1.1 Places.

Convention: When we say a **ring** we mean a commutative ring with 1.

Let K be a field. Adjoin the symbol ∞ to K together with the following rules for $a \in F$:

$$a + \infty = \infty, \quad a \cdot \infty = \infty \text{ if } a \neq 0,$$

$$\infty \cdot \infty = \infty, \quad 1/0 = \infty, \quad , \text{ and } 1/\infty = 0.$$

The expressions $\infty + \infty$, $0 \cdot \infty$ and ∞/∞ are undefined.

A **place** φ of a field F into a field K is a mapping φ: $F \to K \cup \{\infty\}$ such that $\varphi(a+b) = \varphi(a) + \varphi(b)$ and $\varphi(ab) = \varphi(a)\varphi(b)$ whenever the expressions on the right sides of these formulas are defined, and such that $\varphi(1) = 1$. The place φ is **trivial** if $\varphi(a) \neq 0$ for every $a \in F$. In this case φ is an embedding of F into K.

Example 1.1: Let R be a unique factorization domain (e.g. $R = \mathbb{Z}$ or $R = K_0[X]$, where K_0 is a field). Denote the quotient field of R by F. To each prime element p of R we attach a place φ_p of F into $\overline{F}_p = R/pR$ by the following rule:

$$\varphi\left(\frac{a}{b}p^m\right) = \begin{cases} 0 & \text{if } m > 0 \\ \bar{a}/\bar{b} & \text{if } m = 0 \\ \infty & \text{if } m < 0. \end{cases}$$

Here $a, b \in R$ are relatively prime to p, $\bar{a}$ is the residue class of a modulo p and $m \in \mathbb{Z}$.

Exercise 1.2: Let φ be a place of a field F and let $x_1, \ldots, x_n$ be elements of F. Prove that there exists i between 1 and n such that φ is finite at $x_1/x_i, \ldots, x_n/x_i$.

A **local ring** is a ring R with a unique maximal ideal $\mathfrak{m}$. The elements of $R - \mathfrak{m}$ are then the units of R. For example, if φ is a place of a field F, then $R = \{x \in F \mid \varphi(x) \neq \infty\}$ is a local ring whose maximal ideal is $\mathfrak{m} = \{x \in F \mid \varphi(x) = 0\}$. It is the **valuation ring** of φ.

Conversely let R be a **valuation ring** of a field F (i.e., $R \neq F$ and for each $x \in F$ either $x \in R$ or $x^{-1} \in R$. Then R is a local ring whose maximal ideal is the set $\mathfrak{p}$ of all nonunits of R. Indeed, if $a \in \mathfrak{p}$ and $r \in R$, then ra is not a unit and therefore belongs to $\mathfrak{p}$. If $a, b \in \mathfrak{p}$ are nonzero, then we may assume that $ab^{-1} \in R$. Hence $a + b = b(ab^{-1} + 1) \in \mathfrak{p}$. Finally each $u \in R - \mathfrak{p}$ is a unit of R. Hence $\mathfrak{p}$ is the unique maximal ideal of R.

The map φ that maps $x \in R$ onto its residue modulo $\mathfrak{m}$ and $x \in F - R$ onto ∞ is a place of F whose valuation ring is R.

If $\mathfrak{p}$ is a prime ideal in an arbitrary ring R, then

$$R_{\mathfrak{p}} = \{\frac{a}{b} \mid a, b \in R \text{ and } b \notin \mathfrak{p}\}$$

is a local ring whose maximal ideal is $\mathfrak{p}R_{\mathfrak{p}}$. It is the **local ring of R at $\mathfrak{p}$**.

Our first goal is to extend homomorphisms of rings to places of fields.

LEMMA 1.3: *Let φ be a homomorphism of an integral domain R into an algebraically closed field K and let x be a nonzero element of a field that contains R. Then φ extends to a homomorphism of at least one of the rings $R[x]$ and $R[x^{-1}]$ into K.*

Proof: Let $\mathfrak{p} = \mathrm{Ker}(\varphi)$. Extend φ to $R_{\mathfrak{p}}$ by

$$\varphi\left(\frac{a}{b}\right) = \frac{\varphi(a)}{\varphi(b)} \text{ for } a \in R,\ b \in R - \mathfrak{p}.$$

So, we may assume that R is a local ring and $\mathfrak{p}$ is its maximal ideal.

We prove that at least one of the rings $\mathfrak{p} \cdot R[x]$ and $\mathfrak{p} \cdot R[x^{-1}]$ of the rings $R[x]$ and $R[x^{-1}]$ respectively is proper. Otherwise there exist positive integers m and n and elements $a_i, b_j \in \mathfrak{p}$ such that

$$(1a) \qquad 1 = a_0 + a_1 x + \cdots + a_m x^m$$

$$(1b) \qquad 1 = b_0 + b_1 x^{-1} + \cdots + b_n x^{-n}$$

Assume that m and n are minimal integers that satisfy (1). Observe that $1 - a_0$ is a unit of R. We may therefore bring a_0 to the left hand side of (1a) and multiply by $(1 - a_0)^{-1}$ to obtain an equation of the form

$$(2a) \qquad 1 = c_1 x + \cdots + c_m x^m, \qquad c_i \in \mathfrak{p}.$$

Similarly (1b) can be transposed to

$$(2b) \qquad 1 = d_1 x^{-1} + \cdots + d_n x^{-n}, \qquad d_j \in \mathfrak{p}.$$

Assume that $m \geq n$. Then multiply (2b) by x^m and substitute in (2a) to obtain an equation of the form (2a) of smaller degree. This contradiction to the minimality of m proves our assertion.

