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Optimal insurance coverage in situations of pure and speculative risk and the risk-free asset *

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The insured's portfolio consists of an insurable (pure) risk, an uninsurable (speculative) risk, a (proportional) insurance policy and a risk-free asset. The optimal insurance policy (i.e., the proportion to be insured) is examined from the insured's point of view, using the reward to variability concept. The importance of the risk-free asset in reaching an exact and explicit solution is analyzed, while emphasizing the possibility of substitution of the risk-free investment and insurance mechanisms. The paper demonstrates possibilities of improving the insured's welfare by the use of the risk-free rate – which is sometimes less expensive than other risk reduction instruments. The analysis leads to a two-step solution, similar to the well-known Hirschleifer investment model and to the famous Capital Assets Pricing Model.

Keywords: Correlation, Risk-free rate, Portfolio, Optimal insurance, Risk loading, Reward to variability, Proportional insurance, Capital assets pricing model.

1. Introduction

The problem of optimal insurance policies has been discussed in quite a number of recent articles. Most have viewed the insurance policy as a contract between two parties, and used expected utility functional analysis to select the optimal form of the contract [Arrow (1965), Adar and Neumann (1978), Doherty and Schlesinger (1983), Raviv (1979), Smith (1968)]. An alternative approach has been to use efficiency criteria [Doherty (1980,1985) and Kroll (1985)] which only implicitly assume a certain admissible group of utility functions. In this paper the second approach is employed. It is assumed that the insured can incorporate in his portfolio a risk-free asset and thus can use the reward to variability criterion in evaluating risky portfolios. Although this measure suffers from the well-known limitations of the mean variance criterion, it is used in the present analysis since it provides a good approximation for decision makers even when the utility functions and the distribution functions do not fulfill all the necessary assumptions [see Levy and Markowitz (1979), Kroll, Levy and Markowitz (1985)].

The insured's decision is examined in the context of a portfolio of risks, which includes an insurable (pure) risk, an uninsurable (speculative) risk, insurance policy, and a risk-free asset. Previous studies of the optimal insurance problem have not emphasized the impact of the risk-free asset on the insured's optimal strategy. It will be shown that by incorporating the risk-free asset into the analysis, considerable insight may be gained; as has been the case when it was incorporated into financial models (e.g., Hirschleifer investment model and Sharpe–Lintner–Mossin Capital Assets Pricing Model).

Similar to recent papers [Buser and Smith (1984), Doherty and Schlesinger (1983), Mayers and Smith (1983)] it is demonstrated that the insurance transaction should not be considered in isolation. We show that the amount of insurance is related to other exposures of the firm, requiring the insurance decision to be made as part of the management of all the risks of the firm. From the insured's point of view, insurance instruments compete with risk-free investments as well as other investments in the portfolio as a means of risk reduction. It is shown that the risk-free asset plays an important role in the analysis of the optimal insurance coverage.

By definition, the return on a risk-free asset is not correlated with returns from risky investments.

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Thus, the decision on the amount of riskless asset in the portfolio can be separated from the decision on the optimal proportions between the risky alternatives. This separation theorem enables the specification of a unique optimal risky portfolio under the Capital Assets Pricing Model (CAPM). Similarly the inclusion of the risk-free asset in our insurance portfolio problem enables us to reach an explicit and exact solution to the optimization problem of determining the amount of insurance that should be purchased. Note that an exact solution cannot be obtained through the stochastic dominance analysis even if a risk-free asset is included in the portfolio. Thus, Kroll (1983), who used the stochastic dominance approach rather than the mean–variance approach, defines only insurability regions.

The importance of the risk-free asset stems from its ability to serve as an instrument for controlling the overall level of risk in portfolios. It may, thus, substitute other risk reduction mechanisms such as insurance. The substitution relationship is an important element in determining the optimal policy in insurance related problems; it has been shown to be an important factor in determining the insurer's financial policy and ratemaking strategies [Kahane (1979)], and in the present article it is shown to be an important element in designing the insured's optimal insurance coverage.

In our analysis we assume that there is an insurable risk and that the insured has to decide what proportion of risk he would like to insure. We focus on the case of property insurance coverages where the loss typically cannot exceed the value of the property. The analysis, however, is extended to deal with the possibility of the insured buying more insurance to cover possible business interruption losses (when the property damage leads to an additional loss of income).

