On the Isochrone problem of Classical Mechanics

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1 1-dimensional problem: potential well

We will begin this discussion by studying the one-dimensional oscillation of a particle, influenced by a potential $V(x)$, which is described by the conservation of energy:

$$\frac{1}{2}m\dot{x}^2 + V(x) = E. \tag{1}$$

We can always carry out a translation of the origin, so that the particle oscillates between the symmetric points $x = \pm a$. Since, at those points the velocity of the particle vanishes, the total energy will just equal the potential, that is:

$$V(\pm a) = E. \tag{2}$$

The period of oscillations can be found by evaluating:

$$T = 2\sqrt{\frac{m}{2}} \int_{-a}^{a} \frac{dx}{\sqrt{E - V(x)}}. \tag{3}$$

The problem we are interested in, is to find the potential $V(x)$ which makes the period T independent from the energy, that is, the potential making the motion isochrone. Everyone has observed that the harmonic oscillator is the obvious solution to our problem. But, suppose that we are not interested in the simple confirmation that the potential we are searching for is the harmonic oscillator. Our aim is to develop a systematical way of determining the potential in question, so that the same method can be used -with the proper generalization- to solve the analogous problems for two-dimensional oscillations (and much more!).

We can assume the potential in question to be symmetric with reference to the origin $V(-x) = V(x)$. We do not worry here as to whether this is a necessary condition for the existence of solutions, nor to its generality. We just restrain ourselves to solutions with even parity.

Therefore, it is easy to see that the whole dependence of the period to the energy is hidden in the following expression:

$$\sqrt{\frac{2}{m}}(T/4) = \frac{1}{\sqrt{E}} \int_0^a \frac{dx}{\sqrt{1 - \frac{V(x)}{E}}}. \tag{4}$$

If we set $q = \frac{V(x)}{E}$ as the new integration variable, the integral becomes:

$$\frac{1}{\sqrt{E}} \int_{V(0)/E}^{1} \frac{[V^{-1}(qE)]'}{\sqrt{1-q}} dq. \tag{5}$$

Even though the lower integration limit seems to depend on the energy, we can always gauge the potential so that $V(0) = 0$. Thus, the integration limits become independent of the problem parameters:

$$T(E) \sim \frac{1}{\sqrt{E}} \int_0^1 \frac{[V^{-1}(qE)]'}{\sqrt{1-q}} dq. \tag{6}$$

It should be stated, that the prime denotes differentiation with respect to q. Therefore, our demanding the period to be independent from the energy, is equivalent to the independence of the quantity $\frac{1}{\sqrt{E}}[V^{-1}(qE)]'$ from the energy:

$$\partial_E \left(\frac{1}{\sqrt{E}} \frac{\partial}{\partial q} \left\{ V^{-1}(qE) \right\} \right) = 0. \tag{7}$$

Since the quantity inside the brackets has no explicit dependence on the energy, it can only be a function of q:

$$\frac{1}{\sqrt{E}} \frac{\partial}{\partial q} \left\{ V^{-1}(qE) \right\} = f(q). \tag{8}$$

The above relation can be "symmetrized" with respect to the argument of the inverse function, if we consider qE to be the new variable:

$$\sqrt{E} \frac{\partial}{\partial (q)} \left\{ V^{-1}(qE) \right\} = f(q). \tag{9}$$

By bringing the equation in the more appropriate form:

$$\frac{\partial}{\partial(q)}\left\{V^{-1}(qE)\right\} = \frac{f(q)}{\sqrt{E}}, \tag{10}$$

one may argue that f should be of the form $f(q) \sim \frac{1}{\sqrt{q}}$. This is quite obvious, from the fact that the left side of (10) is only a function of the product (qE), and so should the right side, as well.

By taking all the abovementioned into account, we have:

$$\frac{\partial}{\partial(q)}\left\{V^{-1}(qE)\right\} = \frac{\lambda}{\sqrt{qE}}. \tag{11}$$

We only have to substitute $qE = z$ in the last equation, before we can have the long-awaited differential equation for the potential:

$$\frac{dV^{-1}(z)}{dz} = \frac{\lambda}{\sqrt{z}}, \tag{12}$$

which means,

$$V^{-1}(z) = 2\lambda\sqrt{z} + \beta. \tag{13}$$

By acting on both sides of (13) with the potential V, we have:

$$V\left(V^{-1}(z)\right) \equiv z = V(2\lambda\sqrt{z} + \beta). \tag{14}$$

By setting $x = 2\lambda\sqrt{z} + \beta$, and substituting, we finally get:

$$V(x) = \left(\frac{x-\beta}{2\lambda}\right)^2. \tag{15}$$

From our initial demand for positive parity, we get $\beta = 0$ (which wasn't really necessary!). We have, thus, found in a natural way the harmonic oscillator potential. If we put $\lambda = 1/\sqrt{2k}$, then eq. (15) will assume its familiar form:

$$V(x) = \frac{1}{2}kx^2. \tag{16}$$

(hardly necessary as well!).

