MAGNETIC PROPERTIES NEAR PERCOLATION

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INTRODUCTION

At the percolation threshold, clusters of connected sites exhibit self-similar fractal geometry, in which many geometrical properties scale as powers of the relevant length scales. This motivated both the construction of fractal models, on which physical problems may be solved exactly,[1,2] and the detailed study of various geometrical characteristics of percolation clusters.[3,4] In this lecture we shall review the effects of this fractal geometry on the magnetic correlations of spins at and near the percolation threshold. The results reflect on neutron scattering experiments from percolation clusters and similar fractals.

Already the first study of magnetic models on fractals[5] showed that their universal exponents depend on various geometrical characteristic parameters, in addition to the fractal dimensionality. This was followed by detailed studies on various fractals,[6-8] which will not be repeated here. More recently, we undertook[10] to investigate the structure factor which is expected when neutrons are scattered from dilute magnets near percolation. There exist many related experimental studies,[11] and it turns out that the theoretical predictions which result from the fractal structure imply significant changes in the functional forms to which these data should be compared.

Since some of the material covered in this lecture was included in Ref. 10, we proceed with a short summary, which is followed by some more recent results.

The magnetic correlations between two spins propagates only through the *backbone* of the cluster connecting them. It is currently believed[4] that on length scales which are short compared to the percolation correlation length ξ_p (i.e. all length scales at the threshold $p=p_c$) this backbone has the shape shown schematically in Fig. 1. There are "blobs" of multiply connected spins, between which there exist chains of "singly connected" bonds. At the threshold there exists no typical length scale. If the distance between the two end points is r then all the dependences on r should be via power laws. In particular, the average number of singly connected bonds scales as

$$L_1(r) = A_1 r^{\tilde{\zeta}_1}, \tag{1}$$

and Coniglio[4] showed that $\tilde{\zeta}_1 = 1/\nu_p$, where ν_p describes the divergence of the correlation length, $\xi_p \propto |p-p_c|^{-\nu_p}$.

Another quantity of interest is the electrical resistance between the two end points, if each bond has unit resistance. The resistance is believed to scale as

$$L_R(r) = A_R r^{\tilde{\zeta}_R}. \tag{2}$$

At dimensions d=2,3 one has $\tilde{\zeta}_1 = 3/4$, 1.14 and $\tilde{\zeta}_R \simeq 0.97$, 1.25.[10]

At and below p_c all the spins belong to finite clusters. The probability that our two end spins belong to the same cluster, G(r), behaves as

$$G(r) \propto r^{-(d-2+\eta_p)} \tag{3}$$

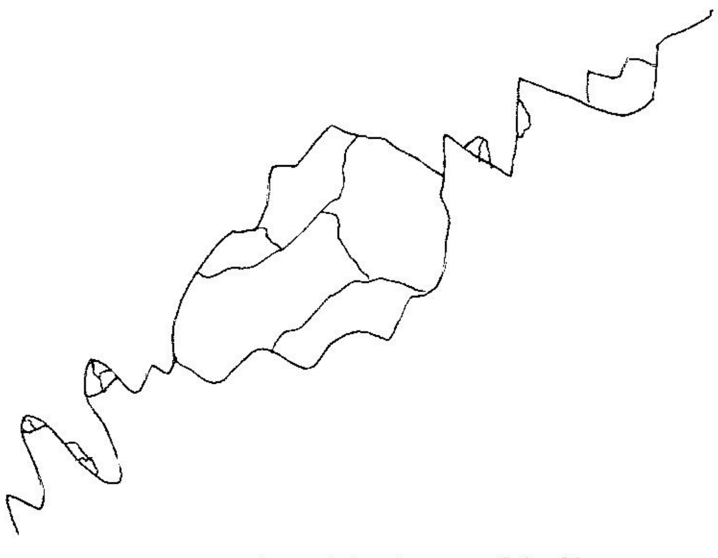

Fig. 1. Link - Blob picture of backbone.

at $p=p_c$ (or $r<<\xi_p$), and as $r^{-(d-1)/2}\exp(-r/\xi_p)$ for $p<p_c$. The exponent η_p has values $\eta_p \approx 0.2$, -0.01 for d=2, 3.

