Optimal Investment-Dividend Policy of an Insurance Firm under Regulation

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Abstract
An insurance decision model including intervention by a regulating agency is defined. The insurance firm’s problem is to establish an investment policy as well as a dividend strategy. Regulation is exercised by a minimal barrier policy for cash holding and penalties for violating this barrier. The joint Insurance Firm-Regulating Agency problem is discussed by using concepts drawn from Stackleberg strategies in game theory. As in the classical model of collective risk theory it is assumed that premium payments are received deterministically from policyholders at a constant rate, while the claim process is determined by a Compound Poisson process. Finally a diffusion approximation is used in order to obtain tractable results for a general claim size distribution.

1. Introduction
Problems of capital adequacy and effective regulation of insurance firms have been raised by both theoreticians and practitioners alike. The objectives of regulation are to protect consumers from defaulting firms as well as to insure the viability of the insurance industry. In a recent paper Borch (1981) investigated the necessity of government regulation in the insurance industry. He found that if the company is primarily interested in making a quick profit some regulation may be required. On the other hand, if the management of the firm takes a long term view, no regulation should be needed.

Most studies of regulation have focused on the effects of regulation on the firms or consumers and less attention has been paid to the regulator’s cost structure. Much attention has been attracted to the effects of regulation of the firm’s capital structure on its probability of ruin (see Black Miller and Posner 1978). Others have examined the effects of the regulation of the insurer’s investment and underwriting policies on its probability of ruin. Such studies show that the regulation may introduce market inefficiencies.

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and may even increase the probability of ruin under certain circumstances (e.g., see Kahane, 1977). Another stream in the literature concentrates on the effects of regulation on the pricing mechanism and on competition (Kareken and Wallace 1978, Meltzer 1976, Stigler 1971, Modigliani and Miller 1958 etc.). The capital assets pricing model has been used in order to derive the profit rates which property—liability insurance firms will earn in a competitive market (Kahane, 1978). The derived formula can be used by regulators to set fair profit rates. The acts of the regulator involve certain social costs in the forms of market inefficiencies, and certain ideological and political compromises. This paper will explicitly incorporate such costs into the optimization model.

In this paper, we assume a basic and well known insurance decision model (e.g., Buhlman 1970, chapter 6), but add several new ingredients to render it more realistic and include in it a regulation mechanism. Throughout the article we distinguish between short term and long term investments. Although we assume a barrier investment strategy (see also Hallin 1977, Borch 1974 and Buhlman 1970), we add an investment conversion feature when the firm's assets reach a minimal, externally imposed, regulation barrier. That is, whenever the firm's holdings fall below a predetermined level, short term investments can be converted into cash to cover claims. Such a procedure is usually employed by insurance firms (see also Tapiero, Zuckerman and Kahane, 1979, Tapiero and Zuckerman 1980).

The problem of the regulating agency is to establish a lower cash holding barrier and penalty costs for violating it. The determination of the above variables is based upon minimization of a cost function consisting of the political and administrative cost associated with the regulator's strategy imposed on the insurance industry as well as the social cost of having insurance firms fail in their payments. That is, the regulating agency cost function is a quantitative description of the protection given to the public from insurance firms failures on one hand, and the inefficiency and lack of competition due to government intervention on the other hand. Once the regulator's strategy is established, the firm's problem is to maximize a long run average profit function in order to determine its investment and dividend policy. Of course, the policy determined by the firm is a function of the strategy used by the regulator. Such a solution implies that the regulating agency is the leader in our problem. A solution concept of this type is defined by a Stackelberg strategy (Stackelberg 1934).

For computational convenience, the profit function of the firm as well as the cost function used by the regulating agency are maintained as simple as possible. Such simplifications enable identification of the essential effects the regulator's strategy has on the firm's policy. Issues of taxes, reinsurance, various gaming situations, etc., are not dealt with here, but could be considered in subsequent extensions of this paper.

Closed form results for our decision model are obtained in the special

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case in which the claim size is exponentially distributed. A diffusion approximation technique is used to achieve tractable results for a general claim size distribution. In this study we show the usefulness of the diffusion approximation theory in handling compound Poisson claim processes in insurance.

