Critical exponents for percolation conductivity in resistor networks*

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The conductivity of two-dimensional and of three-dimensional cubic binary random resistor networks is shown to obey a power-law dependence on the conductivity ratio at the percolation threshold. The relation recently derived by Straley between the exponent of this power law and the other two critical exponents of the conductivity above and below the percolation threshold is accurately obeyed. Extension of the scaling laws for a complex dielectric function of a binary network is provided.

The macroscopic conductivity of a random binary medium of local conductivity values σ_0 and σ_1 in the case where $\sigma_1 \ll \sigma_0$ is characterized by the following power law[1,2] above the percolation threshold.

$$\sigma \propto \sigma_0 (C - C^*)^p \ , \tag{1}$$

where C is the volume fraction of the high-conductivity component and C^* is the percolation threshold. This behavior was first established by Kirkpatrick[1] for simple-cubic resistor networks undergoing bond percolation and site percolation transitions and, more recently, by the present authors[2] for correlated resistor networks in which a continuous percolation occurs. The conductivity of a binary medium for which $\sigma_1 \ll \sigma_0$ rises steeply as C approaches C^* from below. We have found[2] that below the percolation threshold the conductivity of uncorrelated and of correlated cubic resistor networks follows the power law

$$\sigma \propto \sigma_1 (C^* - C)^{-q} \ . \tag{2}$$

The power laws, Eqs. (1) and (2) in the limit $\sigma_1 \ll \sigma_0$, also hold for the effective-medium theory (EMT)[1,3] throughout the entire range of C, and for Bethe-lattice networks[4,5] in the percolation transition region. The exponents p and q for various systems, together with the corresponding values of C^*, are summarized in Table I. Relying on the analogy between critical phenomena and percolation processes[6] Straley[5] has recently proposed that the conductivity of a binary random medium near the percolation threshold can be expressed as a generalized homogeneous function of $\epsilon = C - C^*$, σ_1, and σ_0.[7] The power laws, Eqs. (1) and (2), can be deduced from this assumption in the limit $\sigma_1/\sigma_0 \to 0$ although no information on the values of q and p can be obtained.[5] Straley[5] also deduces a power law dependence of σ on σ_0 and σ_1 at $C = C^*$, which can be recast as a power law in terms of the conductivity ratio σ_1/σ_0

$$\sigma \propto \sigma_0 x^r \ , \tag{3}$$

where r is related to q and p by

$$r = p/(q+p) \ . \tag{4}$$

In this paper, we shall present an alternative derivation of the power laws (1)–(3) and of the scaling relation[5] (4) and test these relations by confronting them with the results of numerical simulations of the conductivity in two-dimensional and in three-dimensional cubic resistor networks.

We shall first present an alternative derivation of Eqs. (1)–(4) based on a scaling assumption which differs somewhat from the scaling assumption used by Straley.[5] We introduce the following scaling assumption concerning the dependence of σ on ϵ, σ_0, and σ_1

$$\frac{\sigma(\epsilon, x)}{\sigma_0} = \lambda \psi(|\epsilon| \lambda^{-\nu}, x\lambda^{-\mu}), \quad \epsilon > 0 \tag{5a}$$

$$\frac{\sigma(\epsilon, x)}{\sigma_0} = \lambda \psi(|\epsilon| \lambda^{\eta}, x\lambda^{-\theta}), \quad \epsilon < 0 \tag{5b}$$

where ν, μ, η, and θ are scaling exponents. Note that if both σ_0 and σ_1 are scaled by the same numerical factor, the conductivity σ, given by Eq. (5), is multiplied by the same numerical factor, since the right-hand side of this equation is invariant under this scaling transformation.