Suppose therefore that $\mathfrak{p} \cdot R[x]$ is a proper ideal. By Zorn's lemma, $R[x]$ has a maximal ideal $\mathfrak{P}$ that contains $\mathfrak{p}$. As $\mathfrak{P} \cap R = \mathfrak{p}$, we may embed $\overline{F} = \varphi(R)$ into $R[x]/\mathfrak{P}$. Let $\bar{x} = x + \mathfrak{P}$. Then $\overline{F}[\bar{x}] = R[x]/\mathfrak{P}$ and the canonical map $R[x] \to \overline{F}[\bar{x}]$ extends φ. Furthermore, $\overline{F}[\bar{x}]$ is a field. Hence $\bar{x}$ is algebraic over $\overline{F}$ and therefore lies in K. ∎

PROPOSITION 1.4 (Chevalley): *Let φ be a homomorphism of an integral domain R into an algebraically closed field K. Let F be a field that contains R. Then φ extends to a place of F into K.*

Proof: Consider the set Φ of all pairs (R_i, φ_i) where R_i is a subring of F that contains R and φ_i is a homomorphism of R_i into K that extends φ. Define a partial ordering on this set by $(R_i, \varphi_i) \leq (R_j, \varphi_j)$ if $R_i \subseteq R_j$ and φ_j extends φ_i. By Zorn's Lemma Φ contains a maximal element (R', φ'). From the maximality, R' is a local ring and $\mathrm{Ker}(\varphi')$ is its unique maximal ideal. By Lemma 1.3, for each $x \in F$ either $x \in R'$ or $x^{-1} \in R'$. If $R' = F$, then φ' is a monomorphism of F into K. Otherwise, the extension of φ' to F that maps each $x \in F - R'$ to ∞ is a place of F into K that extends φ. ∎

EXERCISE 1.5: Let φ be a monomorphism of a field E into a field K. Let F be an algebraic extension of E. Then every place of F that extends φ is trivial.

2. Transcendence bases.

Let F/K be a field extension. Elements $x_1, \ldots, x_n$ are said to be **algebraically independent** over K if $f(x_1, \ldots, x_n) \neq 0$ for each nonzero polynomial $f \in K[X_1, \ldots, X_n]$. A subset B of F is **algebraically independent** over K if every finite subset of B is algebraically independent over K. If in addition, F is algebraic over $K(B)$, then B is a **transcendence base** for F/K.

The existence of transcendence bases follows from Zorn's lemma. Moreover, if S is a set of generators of F/K and T_0 is a subset of S which is algebraically independent over K, then a maximal subset T of S which contains T_0 and which is algebraically independent over K is a bases of F/K.

The cardinality of the transcendence bases is an invariant of F/K. As in the case of linear bases for vector spaces this result depends on an exchange lemma:

LEMMA 2.1: *Let F/K be a field extension. Consider algebraically independent elements $w_1, \ldots, w_r \in F$ over K and let $x_1, \ldots, x_s \in F$. If w_r is algebraic over $E_1 = K(w_1, \ldots, w_{r-1}, x_1, \ldots, x_s)$, then one of the x_i's can be exchanged by w_r. That is, there exists i between 1 and s such that x_i is algebraic over $E_2 = K(w_1, \ldots, w_r, x_1, \ldots, \hat{x}_i, \ldots, x_s)$ (where the hat over x_i means that this element is omitted). In particular $\widetilde{E}_1 = \widetilde{E}_2$.*

Proof: By assumption there exists a nonzero polynomial f with coefficients in K such that

$$f(w_1, \cdots, w_r, x_1, \ldots, x_s) = 0.$$

Since $w_1, \ldots, w_r$ are algebraically independent over K there exists i such that x_i appears in f. Then x_i is algebraic over E_2. ∎

LEMMA 2.2: *Any two transcendence bases of a field extension F/K have the same cardinality.*

Proof: We prove only that if there exists one finite transcendence base, say $\{x_1, \ldots, x_s\}$, $s \geq 1$, then any other transcendence base must also have s elements. Using symmetry, it suffices to prove: If $w_1, \ldots, w_r$ are elements of F which are algebraically independent over K, then $r \leq s$. But this follows by induction from Lemma 2.1.