2. The Firm Insurance Model with Regulation

We consider first the problem as seen from the insurance firm’s point of view. Let $K$ be the minimum requirement funds level (capital) imposed by the regulating agency. Claims occur at a Poisson rate $\lambda$ with successive claims assumed positive, independent and identically distributed random variables $Y_1, Y_2, \ldots$, having a known distribution function $H(\cdot)$ with mean $\xi$. The premium rate is assumed to be a constant and is given by $\pi = \lambda \xi (1 + \delta)$, where $\delta > 0$, is the loading factor. For simplicity, $\delta$ is assumed to be a quantity externally imposed on the firm by competitive conditions. The firm’s policy is given by establishing an investment and dividend strategies. Before proceeding, we describe in detail the firm’s policy: all incoming premiums above the minimum requirement level $K$, are directly transferred into short term investments having a rate of return of $r_1$. When the amount of short term investments reaches a certain level $S$, determined by a managerial decision, all additional premiums are converted into long term investments with rate of return of $r_2$. Further, when short term investments reach the level $S$ and long term investments reach a control level $L$ determined by the firm, all incoming premiums are distributed as a dividend among the shareholders. When the cash level process jumps below the regulation barrier, $K$, the firm converts short term investments into cash to cover claims. If the amount of short term investment is zero, the firm can borrow money at an interest rate of $C$ to cover claims when the cash level process jumps below zero. It is assumed that the conversion time from short term investment into cash and from cash into short term investments is negligible. Furthermore, there is no transaction cost associated with the above conversions. In the general case, it is possible to convert long term investments into cash at a fixed transaction cost of $l_0$ plus a variable cost of $l_1$. Throughout the paper we shall restrict attention to the interesting special case in which $l_0 = \infty$, i.e., long term investments cannot be converted into cash in order to cover claims. More explicitly, let $X(t)$ be the amount of cash plus short term investments at time $t$. The state space of the stochastic process $X$ is defined over the set $(-\infty, K+S]$. If $K \leq X(t) \leq K+S$, then the cash level at time $t$ is $K$ and $X(t)-K$ monetary units are maintained in short term investments. If $X(t) < K$, then no funds are maintained in short term investments and cash reserves fall below the minimum requirement funds level. A penalty cost of rate $R(K-X(t))$ is incurred for violating the regulation barrier $K$, where $R$ is determined by the

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regulator. When the cash level falls below the zero barrier by \( y \) units, an additional (borrowing) cost rate of \( C_y \) is incurred. Note that when \( C \) approaches infinity, a ruin model is obtained.

Let \( Y(t) \) be the amount of long term investments. From our description of the investment policy employed by the firm, it can be seen that the state space of \( Y \) is defined over the set \([0, L]\), where \( L \) is an absorbing state.

The optimality criterion used by us is to maximize long run average profit per unit time. The firm’s profit is defined by earned premiums plus income from short term and long term investments less claims and costs. Two types of costs are distinguished: (a) a penalty cost when violating the regulating barrier \( K \), and (b) a borrowing cost incurred when falling below the zero barrier. Implications of these costs are important and will be dealt with in greater detail in the sequel. We assume that the administrative and commissions costs are independent of the investment-dividend policy adopted by the firm and therefore these costs are not included in the profit function of the insurance firm. In order to obtain a better insight into the nature of our profit function, it is important to note that by employing a simple flow argument it can be seen that the limiting average rate of earned premiums less claims is precisely the conversion rate of cash into short and long term investments plus the expected dividend per unit time. Let \( \Psi(S, L|K, R) \) denote the long run average firm profit when a \((S, L)\) investment-dividend policy is employed given that the minimum cash requirement is \( K \) and the penalty cost rate is \( R \).

