of a cycloid's evolute.