The inclusion of a risk-free asset in the permissible portfolio expands some of the conclusions of previous studies, which extensively analyzed the relationship between optimal coverage to parameters such as the loading factors, the risk levels and the correlation between insurable and uninsurable risks.

Since the main focus of the paper lies with the effects of the risk-free asset, we prefer to concentrate on the simple case of proportional insurance coverage and to avoid the analysis of the question of the existence of an optimal form of insurance contracts, which has been dealt with extensively in the insurance literature.

In the second section, the model is presented and is supported by a graphical exposition that demonstrates its main features. Section 3 analyzes the effects of the parameters on the optimal solution and examines the special cases of non-correlation and negative correlation between the insurable and the uninsurable risks, the latter representing situations of moral hazard and business interruption. The analysis is supported by several numerical examples in Section 4. Some concluding remarks are presented in Section 5.

2. The model

Let Z be the end of period return on one dollar which is invested in an asset.[1] Let us further assume that the return on this asset is composed of two elements, X, the rate of return on the asset from regular activities (uninsurable risk) and Y the possible rate of losses due to damage to the asset itself (insurable risk). Accordingly, the end of period return can be expressed as follows:

$$Z = 1 + X - Y, \quad 0 \le Y \le 1 + X. \tag{1}$$

In the insurance literature terminology X is the 'speculative' risk, and Y represents the 'pure' risk. In a case of total loss, $Y = 1$, and in a case that no loss occurs, $Y = 0$.[2] The case of $Y > 1 + X$ results in bankruptcy. Since we assumed liability, Y is constrained by $1 + X$.

Let us assume that it is possible to buy proportional insurance coverage for the insurable risk, Y. The decision variable – the proportion of Y which is insured – is denoted by p $(0 \le p \le 1)$. The coverage is purchased for a premium rate $S(p)$ which is stated in terms of percentage of the investment value. The premium rate is assumed to cover the expected losses $E(Y)$ plus a loading. For the sake of simplicity we assume that the insurer charges a certain fixed loading, b, and a propor-

[1] This asset may be a portfolio of individual risks.

[2] It is assumed that Y is a stochastic variable with a continuous or non-continuous density function. In practice, it is possible that the extreme points ($Y = 0$ and $Y = 1$) are points of discontinuity since they typically have a non-zero probability mass.

tional loading factor, c.

$$S(p) = b + c \cdot p \cdot \mathrm{E}(Y), \quad b \geq 0,\ c \geq 1. \tag{2}$$

The rate of return on the portfolio consisting of the asset and the insurance policy is Z,

$$Z = 1 + X - Y + pY - S(p). \tag{3}$$

At this point we deviate from previous models by assuming that the decision maker may use a risk-free asset to diversify his portfolio. This asset has a constant (risk-free) rate of return, R. No short positions on the risky assets are allowed. If the individual invests a proportion q of his wealth in the risky assets (including insurance), and a proportion $(1-q)$ in the risk-free asset, his overall final return Zq will be

$$Z_q = q \cdot Z + (1-q) \cdot (1+R). \tag{4}$$

Inserting (3) into (4) we get

$$Z_q = q[1 + X - Y + pY - S(p)] + (1-q)(1+R). \tag{5}$$

The expected return on the overall portfolio is given in (6)

$$\mathrm{E}(Z_q) = q[1 + \mathrm{E}(X) - (1-p)\mathrm{E}(Y) - S(p)] + (1-q)(1+R). \tag{6}$$

The standard deviation is

$$\sigma_{Zq} = q \cdot \left\{ \sigma_x^2 + (1-p)^2 \sigma_y^2 - 2(1-p) \operatorname{cov}(XY) \right\}^{1/2}. \tag{7}$$

The reward to variability ratio provides the optimal criterion for a mean–variance investor who can diversify his risky assets with a riskless one. Thus each risky venture can be evaluated by this measure. [3]

In our case the reward to variability, denoted by M, equals

$$M = \left[\mathrm{E}(Z_q) - (1+R)\right] / \sigma_{zq}. \tag{8}$$

The insured wishes to maximize M. Inserting (2), (6) and (7) into (8) and rearranging leads to

$$M = \left[1 + \mathrm{E}(X) - \mathrm{E}(Y) - p(c-1)\mathrm{E}(Y) - b - (1+R)\right] / \left[\sigma_x^2 + (1-p)^2 \sigma_y^2 - 2(1-p) \operatorname{cov}(XY)\right]^{1/2}. \tag{9}$$

Note that this slope M is independent of the proportion of the risky investment q. This result is the parallel to 'separation theorem' which is the corner stone of the CAPM in finance theory.