MAGNETIC CORRELATIONS

Consider first the Ising case. For a linear chain of $L_1(r)$ bonds in series it is easy to see that the spin-spin correlation function is[10]

$$\langle S(0)S(r)\rangle = (\tan h\, K)^{L_1(r)} = e^{-L_1(r)/\xi_1}, \tag{4}$$

with $K=J/k_BT$, where J is the nearest neighbor exchange, T is the temperature and k_B is the Boltzmann constant. The one-dimensional thermal correlation length is given by

$$\xi_1 = (-\ell n \tan h\, K)^{-1} \simeq \tfrac{1}{2}\, e^{2K}. \tag{5}$$

If two spins are connected via two chains, we can identify pairs of bonds such that disconnecting the two bonds in a pair disconnects the two spins. If the number of these pairs is $L_2(r)$ then to leading order in e^{-2K} one has[10]

$$\langle S(0)S(r)\rangle \simeq 1 - 2L_2 e^{-4K} \simeq e^{-\frac{1}{2}L_2/\xi_1^2} \tag{6}$$

Thus, to leading order in e^{-2K} one may say that the multiply connected bonds create rigid correlations; the spin correlation function decays only on the singly connected spins, and[4]

$$\langle S(0)S(r)\rangle \simeq \exp\,[-L_1(r)/\xi_1]. \tag{7}$$

However, this is true only if $L_1/\xi_1<<L_2/\xi_1^2$. On the percolation clusters, it turns out[10] that $L_2(r)\propto L_1(r)^2$, and therefore Eq. (7) holds only for $L_1(r)<\xi_1$. At larger distances one encounters a faster decay. For practical calculations we shall use a sharp cutoff, i.e. assume that (7) holds for $L_1(r)<u\xi_1$, while $\langle S(0)S(r)\rangle = 0$ for larger distances.

For Heisengerg spins on a linear chain one still has $\langle \vec{S}(0)\cdot\vec{S}(r)\rangle \simeq \exp(-L_1/\xi_1)$, but now (for K>>1)

$$\xi_1 \propto K = J/k_BT. \tag{8}$$

If two chains, of lengths L_1 and L_2, connect the same two spins, then $\langle \vec{S}(0)\cdot\vec{S}(r)\rangle \simeq [1-a/(K_1+K_2)]$, with $(1-a/K_i) = (1-a/K)^{L_i}$, i.e. $K_i \simeq K/L_i$. Thus $K_1+K_2 \simeq K(1/L_1+1/L_2)$, and $\langle \vec{S}(0)\cdot\vec{S}(r)\rangle \simeq \exp(-L_R/\xi_1)$, with $L_R = L_1L_2/(L_1+L_2)$. More generally, $L_R(r)$ is equivalent to the resistance of the bonds connecting the two spins. Again, this should hold only when $L_R<\xi_1$, and we expect a cutoff at larger distances.

The correlations are given as $\exp[-L_i(r)/\xi_1]$ only if the two spins belong to the same cluster. In a real experiment one should average over all configurations, i.e. multiply $\langle \vec{S}(0)\cdot\vec{S}(r)\rangle$ by G(r). At $p=p_c$ we thus expect that

$$\overline{\langle \vec{S}(0)\cdot\vec{S}(r)\rangle} \propto \frac{1}{r^{d-2+\eta_p}} \exp\left(-Ar^{\tilde{\zeta}}/\xi_1\right) \tag{9}$$

for $Ar^{\tilde{\zeta}} < u\xi_1$, and zero otherwise.

We note that a similar result is expected on any fractal structure. The exponent $(d-2+\eta_p)$ is then replaced by $2(d-D)$, where D is the fractal dimensionality of the fractal.

STRUCTURE FACTOR AT PERCOLATION

The structure factor is defined as Fourier transform of the average spin correlation function. Using (9), we write (at $p=p_c$)

$$S(q,T) = \int d^d r\, r^{-(d-2+\eta_p)}\, e^{-A|\vec{r}|^{\tilde{\zeta}}/\xi_1}\, e^{i\,\vec{q}\cdot\vec{r}}, \tag{10}$$

with $A|\vec{r}|^{\tilde{\zeta}} < u\xi_1$. It is easy to check that S has the scaling form

$$S(q,T) = q^{-(2-\eta_p)}\, f(q\xi_1^{\nu_T}), \tag{11}$$

with $\nu_T = 1/\tilde{\zeta}$. For small $y = q\xi_1^{\nu_T}$, $f(y)$ behaves as

$$f(y) = Cy^{(2-\eta_p)}/[1+y^2+\Sigma_4 y^4+\ldots]. \tag{12}$$

The structure factor (12) would be a Lorentzian only if $\eta_p = 0$ and $\Sigma_4 = \Sigma_6 = \ldots = 0$. For an infinite cutoff ($u = \infty$) we find $\Sigma_4 = -0.32$, -0.22 for $d = 2,3$ Ising systems and $\Sigma_4 = 0.09$, 0.19 for $d=2,3$ Heisenberg systems. Since at $d=3$ one also has $\eta_p \simeq -0.01 \pm 0.09$,[12] it seems that a Lorentzian is a good approximation. However, two dimensional systems show large deviations from a Lorentzian.

