Solution of the problem "Ball in a Box", proposed in 04/01

Dr. Eduardo Aoun Tannuri, University of São Paulo, Brazil
(eduat@usp.br)

The question was:

A hollow cylinder, with its both ends closed, is filled with a fluid and is at rest in the space. Inside the cylinder there is a small hard ball with a density equal to the density of the fluid. The ball is initially at rest and is close to the center of one of the lids (let's call it a "front lid"). The cylinder suddenly gains an acceleration *a* and then moves with that constant acceleration (the motion is non-relativistic although the magnitude of the acceleration can be large). The direction of the acceleration is along the axis of the cylinder pointing from the "rear lid" to the "front lid". If viewed from the reference frame of the cylinder at the very first moment after the acceleration appears the ball will, of course, start gaining speed in the direction toward the "rear lid". The question is: will the ball hit the rear lid?

Solution:

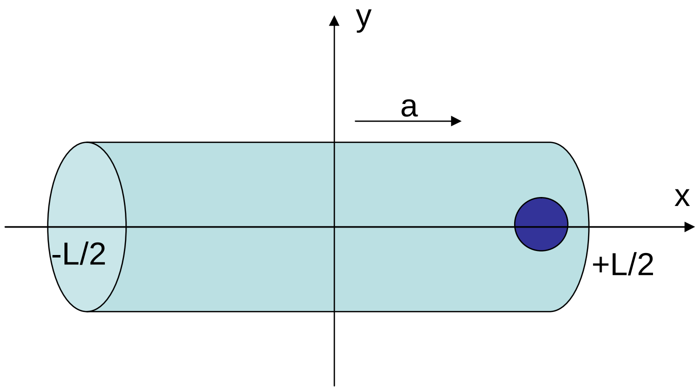

Figure 1. Problem definition

Let's consider a cylinder with length L, sectional area A and a ball with radius R (Figure 1). The density of the ball and of the fluid (at atmosphere pressure) is r_0 and the acceleration of the system is given by a.

Due to the small compressibility, the fluid density will slightly change when it is subjected to a pressure p (related to the atmosphere pressure), following a linear dependence:

$$r = r_0 + k.p \quad (1)$$

where k is a constant that depends on fluid properties. The dynamical equation of an infinitesimal cylinder (Figure 2) is:

$$p(x).A - (p(x) + dp).A = A.dx.r(x).a \tag{2}$$

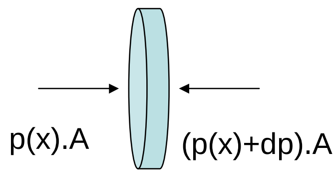

Figure 2. Forces acting on a infinitesimal cylinder

Equations (1) and (2) furnishes the following differential equation:

$$\frac{dp}{dx} = -a.r_0 - a.k.p(x) \tag{3}$$

with the following solution:

$$p(x) = C.\exp(-k.a.x) - r_0 / k \tag{4}$$

being C a constant that depends on boundary conditions. In the present problem, the resulting force action on the whole fluid body given by $r_0.L.A.a$, and the boundary condition can be written as:

$$\left[p(-L/2) - p(L/2)\right].A = r_0 L.A.a \tag{5}$$

Equations (4) and (5) give:

$$C = \frac{r_{0.}L.a}{\exp(k.a.L/2) - \exp(-k.a.L/2)} \tag{6}$$

Fluid density inside the cylinder is then given by:

$$r(x) = \frac{k.a.L}{\exp(k.a.L/2) - \exp(-k.a.L/2)} \exp(-k.a.x).r_0 \tag{7}$$

Considering that the variation of density should be small, due to the small compressibility of the fluid, the inequality $k.a.x << 1$ must be satisfied for $-L/2 \leq x \leq L/2$. So, the exponential functions in equations (4), (6) and (7) will be approximated by its second order Taylor expansions:

$$\begin{aligned}
&\exp(-k.a.x) \cong 1 - k.a.x + (k.a.x)^2 \\
&x = -L/2 \Rightarrow \exp(kaL/2) \cong 1 + kaL/2 + (kaL)^2/4 \\
&x = +L/2 \Rightarrow \exp(-kaL/2) \cong 1 - kaL/2 + (kaL)^2/4
\end{aligned} \tag{8}$$