The problem is to find the optimal proportion of the insurance coverage, p^*, which maximizes the reward to variability measure [equation (9)]. From the first-order condition it may be found that the optimal value of p is

$$p^* = 1 - \left\{ \sigma_x^2 \mathrm{E}(Y)(c-1) + \operatorname{cov}(XY)[\mathrm{E}(X) - c\mathrm{E}(Y) - b - R] \right\} / \left\{ \sigma_y^2 [\mathrm{E}(X) - c\mathrm{E}(Y) - b - R] + \operatorname{cov}(XY)\mathrm{E}(Y)(c-1) \right\}. \tag{10}$$

The development of equation (10) can be found in the appendix.

A graphical presentation of the optimization problem for the simplified case where X and Y are uncorrelated is quite instrumental for both the analysis and understanding of the solution. Figure 1 presents the mean–standard deviation plane. Point Z_0 reflects the expected return and standard deviation on the uninsured project (the combination of pure and speculative risks). Point Z_1 repre-

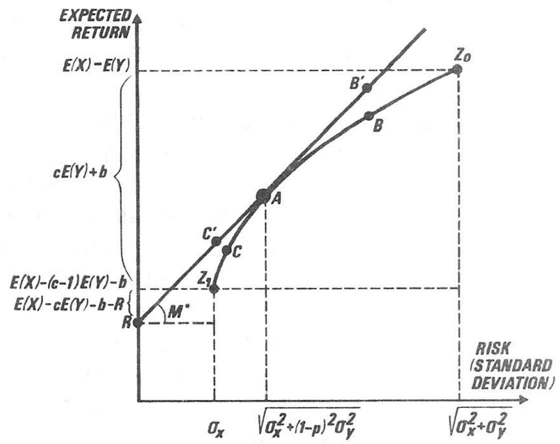

Fig. 1. *Note*: The standard deviations are presented for the case $\gamma = 0$.

[3] It has been recently shown that this measure provides a good approximation for decision makers [see Kroll, Levy and Markowitz (1985), and Levy and Markowitz (1979)]. Moreover, this concept has recently been used in the insurance context by another researcher [Buser and Smith (1984), Doherty (1980)].

sents the expected return and standard deviation on the portfolio which includes full insurance coverage. If the risk-free option is unavailable, the insured may determine the risk–return combination by holding partial insurance coverage. All the possible risk–return combinations obtained by partial insurance are on the locus Z_0Z_1.

Adding the possibility of lending and borrowing at a risk-free rate, R (which would typically be lower than the expected value of Z_1) alters the set of the insured's choices: he may now choose a portfolio of a risk-free project plus one of the combinations on the risky locus Z_0Z_1. According to the reward to variability criterion, the best choice would be a combination of the risk-free project and a risky combination, A. The exact composition of portfolio A, i.e., the amount of insurance to be purchased, is determined by the tangency point between the ray leaving R and the curve Z_0Z_1 (the reward to variability M^* is the slope of this ray), and its explicit solution is expressed by equation (10).

Similar to the 'separation' idea in the well-known CAPM and in Hirschleifer's investment model, the solution to the optimal insurance problem can be divided into two stages: During the first stage the optimal coinsurance proportion, p^*, is selected (point A). The optimal level of coinsurance is uniform for all insureds, regardless of their tastes. During the second step, lending or borrowing at the risk-free rate takes place – until the insured maximizes his specific utility. An insured who, in the absence of the risk-free rate, would prefer to have only little insurance protection (e.g., would choose point B in the region AZ_0) would now probably prefer to be at a point like B' on the ray RA. Portfolio B' is attained by buying more insurance than would be purchased in the absence of the risk-free option – thereby reaching point A – and then discarding the excessive insurance protection by assuming a higher level of financial risk (borrowing at the risk-free rate). The two-step operation is profitable since the investor buys extra insurance coverage at a low insurance premium, and then divests himself of the excessive protection by using the financial market mechanisms which are sold at a lower marginal cost (the risk-free rate).