The case where F/K has an infinite transcendence base can be dealt with Zorn's lemma and is left to the reader. ∎

3. Algebraic sets.

For the rest of this chapter (unless otherwise is specified) K will denote an algebraically closed field, known as the **ground field**. To each subset $\mathfrak{a}$ of $K[\mathbf{X}] = K[X_1, \ldots, X_n]$ we associate the set

$$V(\mathfrak{a}) = \{\mathbf{a} \in K^n |\ f(\mathbf{a}) = 0 \text{ for each } f \in \mathfrak{a}\}.$$

We call it an **algebraic set**. Each $\mathbf{a} \in V(\mathfrak{a})$ is a K**-rational zero** of $\mathfrak{a}$. If $\mathfrak{a}$ is the ideal generated by a subset $\mathfrak{a}_0$ of $K[\mathbf{X}]$, then $V(\mathfrak{a}) = V(\mathfrak{a}_0)$. To each subset A of K^n we associate an ideal of $K[X_1, \ldots, X_n]$:

$$I(A) = \{f \in K[X_1, \ldots, X_n] |\ f(\mathbf{a}) = 0 \text{ for each } \mathbf{a} \in A\}.$$

This correspondence between algebraic sets and ideals is compatible with the lattice structures:

LEMMA 3.1: *For ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{a}_i$ in $K[X_1, \ldots, X_n]$ and subsets A, B of K^n we have:*

(a) $\mathfrak{a} \subseteq \mathfrak{b}$ *implies* $V(\mathfrak{a}) \supseteq V(\mathfrak{b})$,

(b) $A \subseteq B$ *implies* $I(A) \supseteq I(B)$,

(c) $V(\langle \mathfrak{a}_i \rangle_{i \in I}) = \bigcap_{i \in I} V(\mathfrak{a}_i)$, *and*

(d) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

(e) $I(A) \cap I(B) = I(A \cup B)$

Proof: All are obvious except possibly (d). But by (a), $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b})$. Conversely, if $\mathbf{x} \notin V(\mathfrak{a}) \cup V(\mathfrak{b})$, then there exist polynomials $f \in \mathfrak{a}$ and $g \in \mathfrak{b}$ such that $f(\mathbf{x}) \neq 0$ and $g(\mathbf{x}) \neq 0$. But then $fg \in \mathfrak{a} \cap \mathfrak{b}$ and $(fg)(\mathbf{x}) \neq 0$, hence $x \notin V(\mathfrak{a} \cap \mathfrak{b})$. ∎

Note that $V(K[\mathbf{X}]) = \emptyset$ and $V(0) = K^n$. We denote K^n also by $\mathbb{A}^n(K)$, and if K is clear from the context also by $\mathbb{A}^n$. The correspondence between algebraic sets and ideals becomes bijective if we restrict the domain of V to ideals that coincide with their "radical". This is a consequence of Hilbert's Nullstellensatz:

PROPOSITION 3.2 (Weak Nullstellensatz): *Each proper ideal of $K[\mathbf{X}]$ has a K-rational zero.*

Proof: By Zorn's lemma it suffices to prove that each maximal ideal $\mathfrak{m}$ has a K-rational zero.

Indeed, with $x_i = X_i + \mathfrak{m}$, $i = 1, \dots, n$, the ring $K[\mathbf{x}]$ is a field. Let $t_1, \dots, t_r$ be a transcendence base for $K[\mathbf{x}]/K$. Each x_i satisfies an equation with coefficients in $K[\mathbf{t}]$:

$$g_{i,m_i}(\mathbf{t})x_i^{m_i} + \cdots + g_{i,0}(\mathbf{t}) = 0 \tag{1}$$

where $g_{i,m_i} \neq 0$. Choose $a_1, \dots, a_n \in K$ such that $g_{i,m_i}(\mathbf{a}) \neq 0$, $i = 1, \dots, n$. Then extend the K-homomorphism $K[\mathbf{t}] \to K$ which maps t_i onto a_i, $i = 1, \dots, n$ to a place φ of $K(\mathbf{x})$ into K (Proposition 1.4). If $\varphi(x_i) = \infty$ for some i, divide (1) by $x_i^{m_i}$ and apply φ to get a contradiction. Thus $\varphi(\mathbf{x}) = \mathbf{x}'$ is a K-rational zero of $\mathfrak{m}$. ∎

COROLLARY 3.3: *If $f_1, \dots, f_m \in K[\mathbf{X}]$ have no common zero, then there exist $g_1, \dots, g_m \in K[\mathbf{X}]$ such that $g_1 f_1 + \cdots + g_m f_m = 1$.*

EXERCISE 3.4: (a) Let $x_1, \dots, x_n$ be elements of an extension of K. Use the weak Nullstellensatz to prove that if $K[\mathbf{x}]$ is a field, then $x_1, \dots, x_n \in K$.
(b) Prove that every maximal ideal of $K[\mathbf{X}]$ has the form $\langle X_1 - x_1, \dots, X_n - x_n \rangle$, where $x_1, \dots, x_n \in K$. Hint: Use Taylor expansion around $(x_1, \dots, x_n)$.