That results

$$r(x) \cong r_0.\left(1 - k.a.x + (k.a.x)^2\right) \tag{9}$$

with the assumption that $k.a.x << 1$, Equation (9) can be simplified by a linear density profile along cylinder axis:

$$r(x) \cong r_0.\left(1 - k.a.x\right) \tag{10}$$

and the pressure p(x) can be approximated to (using Equations (4) and (6)):

$$p(x) \cong r_0.a.(-x + k.a.x^2) \tag{11}$$

The fluid force acting on the ball can be evaluated by:

$$F(x) = -\iint_S p(x).\vec{n}dS \tag{12}$$

where $\vec{n}$ is the normal vector across the surface element dS. Using Gauss Theorem, Equation (12) can be rewritten:

$$F(x) = -\iiint_V \nabla p(x).dV \tag{13}$$

with $\nabla p(x) = r_0.a.(-1 + 2.k.a.x)\vec{i}$, being $\vec{i}$ the versor along x axis. The volume element in the ball is represented in Figure 3, and can be written as:

$$dV = p(R^2 - r^2).dr \quad \text{, with } -R \le r \le +R \tag{14}$$

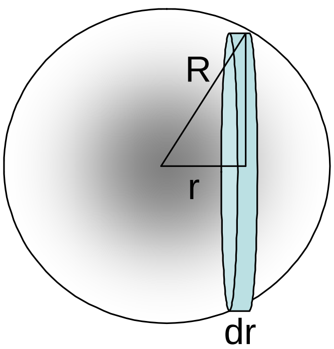

Figure 3. Volume element in the ball

Since x is the position of the center of the ball, the position of the volume element is given by *x+r*. So, the force F(x) can be evaluated by:

$$F(x) = -r_0.a.p.\vec{i}\int_{-R}^{+R}\left(-R^2 + r^2 + 2.k.a.R^2.(x+r) - 2.k.a.r^2.(x+r)\right)dr \tag{15}$$

that gives:

$$F(x) = \frac{4}{3}pR^3.r_0.a.(1-2.k.a.x).\vec{i} \tag{16}$$

Furthermore, there is another pressure-induced force that must be considered, known as added-mass forces. Such forces are proportional to the acceleration of the body related to the fluid, and may be mathematically considered as an “extra” mass “attached” to the body. Indeed, this effect is caused by the energy transfer to the fluid, which is perturbed by the moving body. The mathematical calculation of the added mass can be found in several hydrodynamics textbooks, and will not be detailed here. For a sphere, the added mass force is given by:

$$F_{added\_mass} = -\frac{2}{3}pR^3.r_0.\ddot{x}.\vec{i}$$

The acceleration of the ball related to a fixed reference system is given by:

$$\vec{a}_{Ball} = (a + \ddot{x}).\vec{i} \tag{17}$$

So, applying the 2nd Newton’s Law to the ball (in the direction of x axis) results:

$$m_{Ball}.a + m_{Ball}.\ddot{x} = \frac{4}{3}pR^3.r_0.a.(1-2.k.a.x) - c.\dot{x} - \frac{2}{3}pR^3.r_0.\ddot{x} \tag{18}$$

where $c\dot{x}$ represents viscous forces acting on the ball. Since the density of the ball is r_0, Equation (18) can be rewritten:

$$\ddot{x} = -\frac{4}{3}.k.a^2.x - \frac{2}{3}c'.\dot{x}\text{, with } c' = c/m_{Ball} \tag{19}$$

Equation (19) represents a damped oscillator, with natural frequency $w = \sqrt{\frac{4}{3}.k.a^2}$. So, the ball will leave the front lid toward the rear lead, but it will not reach the rear lead due to damping forces (see Figure 4). The higher the acceleration or compressibility, the higher will be the frequency of the oscillatory motion.

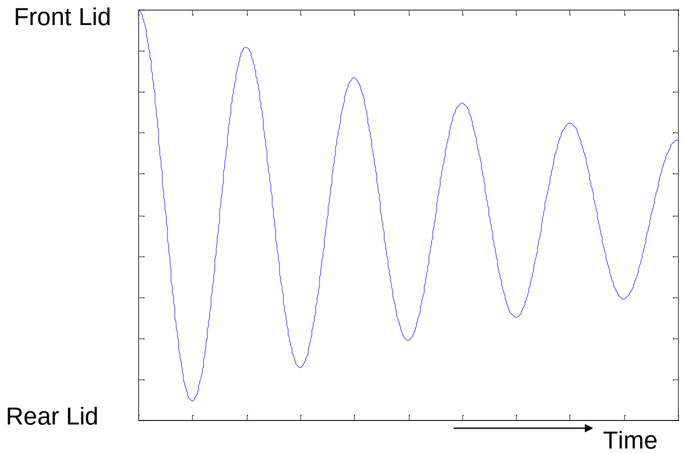

Figure 4. Motion of the ball

Of course, disregarding viscous forces the ball could reach the rear lid, but such assumption is not realistic.