Another investor may, in the absence of a risk-free rate, prefer to hold a portfolio C in the region Z_1A (i.e., he prefers being heavily protected). He, too, may benefit from the perfect capital market; instead of buying as much insurance as he originally intended, he may add less insurance to his uninsured portfolio Z_0, and reach point A. At this point he is still in a too risky situation, and may decrease his excessive risk by lending (investing) money at the risk-free rate until reaching point C'. Point C' dominates the original portfolio C. The superior performance has been achieved by buying only a limited amount of expensive commercial insurance (point A) and then getting the desired risk reduction effect at a lower cost through the market risk-free investment.

3. Analysis of the optimal solution

The optimal solution, which is uniform for all investors, is expressed in equation (10), which has several interesting implications.

3.1. The effects of R and b

The effects of the risk-free investment on the optimal insurance decision have not been studied in the earlier models (which ignored this parameter). From equation (10) we can learn the interesting point that the risk-free rate R and the constant loading factor b, have a similar impact on the optimal insurance proportion p^*. Earlier studies have shown that a higher loading factor generates a disincentive for purchasing insurance. We obtain similar results in our model for both R and b; by differentiating p^* in (10) with respect to the loading factor b or with respect to the risk-free rate, R, we find that the higher these parameters, the smaller the proportion p^*. This result is quite clear, since the higher the fixed loading factor is, and the better the risk-free opportunities are in the market, it is more advantageous to reduce risk by holding more of the risk-free assets in the portfolio than by using the insurance mechanism. In other words, there is a substitution effect between investment in the risk-free asset and the insurance policy.

3.2. The effect of ρ_{xy}

The speculative (uninsurable) risk, X, and the pure (insurable) risk, Y, may be correlated. Consider first the effect of the covariance on p^* for the special and simple case where $c=1$, i.e., when the insurer charges only the expected loss plus a fixed loading, b. In this case equation (10) is replaced by equation (11).

$$p^* = 1 - \frac{\operatorname{cov}(X, Y)}{\sigma_y^2} = 1 - \frac{\sigma_x}{\sigma_y} \rho_{xy}. \tag{11}$$

From (11) we learn that the higher the covariance between x and y, the smaller the optimal proportion p^*. Also note that for $\rho_{xy} < 0$ the amount of the optimal insurance is at $p^* > 1$.[4]

For a loading factor $c > 1$, the effects of the covariance between X and Y on the optimal coverage p^* may be studied by differentiating (10) with respect to the covariance. It may be found that $\partial p^* / \partial \operatorname{cov}(X, Y) > 0$ if

$$\frac{\mathrm{E}(X) - c\mathrm{E}(Y) - b - R}{(c-1)\mathrm{E}(Y)} < \frac{\sigma_x}{\sigma_y}. \tag{12}$$

In other words, the covariance effect is not uniform, and depends on the direction of the inequality sign in (12). The numerator in (12) is the expected rate of the speculative return on the asset, less the deduction of the full insurance premium $c\mathrm{E}(Y) + b$, and the risk-free rate. Thus, $\mathrm{E}(X) - c\mathrm{E}(Y) - b - R$ can be viewed as the 'risk premium' for the speculative risk. The denominator, $(c-1)\mathrm{E}(Y)$, is equal to the portion of the loading factor which is affected by the expected loss $\mathrm{E}(Y)$. Thus, the ratio on the left-hand side of (12) is simply the premium on the speculative risk over the marginal loading factor. Inequality (12) tells us that the optimal proportion of insurance p^* is an increasing function of the covariance when the ratio of speculative risk to pure risk is higher than the premium on speculative risk over marginal cost loading.

From the points of view of the economist and the insurance experts there are three interesting relationships between the speculative and the pure risks.