Let $\mathfrak{a}$ be an ideal in a ring R. Define

$$\sqrt{\mathfrak{a}} = \{f \in R \mid f^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}\}.$$

The ideal $\sqrt{\mathfrak{a}}$ is the **radical** of $\mathfrak{a}$.

EXERCISE 3.5: Prove that $\sqrt{\mathfrak{a}}$ is the intersection of all prime ideals that contain $\mathfrak{a}$. Hint: If no power of an element f of R belongs to $\mathfrak{a}$ use Zorn's Lemma to find a maximal element among all ideals that contain $\mathfrak{a}$ and disjoint to the set of powers of f.

PROPOSITION 3.6 (Strong Nullstellensatz): *Let $\mathfrak{a}$ be an ideal of $K[X_1, \dots, X_n]$. If a polynomial $f \in K[\mathbf{X}]$ vanishes on $V(\mathfrak{a})$, then $f \in \sqrt{\mathfrak{a}}$.*

Proof: We use the, so called, "Rabinovich trick". Consider the ideal $\mathfrak{b}$ of $K[X_1, \dots, X_{n+1}]$ generated by $\mathfrak{a}$ and $1 - X_{n+1}f$. If $\mathfrak{b}$ were a proper ideal, then by the weak form of Hilbert's Nullstellensatz it would have a K-rational zero $(a_1, \dots, a_{n+1})$. But then

$(a_1, \dots, a_n) \in V(\mathfrak{a})$ and $a_{n+1} f(a_1, \dots, a_n) = 1$. Hence, $f(a_1, \dots, a_n) \neq 0$. This contradiction to the assumption of the Proposition implies that $1 \in \mathfrak{b}$. Thus, there exist $f_1, \dots, f_m \in \mathfrak{a}$ and $h_1, \dots, h_{m+1} \in K[X_1, \dots, X_{m+1}]$ such that

$$1 = \sum_{i=1}^{m} h_i(X_1, \dots, X_{n+1}) f_i + h_{m+1}(X_1, \dots, X_{n+1})(1 - X_{n+1} f).$$

Substitute $X_{n+1} = f^{-1}$ in this formula to get:

$$1 = \sum_{i=1}^{m} h_i(X_1, \dots, X_n, f^{-1}) f_i(X_1, \dots, X_n).$$

Clearing denominators this gives:

$$f(X_1, \dots, X_n)^r = \sum_{i=1}^{m} h_i'(X_1, \dots, X_n) f_i(X_1, \dots, X_n)$$

for some $h_i' \in K[X_1, \dots, X_n]$. This means that $f \in \sqrt{\mathfrak{a}}$. ∎

COROLLARY 3.7: *If $\mathfrak{a}$ is an ideal of $K[\mathbf{X}]$, then $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. In particular, if $\mathfrak{p}$ is a prime ideal of $K[\mathbf{X}]$, then $I(V(\mathfrak{p})) = \mathfrak{p}$.*

PROBLEM 3.8: Prove for each subset $\mathfrak{a} \subseteq K[X]$ and each subset $A \subseteq K^n$: If $A = V(\mathfrak{a})$, then $A = V(I(A))$. If $\mathfrak{a} = V(A)$, then $\mathfrak{a} = I(V(\mathfrak{a}))$.

4. Zariski topology.

A topological space X is called **irreducible** if for any decomposition $X = A_1 \cup A_2$ with closed subsets A_i of X we have $X = A_1$ or $X = A_2$. A subset X' of a topological space X is called **irreducible** if X' is irreducible as a space with the induced topology.

LEMMA 4.1: *For a topological space X the following statements are equivalent.*

(a) *X is irreducible.*

(b) *If U_1, U_2 are nonempty open subsets of X then $U_1 \cap U_2 \neq \emptyset$.*

(c) *Any nonempty open subset of X is dense in X.*

Proof: The equivalence of (a) and (b) follows from the definition by taking complements. the equivalence of (b) and (c) is clear by the definition of density. ∎

COROLLARY 4.2: *For a subset X' of a topological space X the following statements are equivalent.*

(a) *X' is irreducible.*

(b) *If U_1, U_2 are open subsets of X with $U_i \cap X' \neq \emptyset$, $i = 1, 2$, then $U_1 \cap U_2 \cap X' \neq \emptyset$.*

(c) *The closure $\overline{X'}$ of X' is irreducible.*

Proof: The equivalence of (a) and (b) is a consequence of Lemma 4.1. That of (b) and (c) follows from the fact that an open set meets X' if and only if it meets $\overline{X'}$. ∎

COROLLARY 4.3: *Every open subset of an irreducible topological space is irreducible.*

An **irreducible component** of a topological space X is a maximal irreducible subset of X.

By Corollary 4.2(c) each irreducible component is closed.

