A Conducting Checkerboard

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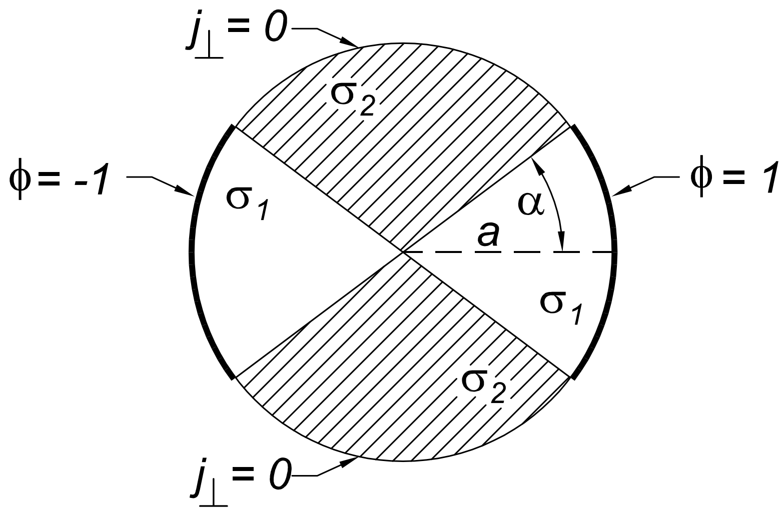
(October 4, 2001)

1 Problem

Some biological systems consist of two "phases" of nearly square fiber bundles of differing thermal and electrical conductivities. Consider a circular region of radius a near a corner of such a system as shown below.



Phase 1, with electrical conductivity σ_1, occupies the "bowtie" region of angle $\pm\alpha$, while phase 2, with conductivity $\sigma_2 \ll \sigma_1$, occupies the remaining region.

Deduce the approximate form of lines of current density $\mathbf{j}$ when a background electric field is applied along the symmetry axis of phase 1. What is the effective conductivity σ of the system, defined by the relation $I = \sigma\Delta\phi$ between the total current I and the potential difference $\Delta\phi$ across the system?

It suffices to consider the case that the boundary arc $(r = a, |\theta| < \alpha)$ is held at electric potential $\phi = 1$, while the arc $(r = a, \pi - \alpha < |\theta| < \pi)$ is held at electric potential $\phi = -1$, and no current flows across the remainder of the boundary.

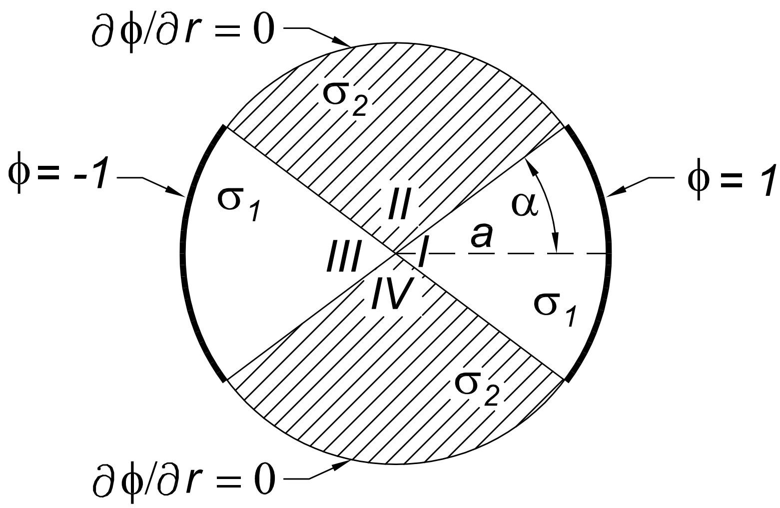
Hint: When $\sigma_2 \ll \sigma_1$, the electric potential is well described by the leading term of a series expansion.

Remarks: The series expansion approach is unsuccessful in treating the full problem of a "checkerboard" array of two phases if those phases meet in sharp corners as shown above. However, an analytic form for the electric potential of a two-phase (and also a four-phase) checkerboard can be obtained using conformal mapping of certain elliptic functions [1]. If the regions of one phase are completely surrounded by the other phase, rather lengthy series expansions for the potential can be given [2]. The present problem is based on work by Grimvall [3] and Keller [4].

2 Solution

In the steady state, the electric field obeys $\nabla \times \mathbf{E} = 0$, so that $\mathbf{E}$ can be deduced from a scalar potential ϕ via $\mathbf{E} = -\nabla\phi$. The steady current density obeys $\nabla \cdot \mathbf{j} = 0$, and is related to the electric field by Ohm's law, $\mathbf{j} = \sigma\mathbf{E}$. Hence, within regions of uniform conductivity, $\nabla \cdot \mathbf{E} = 0$ and $\nabla^2\phi = 0$. Thus, we seek solutions to Laplace's equations in the four regions of uniform conductivity, subject to the stated boundary conditions at the outer radius, as well as the matching conditions that ϕ, $E_\parallel$, and $j_\perp$ are continuous at the boundaries between the regions.