(a) *X and Y are negatively correlated* ($\rho_{xy} < 0$). Such a situation is of special practical interest since it may have two important interpretations in the insurance context:

(1) Moral hazard: In periods of low returns on the investment there is a higher chance for insurable damage.

(2) Business interruption: Damage to the property causes an additional loss of profit in the form of loss of income or incremental costs.

When $\rho_{xy} < 0$ the optimal insurance coverage, p^*, resulting from equation (10) may often be greater than 1 because the insured is willing to purchase coverage for an amount larger than the value of the property in order to compensate for the consequential losses.

(b) *X and Y are independent* ($\rho_{xy} = 0$). In this case the damaged property may be replaced instantaneously without causing any further loss of income, and only the insurance against physical damage to the property is considered. Under this assumption the optimal coverage is

$$p^* = 1 - \left(\frac{\sigma_x}{\sigma_y}\right)^2 \frac{\mathrm{E}(Y)(c-1)}{\mathrm{E}(x) - b - R - c\mathrm{E}(Y)}. \tag{13}$$

The optimal coverage is partial ($p^* < 1$), if

$$\mathrm{E}(X) - R \geq c\mathrm{E}(Y) + b. \tag{14}$$

Condition (14) can be easily explained: the left-hand side represents the excess expected return from the asset, above the risk-free rate. The right-hand side represents the full insurance premium. It is necessary that the risk premium that the insured earns on his speculative activities (about the risk-free rate) exceeds the premium he is required to pay for the insurance coverage. This is most probably the case in reality.

(c) *X and Y are positively correlated* ($\rho_{xy} > 0$). This can happen, for example, if high profitability induces more crime against the insured property (theft, sabotage, vandalism, etc.). Under the natural condition $\mathrm{E}(X) - c\mathrm{E}(Y) - b - R > 0$ partial coverage ($p^* < 1$) is the optimal strategy. This result is explained by means of a graphical exposition (see Figure 2). The curve describing the combination of uninsured project plus insurance does not remain in the same location as in Figure 1: the point Z_0 is located at a different location, Z_0', and the new frontier has a kink. The tangency between the ray leaving R and the frontier $Z_0' Z_1$ cannot be at Z_1 due to this kink, and therefore the optimal solution is always one with partial insurance

[4] The desire to buy more than full coverage regardless of existence of the positive constant loading looks peculiar. However, one should bear in mind that equation (11) is constrained to the case of $c=1$. In all other cases also $\mathrm{E}(Y)$, $\mathrm{E}(X)$, b and R are important elements in determining p^* see equation (10).

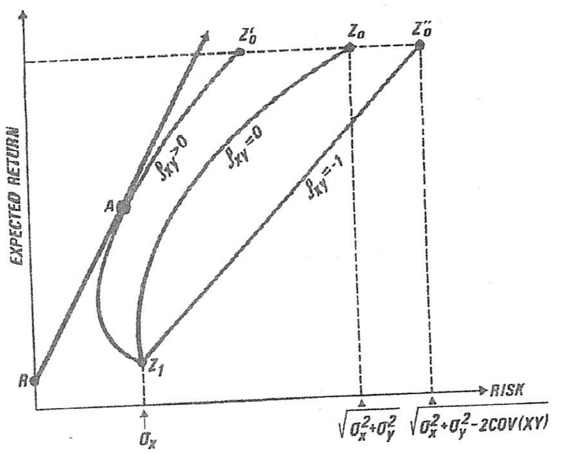

Fig. 2.

coverage. It is noteworthy that unlike the usual case in the Capital Assets Pricing Model, when the risks are uncorrelated ($\text{cov}(X, Y)=0$) there is no kink in the frontier Z_0Z_1, and the lowest standard deviation would be at Z_1.

3.3. The effect of the expected insurable loss

By differentiating (10), we find that p^* is decreasing with respect to the expected loss, $E(Y)$, (for $c>1$).[5] Such a relationship between the expected loss, the loading factor and the optimal insurance coverage has already been discovered in much earlier studies [see e.g., Arrow (1965), Smith (1968)].