To obtain a quantitative feeling for S at $d=2$, we calculated the integral in (10) with $\tilde{\zeta}=1$ and $u=\infty$. We find[10]

$$f(y) = \tilde{C}y^{(2-\eta_p)}\, P^{o}_{1-\eta_p}[(1+y^2)^{-\frac{1}{2}}]/(1+y^2)^{(2-\eta_p)/2}, \tag{13}$$

where P^{o}_{μ} is the associated Legendre function.

For large y we thus have

$$f(y) \propto P^{o}_{1-\eta_p}(y^{-1}) \simeq \frac{\eta_p}{4\pi} + y^{-1} + \ldots . \tag{14}$$

Since $4\pi/\eta_p \simeq 60$, we expect significant deviations from the leading $q^{-(2-\eta_p)}$ already for $q\xi_1^{\nu_T} \lesssim 60$, where $q^{-(3-\eta_p)}$ becomes larger.

Below p_c, $G(r)$ behaves as $r^{-(d-1)/2}\exp(-r/\xi_p)$. Multiplying by Eq. (9), the average correlation function becomes

$$\overline{\langle\vec{S}(0)\cdot\vec{S}(r)\rangle} \propto r^{-(d-1)/2}\exp(-r/\xi_p - Ar^{\tilde{\zeta}}/\xi_1) \tag{15}$$

with some cutoff on r. If $\tilde{\zeta}=1$, as is almost the case for Ising at d=3 or Heisenberg for d=2, and if the cutoff is infinite, then the Fourier transform of (15) becomes a Lorentzian, $S(q,T,p) \propto (\kappa^2+q^2)^{-1}$, with $\kappa=\xi_p^{-1}+\xi_1^{-1}$. Indeed, this expression for κ is exact in one dimension. More generally, the correlation length is given by

$$\xi^2 = \int d^d r\, r^2\, \overline{\langle\vec{S}(0)\cdot\vec{S}(\vec{r})\rangle} \Big/ \int d^d r\, \overline{\langle\vec{S}(0)\cdot\vec{S}(\vec{r})\rangle}, \tag{16}$$

and we expect the scaling form

$$\xi(p,T) = \xi_1^{1/\tilde{\zeta}}\; X\,(\xi_1^{1/\tilde{\zeta}}/\xi_p). \tag{17}$$

Numerical integration, using (15), with A=1 and u=1, yielded the results shown in Fig. 2. For this cutoff we find that the results for Heisenberg ($\tilde{\zeta}$=1.0, 1.25 at d=2,3) and Ising ($\tilde{\zeta}$=3/4, 1.14) are indistinguishable. Note that for large $\xi_1^{-1/\tilde{\zeta}}$, i.e. high temperature, $1/\xi$ is linear in $\xi_1^{-1/\tilde{\zeta}}$, as indeed fitted in many experiments.[11] However, important deviations occur at low temperatures. In this regime, Eq. (17) may be expanded as

$$\xi^{-1} = a_o \xi_1^{-1/\tilde{\zeta}}\,\Big(1+\sum_{n=1} a_n\,(\xi_1^{1/\tilde{\zeta}}/\xi_p)^n\Big), \tag{18}$$

with the a_n's being combinations of incomplete gamma functions. For A = u = 1, we find $a_1 \simeq 0.11$, 0.08 and $a_2 \simeq 0.015$, 0.011 for d = 2, 3. For u = ∞ the results depend on $\tilde{\zeta}$, and we find $a_1 = 5.81$, 1.15, $a_2 = -0.48$, 0.09 for the Ising case, $a_1 = 1.8$, 0.86, $a_2 = 10^{-5}$, 0.86 for the Heisenberg case. The strong dependence on the cutoff means that these results should be taken only qualitatively. However, it is clear that the usual approximation

$$\xi^{-1} = a_o\,(\xi_1^{-1/\tilde{\zeta}} + a_1\xi_p^{-1}) \tag{19}$$

is wrong.

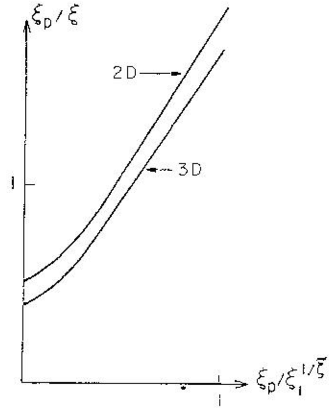

Fig. 2. Inverse Correlation Lengths.

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Scaling Phenomena in Disordered Systems

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