LEMMA 4.4: *Let X be a topological space.*

(a) *Any irreducible subset of X is contained in an irreducible component.*

(b) *The space X is the union of its irreducible components.*

Proof: Since each point $x \in X$ is irreducible, (b) follows from (a). Statement (a) follows from Zorn's Lemma:

For an irreducible subset X_0 of X consider the set $\mathcal{X}$ of all irreducible subsets of X that contain X_0. It is non empty, and if $\{X_\lambda\}_{\lambda \in \Lambda}$ is a linearly ordered family of elements of $\mathcal{X}$, then $Y = \bigcup_{\lambda\in\Lambda} X_\lambda$ is also a member of $\mathcal{X}$. Indeed, if U_1, U_2 are open subsets of X that intersect Y, then $U_i \cap X_{\lambda_i} \neq \emptyset$, for some $\lambda_i \in \Lambda$, $i = 1, 2$. If, say, $X_{\lambda_1} \subseteq X_{\lambda_2}$, then $U_1 \cap U_2 \cap X_{\lambda_2} \neq \emptyset$. Hence $U_1 \cap U_2 \cap Y \neq \emptyset$. Conclude that Y is irreducible.

By Zorn's Lemma $\mathcal{X}$ has a maximal element. It is an irreducible component of X that contains X_0. ∎

A topological space X is called **Noetherian** if every descending sequence $A_1 \supseteq A_2 \supseteq \cdots$ of closed subsets of X is stationary. It is clear that X is Noetherian if and only if every ascending sequence of open subsets is stationary. Also, X is Noetherian if and only if every family of closed subsets has a minimal element, or equivalently, every family of open subsets has a maximal element. In particular every Noetherian space is compact (but not necessarily Hausdorff. Some authors call such spaces quasi compact.) Any subset Y of a Noetherian spade X is Noetherian in its induced topology. Indeed, if $U_1, U_2, U_3, \ldots$ is a sequence of open subsets of X such that $U_1 \cap Y \subseteq U_2 \cap Y \subseteq U_3 \cap Y \subseteq \cdots$, then $U_1 \subseteq U_1 \cup U_2 \subseteq U_1 \cup U_2 \cup U_3 \subseteq \cdots$ is stationary and therefore so is $U_1 \cap Y \subseteq U_2 \cap Y \subseteq U_3 \cap Y \subseteq \cdots$. Also, a finite union of Noetherian spaces is Noetherian.

LEMMA 4.5: *A Noetherian topological space X has only finitely many irreducible components. No component is contained in the union of the others.*

Proof: Let $\mathcal{X}$ be the family of all closed subsets of X that cannot be written as a finite union of irreducible subsets of X. Suppose $\mathcal{X}$ is not empty.

By the minimal condition there is a minimal element $Y \in \mathcal{X}$. The set Y is not irreducible, so $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 are proper closed subsets of Y. But then $Y_i \notin \mathcal{X}$ and therefore Y_i is a finite union of irreducible subsets of X, $i = 1, 2$, and therefore so is Y. This contradiction proves that $\mathcal{X}$ is empty.

In particular X is a finite union of irreducible subsets. By Lemma 4.4(a), each of them is contained in an irreducible component of X. So $X = X_1 \cup \cdots \cup X_n$ with irreducible components X_i of X, and $X_i \neq X_j$ for $i \neq j$.

If Y is an arbitrary irreducible component, it follows from the relation $Y = \bigcup_{i=1}^n (X_i \cap Y)$ that $Y = X_i \cap Y$ for some i, and therefore $Y = X_i$. Thus, all components occur among the X_i. The same argument implies that $X_i \not\subseteq \bigcup_{j\neq i} X_j$. This proves the Lemma. ∎

We would like to equip algebraic sets with a Noetherian topology.

LEMMA AND DEFINITION 4.6: *A ring R is said to be* **Noetherian** *if one of the following equivalent conditions holds:*

(a) *Every ideal of R is finitely generated.*

(b) *Any ascending sequence of ideals of R becomes stationary.*

(c) *Any nonempty set of ideals of R has a maximal element.*

Examples of Noetherian rings are the principal ideal rings, in particular all fields, as well as $\mathbb{Z}$ and $K[X]$, if K is a field. Any homomorphic image of a Noetherian ring is Noetherian.

PROPOSITION 4.7 (Hilbert's basis theorem): *If R is a Noetherian ring, so is $R[X]$.*

Proof: Assume that I is an ideal of $R[X]$ which is not finitely generated. Let $f_1 \in I$ be a polynomial of least degree. If f_k $(k \geq 1)$ has already been chosen, let f_{k+1} be a polynomial of least degree in $I - \langle f_1, \ldots, f_k \rangle$. Let n_k be the degree and $a_k \in R$ the leading coefficient of f_k, $k = 1, 2, \ldots,$. Then $n_1 \leq n_2 \leq \cdots$. Moreover, $\langle a_1 \rangle \subseteq \langle a_1, a_2 \rangle \subseteq \cdots$ is a chain of ideals that does not become stationary. For suppose $\langle a_1, \ldots, a_k \rangle = \langle a_1, \ldots, a_{k+1} \rangle$. Then $a_{k+1} = \sum_{i=1}^{k} b_i a_i$ with $b_i \in R$ and

$$g = f_{k+1} - \sum_{i=1}^{k} b_i X^{n_{k+1} - n_i} f_i \in I - \langle f_1, \ldots, f_k \rangle$$

is of lower degree than f_{k+1}. This contradiction to the choice of f_{k+1} proves that $R[X]$ is Noetherian. ∎