Rewriting (13) we obtain the optimal proportion for the special case where $\rho_{xy}=0$,

$$p^* = 1 - \left(\frac{\sigma_x}{\sigma_y}\right)^2 \cdot \frac{c-1}{\dfrac{E(X)-R-b}{E(Y)} - c}. \tag{15}$$

It is easy to see that $\partial P^*/\partial E(Y) < 0$ (for $c > 1$) when $E(X) > b + R$, i.e., as long as the expected profits from the activities are higher than the fixed charges (risk-free interest plus the fixed cost involved in insurance).[6] This is a reasonable assumption in the practice, since the firm will not buy insurance when it is unable to cover the fixed costs.

3.4. The effect of the expected speculative profit, $E(X)$

The expected speculative profit, $E(X)$, has an opposite effect than that of the pure risk $E(Y)$. Namely, as long as $c > 1$, then the optimal proportion of insurance, p^*, is higher when the expected return on the speculative risk is higher. This result can be interpreted as the combined result of the 'income effect' and the 'substitution effect'. when the investment project is more profitable, the insured is willing to buy less insurance coverage – provided that insurance is an inferior good.[7] To some people this result may seem to be counterintuitive. However, in the portfolio context we can understand this phenomenon as follows: the reward to variability with respect to the speculative risk can be expressed as $(E(X)-cE(Y)-b-R)/\sigma_x$ [see the explanation for inequality (12)]. The higher the expected value of the speculative risk $E(X)$, the higher this ratio. It is well known in finance that in such a case the optimal proportion of the risk-free asset in the portfolio may decrease. In our case, the higher $E(X)$ is, the risk-free asset held in the portfolio decreases and the risk reduction is obtained by increasing the amount of insurance p^*, i.e., we face a substitution effect between the risk-free asset and the insurance policy. This substitution effect, in our case, offsets the above mentioned income effect.

3.5. The loading factor c

The amount of insurance purchased decreases with the loading factor c. Since this is not a new result, we shall demonstrate this relationship only for the special case of $\rho_{xy}=0$. The partial derivative of (13) gives

$$\frac{\partial p^*}{\partial c} = -\frac{\sigma^2 x}{\sigma_y^2} \cdot \frac{E(Y)\cdot[E(X)-E(Y)-b-R]}{[E(X)-cE(Y)-b-R]^2}. \tag{16}$$

[5] This claim is correct under the additional, and natural, assumption that $E(X) > B + R$.

[6] Note that if (14) holds then $E(X) > b + r$. However, the converse does not hold. Thus (14) is sufficient for $p^* < 1$ and also for $\partial p^*/\partial E(Y) < 0$.

[7] If we assume Decreasing absolute Risk Aversion (DARA) then insurance is an inferior good, and the highest the income, the less insurance is purchased.

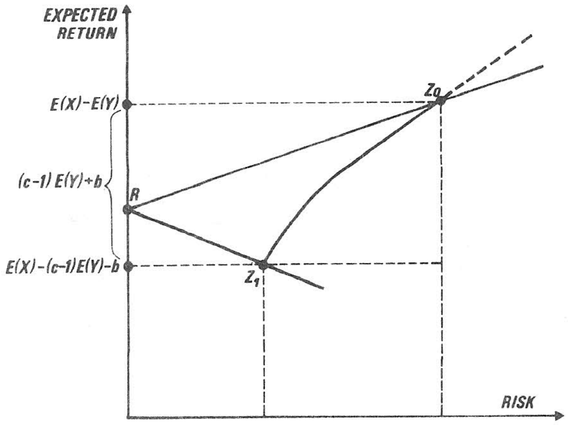

Fig. 3.

The derivative in (16) is negative under the realistic assumption that $c > 1$ and $E(x) - R > cE(y) + b$. Therefore, in general a higher loading factor, c, will lead the insured to cover a smaller proportion of the exposed asset's value.

Figure 3 depicts the different assumption of $E(X) - cE(Y) - b - R < 0$. In this case the insurance premium exceeds the excess return on the uninsured project. Thus, the optimal proportion of insurance would be at point Z_0 where no insurance is purchased. Actually, the firm would be interested in reaching a negative proportion; i.e., would like to sell insurance. Note that the situation of full insurance would represent a negative return to variability. (See the ray RZ_1 in Figure 3.)