We analyze this two-dimensional problem in a cylindrical coordinate system (r, θ) with origin at the corner between the phases and $\theta = 0$ along the radius vector that bisects the region whose potential is unity at $r = a$. The four regions of uniform conductivity are labeled I, II, III and IV as shown below.



Since $\mathbf{j}_\perp = j_r = \sigma E_r = -\sigma\partial\phi/\partial r$ at the outer boundary, the boundary conditions at $r = a$ can be written

$$\phi_I(r = a) = 1, \tag{1}$$

$$\frac{\partial\phi_{II}(r = a)}{\partial r} = \frac{\partial\phi_{IV}(r = a)}{\partial r} = 0, \tag{2}$$

$$\phi_{III}(r = a) = -1. \tag{3}$$

Likewise, the condition that $j_\perp = j_\theta = \sigma E_\theta = -(\sigma/r)\partial\phi/\partial\theta$ is continuous at the boundaries between the regions can be written

$$\sigma_1 \frac{\partial\phi_I(\theta = \alpha)}{\partial\theta} = \sigma_2 \frac{\partial\phi_{II}(\theta = \alpha)}{\partial\theta}, \tag{4}$$

$$\sigma_1 \frac{\partial\phi_{III}(\theta = \pi - \alpha)}{\partial\theta} = \sigma_2 \frac{\partial\phi_{II}(\theta = \pi - \alpha)}{\partial\theta}, \tag{5}$$

etc.

From the symmetry of the problem we see that

$$\phi(-\theta) = \phi(\theta), \tag{6}$$

$$\phi(\pi - \theta) = -\phi(\theta), \tag{7}$$

and in particular $\phi(r = 0) = 0 = \phi(\theta = \pm\pi/2)$.

We recall that two-dimensional solutions to Laplace's equations in cylindrical coordinates involve sums of products of $r^{\pm k}$ and $e^{\pm ik\theta}$, where k is the separation constant that in general can take on a sequence of values. Since the potential is zero at the origin, the radial function is only r^k. The symmetry condition (6) suggests that the angular functions for region I be written as $\cos k\theta$, while the symmetry condition (7) suggests that we use $\sin k(\pi/2 - \theta)$ in regions II and IV and $\cos k(\pi - \theta)$ in region III. That is, we consider the series expansions

$$\begin{aligned}
\phi_I &= \sum A_k r^k \cos k\theta, & (8)\\
\phi_{II} = \phi_{IV} &= \sum B_k r^k \sin k\left(\frac{\pi}{2} - \theta\right), & (9)\\
\phi_{III} &= -\sum A_k r^k \cos k(\pi - \theta). & (10)
\end{aligned}$$

The potential must be continuous at the boundaries between the regions, which requires

$$A_k \cos k\alpha = B_k \sin k\left(\frac{\pi}{2} - \alpha\right). \tag{11}$$

The normal component of the current density is also continuous across these boundaries, so eq. (4) tells us that

$$\sigma_1 A_k \sin k\alpha = \sigma_2 B_k \cos k\left(\frac{\pi}{2} - \alpha\right). \tag{12}$$

On dividing eq. (12) by eq. (11) we find that

$$\tan k\alpha = \frac{\sigma_2}{\sigma_1} \cot k\left(\frac{\pi}{2} - \alpha\right). \tag{13}$$

There is an infinite set of solutions to this transcendental equation. When $\sigma_2/\sigma_1 \ll 1$ we expect that only the first term in the expansions (8)-(9) will be important, and in this case we expect that both $k\alpha$ and $k(\pi/2 - \alpha)$ are small. Then eq. (13) can be approximated as

$$k\alpha \approx \frac{\sigma_2/\sigma_1}{k(\frac{\pi}{2} - \alpha)}, \tag{14}$$

and hence

$$k^2 \approx \frac{\sigma_2/\sigma_1}{\alpha(\frac{\pi}{2} - \alpha)} \ll 1. \tag{15}$$

Equation (11) also tells us that for small $k\alpha$,

$$A_k \approx B_k k\left(\frac{\pi}{2} - \alpha\right). \tag{16}$$

Since we now approximate ϕ_I by the single term $A_k r^k \cos k\theta \approx A_k r^k$, the boundary condition (1) at $r = a$ implies that

$$A_k \approx \frac{1}{a^k}, \tag{17}$$

and eq. (16) then gives

$$B_k \approx \frac{1}{ka^k(\frac{\pi}{2}-\alpha)} \gg A_k. \tag{18}$$

The boundary condition (2) now becomes

$$0 = kB_k a^{k-1}\sin k\left(\frac{\pi}{2}-\theta\right) \approx \frac{k(\frac{\pi}{2}-\theta)}{a(\frac{\pi}{2}-\alpha)}\,, \tag{19}$$

which is approximately satisfied for small k.