It can be seen that the expected earned premiums less claims per unit time is given by \( \lambda \xi \delta \) where \( \lambda \) is the arrival rate of claims, \( \xi \) is the expected claim size and \( \delta \) is the loading factor. The firm’s objective function can now be expressed as follows:

\[
\Psi(S, L|K, R) = \lambda \xi \delta + \lim_{t \to \infty} \frac{1}{t} \left\{ \int_0^t Y(t) \, dt + \int_0^t r_1(X(t) - K)I_{(X(t) \geq K)} \, dt \right. \\
- \left. \int_0^t R(K - X(t))I_{(X(t) < K)} \, dt + \int_0^t C(t)I_{(X(t) < 0)} \, dt \right\}. 
\]  

(1)

Where \( I_E \) is the indicator function of the event \( E \).

Without loss of generality, state that

\( X(0) = K + S \), and \( Y(0) = L \).

In words, the initial state of the stochastic process \( X \), representing the amount of cash plus short term investments, is \( K + S \), while the amount of long term investments at time zero is assumed to be \( L \).

Let

\[
Q = \inf \{ t > 0; X(t) < K + S \}. 
\]  

(2)

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The quantity $Q$ is a random variable indicating the first instant at which the stochastic process $X$ falls below the initial state $K+S$.

Similarly, define a stopping time $T$

$$T = \inf \{ t > Q; X(t) = K+S \}$$

indicating the time in which $X$ returns to state $K+S$ after time $Q$.

We will refer to the time period $[0, T]$ as a cycle. It can be seen that at time $T$, the bivariate process $(X, Y)$ regenerates itself. Applying renewal argument, the long run average profit can be expressed as follows:

$$\Psi(S, L|K, R) = \lambda \xi \delta + r_2 L + \frac{1}{E[T]} E \left[ \int_0^T f(X(t)) \, dt \right]$$

where

$$f(x) = \begin{cases} 
  r_1 (x-K) & \text{if } K \leq x \leq K+S, \\
  -R(K-x) & \text{if } 0 \leq x < K, \\
  -R(K-x) + Cx & \text{if } x < 0. 
\end{cases}$$

Maximization of $\Psi(S, L|K, R)$ with respect to $S$ and $L$ provides an optimum investment-dividend policy. We proceed to determine an explicit expression for $\Psi(S, L|K, R)$. To do so it is necessary to determine the stationary distribution of the $X$ process. Define

$$W(x, t) = P(X(t) \geq K+S-x), \quad (x > 0).$$

By a “backward” argument we obtain for $\Delta t \to 0$

$$W(x, t+\Delta t) = (1-\lambda \Delta t) W(x+\mu(\delta) \Delta t, t) + \lambda \Delta t \int_0^{\xi} W(x-y, t) \, dH(y) + o(\Delta t)$$

where

$$\mu(\delta) = (1+\delta) \lambda \xi.$$

From (6) we obtain

$$\frac{1}{\mu(\delta)} \frac{\partial}{\partial t} W(x, t) = -\frac{\lambda}{\mu(\delta)} W(x, t) + \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial x} W(x, t) + \frac{\lambda}{\mu(\delta)} \int_0^{\xi} W(x-y, t) \, dH(y) \right].$$

Let

$$W(x) = \lim_{t \to \infty} W(x, t).$$

The limiting distribution $W(x)$ satisfies (7) with the condition $\partial W(x, t)/\partial t = 0$. Hence, $W(x)$ satisfies the following integro-differential equation

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\[
-\frac{\lambda}{\mu(\delta)} W(x) + \frac{\delta}{\mu(\delta)} W(x) + \frac{\lambda}{\mu(\delta)} \int_0^x W(x-y) dH(y) = 0.
\] (8)

The solution for (8) is known (see for example Takač 1962 and Cohen 1969). To determine the solution we define

\[
F(w) = \begin{cases} \frac{\lambda}{\mu(\delta)} \int_0^w \frac{1-H(u)}{1-H(0)} du & \text{for } w > 0, \\ 0 & \text{for } w \leq 0. \end{cases}
\] (9)

Observing that \( \int_0^w [1-H(u)] du = \xi \), it is readily seen that \( F(\cdot) \) is a defective distribution function.