COROLLARY 4.8: *Let R be a Noetherian ring and $S = R[x_1, \ldots, x_n]$ an extension ring of R which is finitely generated. Then S is also Noetherian.*

Proof: As S is a homomorphic image of a polynomial ring $R[X_1, \ldots, X_n]$ is suffices to prove that the later is Noetherian. But this follows by induction on n from Proposition 4.7. ∎

In particular, for a principal ideal ring R the polynomial ring $R[X_1, \ldots, X_n]$ and its homomorphic images are Noetherian. In particular $\mathbb{Z}[X_1, \ldots, X_n]$ and $F[X_1, \ldots, X_n]$ for any field K are Noetherian. The last fact has the following consequence for algebraic sets in $\mathbb{A}^n$.

COROLLARY 4.9: *Every decreasing chain $A_1 \supseteq A_2 \supseteq \cdots$ of algebraic subsets of $\mathbb{A}^n$ is stationary.*

Since finite unions and arbitrary intersections of algebraic subsets of $\mathbb{A}^n$ are algebraic sets (Lemma 3.1), the algebraic sets form the closed sets of a topology on $\mathbb{A}^n$ known as the **Zariski topology**. Equipped with the Zariski topology $\mathbb{A}^n$ is the **affine space of dimension** n. Corollary 4.9 implies that $\mathbb{A}^n$ and therefore each algebraic set is Noetherian space. We may therefore apply Lemma 4.5 on these spaces:

PROPOSITION 4.10: *Every algebraic subset V of $\mathbb{A}^n$ has only finitely many irreducible components $V_1, \ldots, V_m$. We have $V = V_1 \cup \cdots V_m$ and no V_i is contained in the union of the others.*

PROBLEM 4.11: *Prove that an algebraic subset V of $\mathbb{A}^n$ is irreducible if and only if $I(V)$ is a prime ideal.*

5. The projective space.

Define $\mathbb{P}^n(K)$ to be the set of $(n+1)$-tuples $(x_0, \ldots, x_n) \in K^{n+1}$ such that some $x_i \neq 0$ modulo the equivalence relation

$$(x_0, \ldots, x_n) \sim (\alpha x_0, \ldots, \alpha x_n), \qquad \alpha \in K^\times.$$

Denote the equivalence class of an $(n+1)$-tuple $(x_0, \ldots, x_n)$ by $x_0 : \cdots : x_n$ or also by $\mathbf{x}$ and call it a **point** of $\mathbb{P}^n(K)$. The $(n+1)$-tuple itself is a **set of homogeneous coordinates** for $\mathbf{x}$. We will also write $\mathbb{P}^n$ for $\mathbb{P}^n(K)$.

The space $\mathbb{P}^n$ can be covered by $n+1$ subsets $U_0, U_1, \ldots, U_n$ where $U_i = \{\mathbf{x} \in \mathbb{P}^n \mid x_i \neq 0\}$. For each i there is a natural bijection $U_i \to \mathbb{A}^n$:

$$x_0 : x_1 : \cdots : x_n \mapsto (\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i}), \qquad \Big(\frac{x_i}{x_i} \text{ omitted}\Big).$$

We may use this correspondence to equip U_i with a topology. Then we need to glue these topologies to a topology of $\mathbb{P}^n$. Instead we take a direct approach, analogous to the one taken for $\mathbb{A}^n$.

Each polynomial $f \in K[X_0, \ldots, X_n]$ can be uniquely written in the form $f = \sum_{i=1}^d f_i$, where f_i is a homogeneous polynomial of degree i called the **homogeneous component** of f of degree i. An ideal $\mathfrak{a}$ of $K[X_0, \ldots, X_n]$ is said to be **homogeneous** if it contains which each f every homogeneous component of f. Any ideal $\mathfrak{a}$ generated by homogeneous ideals is homogeneous. For let h_i, $i \in I$, be homogeneous generators of $\mathfrak{a}$, and set $d(i) = \deg(h_i)$. Let $f = \sum_i g_i h_i$ be in $\mathfrak{a}$. Write $f = \sum_k f_k$, with f_k homogeneous of degree k. Write $g_i = \sum_j g_{ij}$ where g_{ij} homogeneous of degree k. Then

$$\sum_k f_k = \sum_k \Big(\sum_{j+d(i)=k} g_{ij} h_i \Big).$$

The kth term on the right hand side is homogeneous of degree k. Hence $f_k = \sum_{j+d(i)=k} g_{ij} h_i \in \mathfrak{a}$. By Hilbert's basis theorem, $\mathfrak{a}$ has a finite set of generators and therefore also a finite set of homogeneous generators.