3.6. The effects of σ_x, σ_y

The optimal insurance coverage is typically a decreasing function of σ_x, and an increasing function of σ_y. [8] The willingness to buy insurance is lower when, and as long as, the variability ('risk') of the speculative project is relatively high, and when the insurable risk is relatively less noticeable. In other words, the optimal insurance p^* is a decreasing function of the ratio σ_x/σ_y.

The special case where $\sigma_x = 0$

When the speculative risk does not exist, (i.e., when the return on the investment is known with certainty), the asset is exposed only to the insurable risk. The reward to variability in this interesting case is

$$M = \frac{E(X) - E(Y) + p(1-c)E(Y) - b - R}{(1-p)\sigma_y} \quad (17)$$

and

$$\frac{\partial M}{\partial p} = \frac{1}{(1-p)\sigma_y^2}\left[E(X) - cE(Y) - b - R\right]. \quad (18)$$

The reward to variability increases with p and reaches a maximum at $p = 1$, as long as the condition $E(X) - R > b + cE(Y)$ is met. We thus have a corner solution with full insurance coverage. On the other hand, if our assumption is not met, and $E(X) - R < b + cE(Y)$, the reward to variability will be a decreasing function of p, and will reach its maximum at $p = 0$ when no insurance is bought. The latter case is quite reasonable since in a competitive capital market a risk-free asset will have an expected return of $E(X) = R$ and the only result of purchasing insurance would be a decrease in expected return (i.e., insurance premium $= b + cE(Y) > 0 = E(X) - R$) without affecting the variability. Only when the insurance premium would be negative (i.e., $E(X) - R = 0 > b + cE(Y)$), would insurance be worthwhile from the insured's point of view.

4. Numerical examples

The solution of the optimization problem is quite sensitive to its parameters. A slight change in the risk-free rate, the correlation coefficient or the standard deviation may cause a significant change in the optimal insurance coverage (p^*). This is shown with simple numerical examples. Assume

$E(X) = 0.25$	expected return on the project,
$E(Y) = 0.04$	expected (insurable) loss, [9]
$\sigma_y = 1$	the insurable risk,
$b = 0.01$	constant insurance loading [10] (1% of the value of the property),
$c = 1.2$	proportional loading factor (20% above the expected loss).

The values in Table 1 state the proportions of the

[8] This conclusion is made under the *ceteris paribus* assumption and especially that the covariance remains constant (independent of σ_x and σ_y).

[9] The expected loss in this example tends to be higher than in most areas of insurance in practice.

[10] The assumed constant loading factor is higher than the common loading in practice.

Table 1
The optimal proportion of insurance. [a]

Line	Risk-free rate R	Correlation coefficient ρ_{xy}	σ_x/σ_y 0.25	0.5	1.0	2.0	3.0	4.0
1	0.05	0	0.99	0.99	0.94	0.78	0.49	0.10
2	0.05	−0.05	1.01	1.01	0.99	0.85	0.64	0.29
3	0.05	−0.1	1.02	1.03	1.04	0.97	0.79	0.49
4	0.05	−0.5	1.12	1.24	1.46	1.82	2.09	2.24
5	0.10	−0.5	1.12	1.23	1.43	1.71	1.82	1.73
6	0.15	−0.5	1.11	1.21	1.34	1.29	0.70	−0.69

[a] In this table $E(X) = 0.25$ and $E(Y) = 0.01$.

property value that should be insured p^* from equation (10). The analysis demonstrates the sensitivity to the following parameters:

σ_x = the risk of the project (speculative risk),
R = the risk-free rate,
ρ_{xy} = the correlation coefficient between the insurable and uninsurable risks.

A comparison of the figures within each of the first two lines in Table 1 shows that when the correlation is close to zero and the uninsured risk increases in relation to the insured risk, less insurance will be purchased. However, when the correlation coefficient is significantly negative, the above conclusion, *ceteris paribus*, may be reversed for a wide range of σ_x/σ_y values. The risk of business interruption may become so great that the insured would be interested in purchasing much more insurance (for proportion exceeding 1– see the fourth line).

Although a higher risk-free interest rate, *ceteris paribus*, may reduce the willingness to buy insurance (compare the fourth and fifth lines), a further increase in the risk-free rate may suddenly turn the insurance mechanism into an unattractive risk reduction mechanism. At that point the individual may elect to be an insurance seller rather than an insurance buyer (compare the fifth and sixth lines).