In the limit $x \to 0$, by choosing $\lambda = |\epsilon|^{1/\nu}$ for $\epsilon > 0$ and $\lambda = |\epsilon|^{-1/\eta}$ for $\epsilon < 0$, we get

$$\sigma(\epsilon, 0) = \sigma_0 \epsilon^p \psi(1,0) \ , \quad \epsilon > 0 \tag{6a}$$

$$\sigma(\epsilon, 0) = \sigma_1 (-\epsilon)^{-q} \psi(1, 0), \quad \epsilon < 0 \tag{6b}$$

where

$$p = 1/\nu \tag{7a}$$

and

TABLE I. Critical exponents of percolation conductivity.

	C^*	q	p	$\frac{p}{q+p}$	r
Three-dimensional Bond percolation[a]	0.25	0.9 ± 0.1	1.6 ± 0.1	0.64 ± 0.04	0.67 ± 0.08
Three-dimensional Continuous percolation[b]	0.145 ± 0.005	0.9 ± 0.1	1.4 ± 0.1	0.61 ± 0.05	0.65 ± 0.05
Two-dimensional Bond percolation[c]	0.48 ± 0.02	1.0 ± 0.1	1.1 ± 0.1	0.52 ± 0.05	0.51 ± 0.01
Three-dimensional EMT	0.33	1	1	0.5	0.5
Two-dimensional EMT	0.5	1	1	0.5	0.5
Bethe lattice of coordination number z[d]	$\frac{1}{z-1}$	1	2	0.66	$\gtrsim 0.6$

[a] References 1, 2, and this work.
[b] Reference 2 and this work.
[c] This work and Ref. 8.
[d] References 4 and 5.

$$q = 1/\eta \ . \tag{7b}$$

At percolation threshold $C = C^*$, we obtain

$$\sigma(0^+, x) = \sigma_0 x^r \psi(0, 1) \ , \tag{8a}$$

$$\sigma(0^-, x) = \sigma_1 x^{1/\theta} \psi(0, 1) \ , \tag{8b}$$

where $r = 1/\mu$.

From the condition of continuity of the conductivity at $\epsilon = 0$, $\sigma(0^+, x) = \sigma(0^-, x)$, the following identity results

$$x^{1/\theta + 1} = x^r \ . \tag{9}$$

Similarly, from the continuity condition of the derivative

$$\left(\frac{\partial \sigma}{\partial \epsilon}\right)_{\epsilon = 0^+} = \left(\frac{\partial \sigma}{\partial \epsilon}\right)_{\epsilon = 0^-}$$

we deduce the identity

$$x^{(1-\nu)r} = x^{1+(1+\eta)/\theta} \ . \tag{10}$$

The power law Eq. (3) results from Eq. (8), while the relation $r = p/(q+p)$, Eq. (4), follows from Eqs. (7), (9), and (10). Numerical simulation methods have been utilized[1,2] to study the x dependence of the conductivity at $C = C^*$ for simple three-dimensional cubic networks ($C^* = 0.25$)[1] and for correlated cubic networks which simulate a continuous percolation transition[2] ($C^* = 0.145 \pm 0.005$). The plots of $\log\sigma/\sigma_0$ vs $\log x$ presented in Fig. 1 reveal that the power law Eq. (3) holds for values of $x \leq 0.1$. The values of q, p, and r summarized in Table I are in agreement with Straley's scaling relation, Eq. (4).

As numerical data for the two-dimensional percolation conductivity are scarce,[8] we have also studied in detail the conductivity of noncorrelated two-dimensional square networks at the percolation region. From the log-log plots of σ/σ_0 vs $(C^* - C)$ and of σ/σ_1 vs $(C - C^*)$ below C^* (Fig. 2) we find that the power laws, Eqs. (1) and (2), hold for this two-dimensional system with $q = 1.0 \pm 0.1$, $p = 1.1 \pm 0.1$, and $C^* = 0.48 \pm 0.02$. Our value for p is in agreement with simulations of Kirkpatrick[8] and with a renormalization-group analysis by Stinchcombe and Watson.[9] Simple symmetry arguments[10] show that for two-dimensional bond percolation $C^* = 0.5$. The discrepancy between our numerical result for C^* and this latter value is due to the small size of our two-dimensional networks (30×30). Our results for q and p given in Table I are in agreement with a theoretical prediction of