Definition 5.1: *A* **closed algebraic set** *in $\mathbb{P}^n$ is a set consisting of all zeros of a finite collection of homogeneous polynomials $f_i \in K[X_0, \ldots, X_n]$, $i = 1, \ldots, m$. We denote it by $V(f_1, \ldots, f_m)$.*

This makes sense because if f is homogeneous, and $(x_0, \ldots, x_n)$, $(\alpha x_0, \ldots, \alpha x_n)$ are two sets of homogeneous coordinates of the same point, then $f(x_0, \ldots, x_n) = 0$ if and only if $f(\alpha x_0, \ldots, \alpha x_n) = 0$. From the preceding discussion $V(f_1, \ldots, f_m)$ is also the set of zeros of the homogeneous ideal $\mathfrak{a} = \langle f_1, \ldots, f_m \rangle$:

$$V(\mathfrak{a}) = \{\mathbf{x} \in \mathbb{P}^n |\ f(\mathbf{x}) = 0 \text{ for each } f \in \mathfrak{a}\}$$

Dually, we define for $A \subseteq \mathbb{P}^n$:

$$I(A) = \{f \in K[X_0, \ldots, X_n] |\ f(\mathbf{x}) = 0 \text{ for all } \mathbf{x} \in A\}.$$

This is a homogeneous ideal. Indeed, if $f \in I(A)$ and $f = \sum f_i$ is a presentation of f as a sum of its homogeneous components and $\mathbf{x} \in \mathfrak{a}$, then

$$\sum_i f_i(\mathbf{x}) t^i = \sum_i f_i(t\mathbf{x}) = f(t\mathbf{x}) = 0$$

for each $t \in K$. Hence $f_i(\mathbf{x}) = 0$. Conclude that $f_i \in I(A)$ for each $i = 1$. Obviously

(1a) $\mathfrak{a} \subseteq I(V(\mathfrak{a}))$ for each homogeneous ideal $\mathfrak{a}$ and

(1b) $A \subseteq V(I(A))$ for each subset A of $\mathbb{P}^n$.

It follows that

(2a) $A = V(\mathfrak{a})$ implies $V(I(A)) = A$ and

(2b) $\mathfrak{a} = I(A)$ implies $I(V(\mathfrak{a})) = \mathfrak{a}$.

Indeed, if $A = V(\mathfrak{a})$, then, by (1), $\mathfrak{a} \subseteq I(V(\mathfrak{a})) = I(A)$ and $A \subseteq V(I(A)) \subseteq V(\mathfrak{a}) = A$. Thus $A = V(I(A))$. Statement (2a) is proved analogously.

Proposition 5.2: *The map I maps the family of all algebraic subsets of $\mathbb{P}^n$ bijectively onto the set of all homogeneous ideals $\mathfrak{a} \subseteq K[X_0, \ldots, X_n]$ such that $\mathfrak{a} \neq \langle X_0, \ldots, X_1 \rangle$ and $\mathfrak{a} = \sqrt{\mathfrak{a}}$.*

Proof: By (2), I maps the family of all algebraic subsets of $\mathbb{P}^n$ bijectively onto the set of all homogeneous ideal $\mathfrak{a}$ that satisfy

(3) $$\mathfrak{a} = I(V(\mathfrak{a})).$$

Each of these ideals certainly equal its radical. Moreover, $V(\langle X_0,\dots,X_n\rangle)=\emptyset$. Hence $1\in I(V(\langle X_0,\dots,X_n\rangle))$. So $\langle X_0,\dots,X_n\rangle$ does not satisfy (3).

To complete the proof we have to show that if $\mathfrak{a}=\sqrt{\mathfrak{a}}$, and $\mathfrak{a}\neq\langle X_0,\dots,X_n\rangle$, then $\mathfrak{a}$ satisfies (3). Indeed, let $V^*(\mathfrak{a})$ be the algebraic set corresponding to $\mathfrak{a}$ in the affine space $\mathbb{A}^{n+1}$. Then $(x_0,\dots,x_n)\in V^*(\mathfrak{a})$ implies $(\alpha x_0,\dots,\alpha x_n)\in V^*(\mathfrak{a})$ for each $\alpha\in K$. Therefore, either

(4a) $V^*(\mathfrak{a})$ is empty,

(4b) $V^*(\mathfrak{a})$ equals the origin only, or

(4c) $V^*(\mathfrak{a})$ is the union of lines through the origin: i.e., it is the **cone** over the subset $V(\mathfrak{a})$ in $\mathbb{P}^n$.