One may readily observe, that the method we used in the above, can be properly generalized to give the isochrone potentials for multidimensional problems. Problems, like bead on a wire, can be solved, because they may be multidimensional to us-obeservers, but they are one-dimensional to the bead! Therefore, no other parameter (except for the energy) arises in the problem.

But, for example, if we consider a small sphere in a bowl, there are two independent parameters in the problem, the total energy and total angular momentum. The problem of finding the "analogous" isochrone (cylindrically symmetric) bowl is then a problem of making the period independent of two parameters and L. Clearly, such a thing as an isochrone potential for two independent parameters cannot be solved (no such solution can exist).

2 Two-dimensional problem: Motion of a bead along a wire

As a "two-dimensional" problem, we discuss the motion of a bead along a wire of shape $y = y(x)$. We assume, again, a symmetric solution $y(-x) = y(x)$ and take into account that $mg\ y\,(a) = E$. The period of oscillation will then be analogous to:

$$\int_0^a \sqrt{\frac{1+[y'(x)]^2}{E - mg\ y(x)}}dx = \frac{1}{\sqrt{E}} \int_0^a \sqrt{\frac{1+(y')^2}{1-\frac{mg}{E}y(x)}} \tag{17}$$

By setting $q = \frac{mg}{E}y(x)$ (again), where $x = y^{-1}\left(\frac{qE}{mg}\right)$, we have:

$$dx = \left[y^{-1}\left(\frac{qE}{mg}\right)\right]' dq. \tag{18}$$

The slope of the wire will be:

$$y'(x) = \frac{E}{mg}\frac{1}{\left[y^{-1}\left(\frac{qE}{mg}\right)\right]'}. \tag{19}$$

We also choose (arbitrarily) $y(0) = 0$. Then, the integral becomes:

$$\frac{1}{\sqrt{E}} \int_0^{mg} \frac{dq}{\sqrt{1-q}} \left[y^{-1}\left(\frac{qE}{mg}\right)\right]' \sqrt{1 + \left(\frac{E}{mg}\frac{1}{\left[y^{-1}\left(\frac{qE}{mg}\right)\right]'}\right)^2}, \tag{20}$$

which should be independent of the energy. This requirement, like in the one-dimensional case, is written:

$$\sqrt{E}\frac{d}{dz}\left[y^{-1}(z)\right] \cdot \sqrt{1 + \frac{1}{\left(\frac{d}{dz}y^{-1}(z)\right)}} = f(q), \tag{21}$$

where we have set $z = qE/mg$. The function $f(q)$ should -again- be chosen so that $f(q) = \frac{\lambda}{\sqrt{q}}$ for the same reasons (as before). Finally, we have:

$$\sqrt{1 + \left(\left[y^{-1}\left(\frac{qE}{mg}\right)\right]'\right)^2} = \frac{\lambda}{\sqrt{z}}, \tag{22}$$

which is easily integrated to yield:

$$y^{-1}(z) = \int dz \sqrt{\frac{\lambda^2}{z} - 1} = z\sqrt{\frac{\lambda^2}{z} - 1} - \lambda^2 \cot^{-1} \sqrt{\frac{\lambda^2}{z} - 1} + \beta. \tag{23}$$

It is quite possible that one does not experience emotions of extreme delight when first facing solution (23). Nevertheless, we shall see that we can make things simpler. Firstly, we can substitute $z = y(x)$, and the equation will become:

$$x = y\sqrt{\frac{\lambda^2}{y} - 1} - \lambda^2 \cot^{-1} \sqrt{\frac{\lambda^2}{y} - 1} + \beta. \tag{24}$$

By setting $x = 0$ in the above eqn. and using $y(0) = 0$, we find $\beta = 0$. What is more, if we set $y = (\lambda^2/2)(1 - \cos\xi)$, where ξ is just a parameter valued from 0 over to 2π, we find $x = -(\lambda^2/2)(\xi - \sin\xi)$. The curve defined by this parametric form is easily identified as a cycloid.