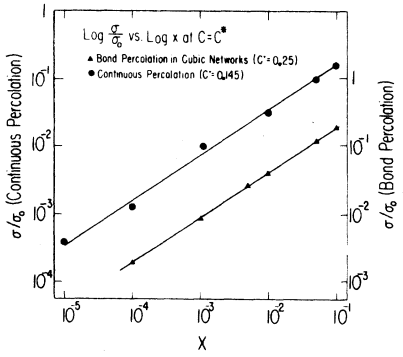

FIG. 1. Log-log plots of the conductivity of three-dimensional cubic resistor networks at the percolation threshold $C = C^*$ vs the conductivity ratio x: (▲) bond percolation, $C^* = 0.25$; (●) continuous percolation, $C^* = 0.145$.

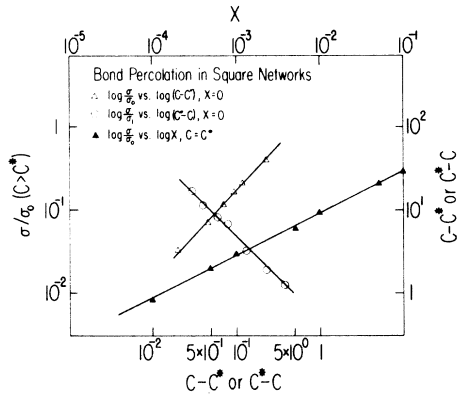

FIG. 2. Conductivity of two-dimensional square resistor networks near the bond percolation threshold, $C^* = 0.48$, network size: 30×30. (Δ) log-log plot of σ/σ_0 vs $(C - C^*)$ for $C > C^*$ and $x = 0$; (○) log-log plot of σ/σ_1 vs $(C^* - C)$ for $C < C^*$ and $x = 0$; (▲) log-log plot of σ/σ_0 vs x at $C = C^*$.

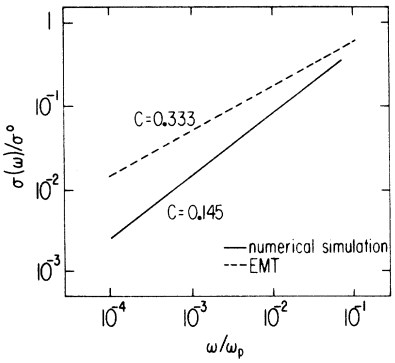

FIG. 3. Results of numerical simulation for the frequency-dependent conductivity of a metal-nonmetal inhomogeneous medium at metallic volume fraction $C = C^*$, the percolation threshold. $\epsilon^0(\omega)$ is given by a Drude-type dielectric function $\epsilon^0(\omega) = 1 + \omega_p^2/[\omega(\omega + i\gamma)]$ with $\gamma = \frac{1}{6}$ and $\omega_p = 1$, while $\epsilon^1(\omega) = 40 + 20i$. The values of $\sigma(\omega)$ are normalized by $\sigma^0 = \sigma(\omega = 0, C = 1)$. (——) numerical simulation on bond correlated networks at $C = C^* = 0.145$; (----) EMT, $C = \frac{1}{3}$.

Straley,[11] that $q = p$ for bond percolation on a square lattice. Straley's argument is based on the identity of the bond percolation problem on a square lattice to the bond percolation problem on the dual square lattice. The result $C^* = 0.5$ also follows from the same duality argument. These relationships were first demonstrated for a continuous two-dimensional system,[12,13] so that $q = p$ should also hold in the case of continuous percolation in two dimensions. The log-log plot of σ/σ_0 vs x at $C = C^*$ is presented in Fig. 2. The values of the exponents summarized in Table I confirm the validity of Eqs. (3) and (4) in the two-dimensional case.