So we accept the first terms of eqs. (8)-(10) as our solution, with k, A_k and B_k given by eqs. (15), (17) and (18).

In region I the electric field is given by

$$E_r = -\frac{\partial\phi_I}{\partial r} \approx -k\frac{r^{k-1}}{a^k}\cos k\theta \approx -k\frac{r^{k-1}}{a^k}\,, \tag{20}$$

$$E_\theta = -\frac{1}{r}\frac{\partial\phi_I}{\partial\theta} \approx k\frac{r^{k-1}}{a^k}\sin k\theta \approx k^2\theta\frac{r^{k-1}}{a^k}\,. \tag{21}$$

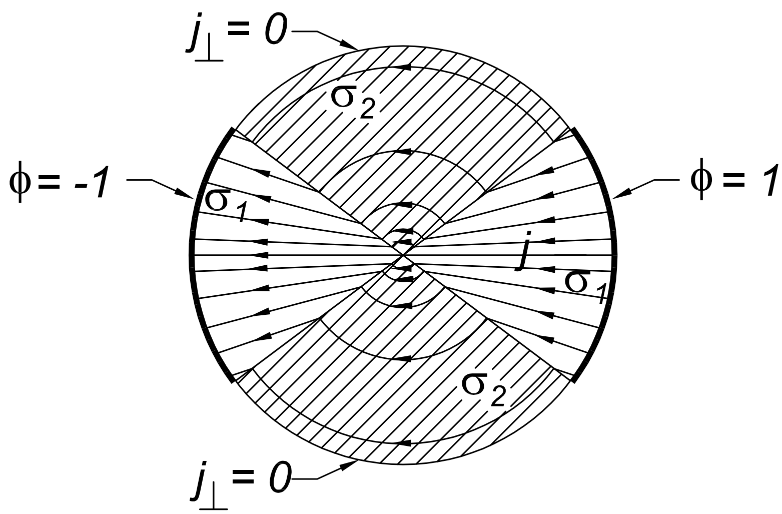
Thus, in region I, $E_\theta/E_r \approx k\theta \ll 1$, so the electric field, and the current density, is nearly radial. In region II the electric field is given by

$$E_r = -\frac{\partial\phi_{II}}{\partial r} \approx -k\frac{r^{k-1}}{ka^k(\frac{\pi}{2}-\alpha)}\sin k\left(\frac{\pi}{2}-\theta\right) \approx -k\frac{r^{k-1}}{a^k}\frac{\frac{\pi}{2}-\theta}{\frac{\pi}{2}-\alpha}\,, \tag{22}$$

$$E_\theta = -\frac{1}{r}\frac{\partial\phi_{II}}{\partial\theta} \approx k\frac{r^{k-1}}{ka^k(\frac{\pi}{2}-\alpha)}\cos k\left(\frac{\pi}{2}-\theta\right) \approx \frac{r^{k-1}}{a^k(\frac{\pi}{2}-\alpha)}\,. \tag{23}$$

Thus, in region II, $E_r/E_\theta \approx k(\pi/2-\theta) \ll 1$, so the electric field, and the current density, is almost purely azimuthal.

The current density $\mathbf{j}$ follows the lines of the electric field $\mathbf{E}$, and therefore behaves as sketched below:



The total current can be evaluated by integrating the current density at $r = a$ in region I:

$$I = 2a\int_0^\alpha j_r d\theta = 2a\sigma_1\int_0^\alpha E_r(r=a)d\theta \approx -2k\sigma_1\int_0^\alpha d\theta = -2k\sigma_1\alpha = -2\sqrt{\frac{\sigma_1\sigma_2\alpha}{\frac{\pi}{2}-\alpha}}\,. \tag{24}$$

In the present problem the total potential difference $\Delta\phi$ is -2, so the effective conductivity is

$$\sigma = \frac{I}{\Delta\phi} = \sqrt{\frac{\sigma_1\sigma_2\alpha}{\frac{\pi}{2} - \alpha}}. \tag{25}$$

For a square checkerboard, $\alpha = \pi/4$, and the effective conductivity is $\sigma = \sqrt{\sigma_1\sigma_2}$. It turns out that this result is independent of the ratio σ_2/σ_1, and holds not only for the corner region studied here but for the entire checkerboard array.

3 References

[1] R.V. Craster and Yu.V. Obnosov, *Checkerboard composites with separated phases*, J. Math. Phys. **42**, 5379 (2001).

[2] Bao Ke-Da, Jörger Axell and Göran Grimvall, *Electrical conduction in checkerboard geometries*, Phys. Rev. B **41**, 4330 (1990).

[3] M. Söderberg and G. Grimvall, *Current distribution for a two-phase material with chequer-board geometry*, J. Phys. C: Solid State Phys. **16**, 1085 (1983).

[4] Joseph B. Keller, *Effective conductivity of periodic composites composed of two very unequal conductors*, J. Math. Phys. **28**, 2516 (1987).