5. Summary and concluding remarks

In this paper we have analyzed the insurance buying decision as part of the portfolio optimization problem; where the insurance mechanism serves as a means of reducing the risk of the entire portfolio. Although earlier papers have already addressed this issue, they ignored the possible risk reduction which may be achieved by the inclusion of a risk-free asset in the portfolio. The incorporation of this option in our model draws attention to the impact the risk-free asset has on the optimal insurance coverage. We were able to show interesting relationships – which may be regarded as 'income' and 'substitution' effects – between the risk-free asset and the insurance mechanism, when used as a means of controlling the portfolio's risk.

The importance of the risk-free asset concept, and its ability to lead to clear and exact solutions is not surprising, however. Early studies of other financial problems (and especially the seminal works of Hirschleifer, Tobin, and the Capital Assets Pricing Model) have already demonstrated the potential of the risk-free asset concept in solving problems of choice for optimal risk–return combinations. It is only surprising to observe that until now such a straightforward step has not been taken in the insurance literature.

Since the major thrust of this paper was the implications of the risk-free rate concept, we did not attempt to examine the broader problems of the optimal form of the insurance contract, or the optimal deductible policy [which were analyzed by Arrow (1965), Raviv (1979) and others]. We constrained the analysis to proportional insurance arrangements, and to the selection of the optimal proportion of the assets value that should be insured.

The paper re-establishes old principles, for example, that a risk-averse insured will prefer to be fully insured when faced with an actuarially fair insurance (no loading above the expected loss), but otherwise would prefer to share the risk.

The analysis led, however, to new findings concerning the substitution effect between the risk-free investment and insurance mechanisms, and the effects of other parameters such as the correlation between the insurable (pure) risk and the uninsurable (speculative) risk.

Appendix: The development of equation (10) in the text

Equation (9) in the text is

$$M=\frac{\mathrm{E}(X)-\mathrm{E}(Y)-p(c-1)\cdot\mathrm{E}(Y)-b-R}{\left(\sigma_x^2+(1-p)^2\sigma_y^2-2(1-p)\sigma_x\sigma_y\rho_{xy}\right)^{1/2}} \tag{A.1}$$

By differentiating M with respect to p and equating to zero we find that the first order conditions for extremum of M is

$$\begin{aligned}&(c-1)\mathrm{E}(Y)\left[\sigma_x^2+(1-p^*)^2\sigma_y^2-2(1-p^*)\sigma_x\sigma_y\rho_{xy}\right]\\&\quad=\left[\mathrm{E}(X)-\mathrm{E}(Y)-p^*(c-1)\mathrm{E}(Y)-b-R\right]\cdot\left[(1-p^*)\sigma_y^2-\sigma_x\sigma_y\rho_{xy}\right].\end{aligned} \tag{A.2}$$

Through a quite tedious manipulation of terms in (A.2) we find that p^* can be written as follows:

$$p^*=\frac{(\mathrm{E}(X)-b-R)\left(\sigma_y^2-\sigma_x\sigma_y\rho_{xy}\right)-(c-1)\mathrm{E}(Y)\sigma_x^2-c\mathrm{E}(Y)\sigma_y^2+2c\mathrm{E}(Y)\sigma_x\sigma_y\rho_{xy}-\mathrm{E}(Y)\sigma_x\sigma_y\rho_{xy}}{\mathrm{E}(Y)\sigma_x\sigma_y\rho_{xy}-\left(\mathrm{E}(Y)\sigma_y^2-\mathrm{E}(Y)\sigma_x\sigma_y\rho_{xy}+\sigma_y^2(\mathrm{E}(X)-b-R\right)}. \tag{A.3}$$

With additional manipulations we obtain

$$p^*=1-\frac{\sigma_x^2\mathrm{E}(Y)(c-1)+\operatorname{cov}(XY)A}{\sigma_y^2A+\operatorname{cov}(XY)\mathrm{E}(Y)(c-1)}, \tag{A.4}$$

where $A=\mathrm{E}(X)-c\mathrm{E}(Y)-b-R$.

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