Define

\[
\tilde{W}(x) = \begin{cases} \int_0^x \left\{ \sum_{n=0}^\infty F^{(n)}(u) \right\} du & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases}
\] (10)

where \( F^{(n)} \) for \( n \geq 1 \) is the \( n \)-fold convolution of \( F \) and \( F^{(0)} \) represents a probability distribution degenerated at zero. The limiting distribution \( W(x) \) is given by

\[
W(x) = \begin{cases} 0 & \text{for } x < 0, \\ \tilde{W}(x) & \text{for } x \geq 0. \end{cases}
\] (11)

By applying a renewal argument (see Stidham 1972) the long run average profit can now be expressed as follows

\[
\Psi(S, L, K, R) = 2\xi\beta + r_2 L + \int_0^S r_1(S-u) dW(u) - \int_0^S R(u-S) dW(u) - \int_{K+S}^S C(u-(K+S)) dW(u).
\] (12)

Optimum investment-dividend policy is then found by maximizing (12) by an appropriate choice of \( S \) and \( L \). Of course a solution can be found by numerical means once the distribution function of the claim size \( H(\cdot) \) is specified.

3. Exponential Claim Size

In this section we examine the case in which the claim size is exponentially distributed with mean \( \xi = 1/\theta \). The distribution function \( H(\cdot) \) is given by

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\[ H(u) = \begin{cases} 0 & \text{for } u \leq 0, \\ 1 - e^{-\theta u} & \text{for } u > 0. \end{cases} \]

Using equations (9), (10) and (11) it can be verified that the limit distribution \( W(x) \) is given by
\[ W(x) = \frac{1 + (1/\delta) \left( 1 - \exp \left( -x \theta \delta / (1 + \delta) \right) \right)}{1 + (1/\delta)}, \quad (x \geq 0). \quad (13) \]

Note that the limiting probability of the event \( X(t) = K + S \) is given by \( \delta / (1 + \delta) \). Therefore, the average return from short term investments is given by
\[ \int_0^S r_i (S-u) dW(u) = \frac{r_i \delta S}{1 + \delta} + \int_0^S r_i (S-u) dW(u). \quad (14) \]

But since
\[ dW(u) = \left[ \delta \theta (1 + \delta)^2 \right] \exp \left( - \theta \delta u (1 + \delta) \right) du \quad (15) \]
using (15), the objective function of the firm can be expressed as follows
\[ \Psi(S, L|K, R) = \lambda \xi \delta + r_2 L + r_1 S - \frac{r_i}{\theta \delta} + e^{-\gamma S} \left[ r_1 - R - Ce^{-\gamma K} \right]. \quad (16) \]

where \( \gamma = \theta \delta / (1 + \delta) \).

To satisfy the consumption requirement of the shareholders, it is necessary to ensure an appropriate stream of dividends. Therefore, in our case, we shall assume that the total amount of investments (short term plus long term) is bounded from above by a certain quantity, say, \( A \). From the structure of the objective function (see (16)), it can be seen that under the optimal policy \( S^* + K^* = A \). Hence, the profit function can be expressed as a function of \( S \) by substituting \( L = A - S \) in (16) as follows
\[ \Psi(S, L|K, R) = \lambda \xi \delta + r_2 (A - (r_2 - r_1) S - \frac{r_i}{\theta \delta} + e^{-\gamma S} \left[ r_1 - R - Ce^{-\gamma K} \right]. \quad (17) \]

We now proceed to compute \( S^* \). Recalling that \( r_2 > r_1 \), it can be seen that if \( r_1 - R - Ce^{-\gamma K} > 0 \), then the profit function is monotonically decreasing in \( S \), and therefore, in this case \( S^* = 0 \) and \( L^* = A \), independently of the return rate, \( r_2 \), from long term investments. On the other hand, if \( r_1 - R - Ce^{-\gamma K} < 0 \), then the profit function is a concave function in \( S \). In this case, let \( S^* \) be the solution for the equation \( \partial \Psi / \partial S = 0 \). It can be seen that
\[ S^* = \frac{1}{\gamma} \ln \left[ \frac{R + Ce^{-\gamma K} - r_1}{(r_2 - r_1)(1 + \delta)} \right]. \quad (18) \]