It can easily be verified that the explicit EMT result[1,3(e)] for $\sigma(\epsilon, x)$ for both two-dimensional and three-dimensional system is a generalized homogeneous function of ϵ and of x being of the form given by Eq. (5) throughout the entire range $0 \leq C \leq 1$. The EMT results in $q = p = 1$ for $x < 0.1$. Accordingly, the power laws Eqs. (1) and (2) hold in this case for $0 \leq C \leq 1$ in the limit of small x. Note that in contrast to the three-dimensional case, in two-dimensional systems the EMT predicts the correct value of C^*. Also the EMT value of $p = 1$ is close to the numerical value $p = 1.1 \pm 0.1$. Thus for a two-dimensional inhomogeneous medium the EMT is quite accurate for all values of C and of x.

The scaling arguments tested herein do not provide information on the magnitude of the critical exponents q, p, and r. It is interesting to note that the exponent q has values that are close to unity for all the systems considered here and, in particular, being independent of the dimensionality of the system. The relation

$$\sigma(C) = \sigma_1 C^*/(C^* - C) \tag{11}$$

can be given the following simple interpretation. If we assume that below the percolation threshold the medium is represented by a one-dimensional channel with a fraction C/C^* of high-conductivity regions and $(C^* - C)/C^*$ of low-conductivity regions, then the conductivity of the channel is given by Eq. (11). The result $q \cong 1$ is thus an indication of the quasi-one-dimensional nature of the conductivity below the percolation threshold.

The foregoing scaling arguments are valid for the more general case of the complex dielectric function of an inhomogeneous medium undergoing a percolation transition.[14] Consider a random binary system with local dielectric functions $\epsilon^0(\omega) = \epsilon_1^0(\omega) + i\epsilon_2^0(\omega)$ and $\epsilon^1(\omega) = \epsilon_1^1(\omega) + i\epsilon_2^1(\omega)$. In the limit $x(\omega) = \epsilon^0(\omega)/\epsilon^1(\omega) \to 0$

$$\epsilon(\omega) \propto \epsilon^0(\omega)(C - C^*)^p, \quad C > C^* \tag{12a}$$

$$\epsilon(\omega) \propto \epsilon^1(\omega)(C^* - C)^{-q}, \quad C < C^* \tag{12b}$$

while at $C = C^*$

$$\epsilon(\omega) \propto \epsilon^0(\omega) x(\omega)^r . \tag{12c}$$

In particular, consider a binary medium made up of a dielectric with $\epsilon^1 = 1$ and a metal characterized by a Drude-type dielectric function $\epsilon^0(\omega)$. For frequencies much smaller than the reciprocal Drude relaxation time, $\epsilon^0(\omega) \propto qi/\omega$ so that at $C = C^*$

$$\sigma(\omega) = \omega\, \epsilon_2(\omega) \propto \omega^r , \tag{13a}$$

$$\epsilon_1^1(\omega) \propto \omega^{r-1} . \tag{13b}$$

Figure 3 presents a plot of $\sigma(\omega)$ vs ω for a Drude metal-dielectric system obtained from numerical simulations of the complex dielectric function[14] of a binary medium at $C = C^*$. The numerical results

are in agreement with Eq. (12c) resulting in the value $r = 0.75 \pm 0.05$, which is consistent with, but may be slightly higher than, the value $r = 0.65 \pm 0.05$ obtained (Table I) for the conductivity $\sigma(\omega = 0)$.

After this work was completed, a manuscript was received from Dr. J. P. Straley covering substantially the same ground with substantially similar results. We are grateful to Dr. Straley for informing us of his work.

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[7]We are greatly indebted to Scott Kirkpatrick who called our attention to Ref. 5 and pointed out the interest in studying the x dependence of the conductivity at the percolation threshold.

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