The optimal value \( S^* \) is given by

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\[ S^* = \begin{cases} \tilde{S} & \text{if } \tilde{S} \leq A, \\ A & \text{if } \tilde{S} > A. \end{cases} \] (19)

We continue to examine the case in which \( r_1 - R - Ce^{-rK} < 0 \). First note that when \( r_2 = r_1 \), then \( S^* = A \) and \( L^* = 0 \). Further, \( S^* \) is monotonically nondecreasing as a function of \( R \) and \( C \). These results are of course both logical and expected. An additional consequence from (18) is that the effect of the borrowing cost \( C \) upon \( S^* \) decreases to zero when \( K \) approaches infinity, which is intuitively clear, since the expected borrowing cost per unit time decreases to zero in this case. Finally, in the ruin model, when \( C \to \infty \), the optimal policy is given by \( S^* = A \) and \( L^* = 0 \), which is clear.

These results depend of course on the claims size distribution being exponential. Below we shall use a diffusion approximation and obtain results that will be appropriate for arbitrary claim size distributions.

4. Diffusion Approximation

A simplification of the insurance problem for a general claim size distribution can be reached by using a diffusion approximation. To obtain our approximation we replace the stochastic process \( X \) by a suitable diffusion process, say, \( Z \) defined over the set \((-\infty, K+S)\). Such an approximation is reached as follows. Let \( \Delta_h(X(t)) \) be an increment in the \( X \) process accrued over the time interval \((t, t+h)\). Thus

\[ \Delta_h(X(t)) = X(t+h) - X(t), \]

and define the limits

\[ \alpha = \lim_{h \downarrow 0} \frac{1}{h} E[\Delta_h(X(t))|X(t)], \] (20)

and

\[ \beta = \lim_{h \downarrow 0} \frac{1}{h} \text{Var}[\Delta_h(X(t))|X(t)]. \] (21)

Using well known properties of the compound Poisson process it can be verified that

\[ \alpha = \lambda \xi \delta, \] (22)

and

\[ \beta = \lambda \int_0^\infty u^2 dH(u). \] (23)

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The diffusion approximation process $Z$ has a drift parameter $\alpha$ and a diffusion parameter $\beta$. Let

$$T_{x,y} = \inf \{ t \geq 0; Z(t) = y|Z(0) = x \}. \quad (24)$$

An approximate expected cycle time (see (2) and (3)) can be expressed as follows

$$E(T) = E[Q] + \int_0^\infty E[T_{K+S-u,u+S}] \, dH(u). \quad (25)$$

The theory of diffusion processes implies that

$$E[T_{x,y}] = \frac{y-x}{\alpha}, \quad \text{for} \ (x < y). \quad (26)$$

Further, recalling that claims arrive according to a Poisson process with rate $\lambda$, it can be seen that

$$E[Q] = 1/\lambda.$$

Hence

$$E(T) = 1/\lambda + \int_0^\infty \frac{u}{\alpha} \, dH(u) = 1/\lambda + \frac{\tilde{T}}{\alpha}. \quad (27)$$

Let

$$b(x) = E \left[ \int_0^T f(Z(u)) \, du | Z(0) = x \right], \quad \text{for} \ x < K+S, \quad (28)$$

where

$$\tilde{T} = \inf \{ t \geq 0; Z(t) = K+S \}. \quad (29)$$

Using well known properties of diffusion processes it can be seen that $b(x)$ satisfies the following differential equation (Kushner, 1971)

$$\alpha \frac{\partial b(x)}{\partial x} + \frac{1}{2} \beta \frac{\partial^2 b(x)}{\partial x^2} + f(x) = 0, \quad (30)$$

with the boundary condition

$$b(K+S) = 0.$$

Now define

$$B = \int_0^\infty b(K+S-u) \, dH(u). \quad (31)$$
Using equations (4), (27), and (31), an approximate long run average profit under an investment-dividend policy $(S, L)$ is given by

$$\Psi(S, L|K, R) = \lambda \xi \delta + r_1 L + \frac{B+(1/\lambda) r_1 S}{1/\lambda + \xi/\alpha}. \quad (32)$$

An explicit solution for (32) can be found however. Using a well known result of Kushner 1971 (see also Tapiero and Zuckerman, 1980a) a solution for $b$ in (30) and its insertion in (31) yields

$$B = \int_0^\infty \int_{K+S-x}^{K+S} \int_{-\infty}^v e^{-2\alpha \beta v(\zeta-z) f(z)} \, dv \, dz \, dH(x) \quad (33)$$

where $f(z)$ is given in equation (4). To compute the integral in (33), it is necessary to break the integration range to be consistent with the definition of $f(z)$ in (4). After cumbersome computations, we find that if $\mu=2\alpha/\beta$, then

$$B = \int_0^S \frac{R}{\mu} \left[ \frac{(S+1/\mu) x - \frac{x^2}{2} + \frac{\mu S(1-e^{\mu x})}{\mu^2}}{R+Kx} \right] \, dH(x) + \int_S^{K+S} \frac{R}{\mu} \left[ \frac{(K+1/\mu) e^{-\mu(S+x)} - 1 + \mu \left( \frac{Sx - x^2}{2} + x \right)}{R+Kx} \right] \, dH(x) + \int_{K+S}^{\infty} \frac{R}{\mu} \left[ \frac{\mu(K+S) x - \frac{x^2}{2} - x}{R+Kx} \right] \, dH(x). \quad (34)$$

Inserting (34) into (32) and assuming that the claim size distribution $dH(x)$ is given, an analytical expression for $\Psi$ is given. Optimal $S$ is given by solving $\partial \Psi/\partial S = 0$ which can be easily found from (34). Similarly, optimization with respect to $\delta$, and other parameters the firm is in control of, can be conducted.

If the regulating agency "knows" the function (32) it can be the leader in the game (see Stackleberg 1934) between the regulating agency and the insurance firm.

4. The Regulating Agency's Problem

In this section, we consider the regulating agency's problem. We shall restrict attention to the problem of determining an optimal cash barrier policy $K$ imposed on firms for a given penalty cost rate $R$. The purpose of the agency is two-fold. On the one hand, it seeks a level $K$ insuring that firms hold enough reserves to satisfy outstanding claims (as expressed by the requirement that firm's reserves remain always above zero). On the other hand, the agency imputes a cost proportional to $K$. Such cost stands for the political and administrative cost incurred by the agency enforcing the requirement level $K$. Although a firm cannot theoretically fail to satisfy
claims (because of the borrowing feature and conversion mechanism), there are various complications, such as delays in payments, decreased insured security, etc., which may be thought as a "social cost". Thus, we associate to firm's temporary insolvency a social cost $M(x)$, when the firm's reserves decrease to $x < 0$. If $h$ is the imputed cost rate associated to the requirement level $K$, we note that the agency's cost function is given by

$$\eta(K) = hK + \int_0^\infty M(K-x) \, dW(x),$$  \hspace{1cm} (35)

where $W(x)$ is the probability measure defined earlier for the case in which $S = 0$, i.e., the firm transfers all incoming premiums above the level $K$ to a long-term investments.

If $h = 0$, the agency's solution would be to make $K$ as large as possible, $(K = \infty)$, and by implication force firms to accumulate funds infinitely. However, for $h > 0$, large $K$, leads also to large cost. Thus, an optimum requirement level $K$ will be a function of the social costs of firm's failure and the cost parameter $h$. In particular, for exponential claim size with parameter $\theta$ we obtain (see (13))

$$\eta(K) = hK + \left( K + \frac{1}{\theta \bar{\delta}} \right) \, m \exp \left( \frac{-K\theta}{1+\bar{\delta}} \right),$$  \hspace{1cm} (36)

where the social cost function is given by $M(x) = -mx$ for $m > 0$. Using (36) an optimal $K^*$ can be obtained.

References


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