Moreover, by Hilbert's Nullstellensatz (Proposition 3.6)

$$\mathfrak{a}=I(V^*(\mathfrak{a})). \tag{5}$$

In case (4a), statement (5) implies that $\mathfrak{a}=K[X_0,\dots,X_n]$ (Proposition 3.2), hence $I(V(\mathfrak{a}))$ –which always contains $\mathfrak{a}$ – must equal $\mathfrak{a}$ since there is no larger ideal. In case (4b), (5) implies that $\mathfrak{a}=\langle X_0,\dots,X_n\rangle$ which we have excluded. In case (4c), if f is a nonzero homogeneous polynomial and f vanishes on $V(\mathfrak{a})=\{x_0{:}\cdots{:}x_n \mid 0\neq(x_0,\dots,x_n)\in V^*(\mathfrak{a})\}$, then f vanishes on a nonempty set of $\mathbb{A}^n$. Hence $\deg(f)>0$ and f vanishes on $V^*(\mathfrak{a})$. Hence, by (5), $I(V(\mathfrak{a}))\subseteq\mathfrak{a}$. The other inclusion is stated in (1a). So, $I(V(\mathfrak{a}))=\mathfrak{a}$ and Proposition 5.2 is proven. ∎

The same lattice-theoretic identities hold as in the affine case:

PROPOSITION 5.3: *For homogeneous ideals $\mathfrak{a},\mathfrak{b},\mathfrak{a}_i$ in $K[X_0,\dots,X_n]$ and algebraic subsets A,B of $\mathbb{P}^n$ we have:*

(a1) $\mathfrak{a}\subseteq\mathfrak{b}$ *implies* $V(\mathfrak{a})\supseteq V(\mathfrak{b})$,

(a2) $A\subseteq B$ *implies* $I(A)\supseteq I(B)$,

(a3) $V(\langle\mathfrak{a}_i\rangle_i)=\bigcap_i V(\mathfrak{a}_i)$, *and*

(a4) $V(\mathfrak{a}\cap\mathfrak{b})=V(\mathfrak{a})\cup V(\mathfrak{b})$.

It follows as in the affine case that the algebraic subsets are the closed sets of a Noetherian topology on $\mathbb{P}^n$. As in the affine case, a basis for this topology are the sets

$$(\mathbb{P}^n)_f=\{\mathbf{x}\in\mathbb{P}^n \mid f(\mathbf{x})\neq 0\}.$$

So we have the notion of irreducible algebraic sets and we may apply Lemma 4.4 again:

Proposition 5.4: *In the bijection of Proposition 5.2, the irreducible algebraic sets correspond exactly to the homogeneous prime ideal (the empty set corresponds to the ideal $\langle X_0, \dots, X_n \rangle$). Moreover, every closed algebraic set V can be written in exactly one way as*

$$V = V_1 \cup \cdots \cup V_m,$$

where the V_i are irreducible algebraic sets and $V_i \not\subseteq V_j$ if $i \neq j$.

Proof: We have to prove the statment about the irreducible sets and prime ideals.

First suppose that $\mathfrak{p}$ is a homogeneous prime ideal of $K[X_0, \dots, X_n]$. We prove that $V(\mathfrak{p})$ is irreducible. In fact suppose that $V(\mathfrak{p}) = A \cup B$ with A, B algebraic sets in $\mathbb{P}^n$. Then $\mathfrak{p} = I(V(\mathfrak{p})) = I(A) \cap I(B) \supseteq I(A)I(B)$. Hence $\mathfrak{p} \supseteq I(A)$ or $\mathfrak{p} \supseteq I(B)$. So, $V(\mathfrak{p}) \subseteq A$ or $V(\mathfrak{p}) \subseteq B$. Conclude that $V(\mathfrak{p}) = A$ or $V(\mathfrak{p}) = B$, and therefore $V(\mathfrak{p})$ is irreducible.

Conversely, suppose that C is an irreducible set in $\mathbb{P}^n$. We prove that $I(C)$ is prime. Indeed, let f, g be polynomials $K[X_0, \dots, X_n]$ such that $fg \in I(C)$. Let $f = \sum_i f_i$ and $g = \sum_j g_j$ be a presentation of f and g as sums of homogeneous polynomials. Assume that $f, g \notin I(C)$. Then at least one of the components of f (resp., g) does not belong to $I(C)$. Let r (resp., s) be the smallest integer such that $f_r \ /I(C)$ (resp., $g_s \notin I(C)$). Then $\sum_k \sum_{i+j=k} f_i g_j = fg \cong 0 \mod I(C)$. Hence $f_r g_s + \sum_{i+j>k} f_i g_j \cong 0 \mod I(C)$. Since $I(C)$ is homogeneous $f_r g_s \in I(C)$.

Thus, without loss, we may assume that f and g are homogeneous. But then $\mathfrak{a} = \langle I(C), f \rangle$ and $\mathfrak{b} = \langle I(C), g \rangle$ are homogeneous ideals and $C = V(\mathfrak{a}) \cup V(\mathfrak{b})$. Since C is irreducible we have, e.g., $C = V(\mathfrak{a})$. In particular f vanishes on C. Hence $f \in I(C)$. Conclude that $I(C)$ is prime.

Problem 5.5: Let $A \subseteq \mathbb{P}^n$ be a closed algebraic set, and let H be the **hyperplane at infinity** $X_0 = 0$. Identify $U_0 = \mathbb{P}^n - H$ with $\mathbb{A}^n$ in the usual way. Prove that $A \cap U_0$ is a closed algebraic subset of $\mathbb{A}^n$ and show that the ideal of $A \cap U_0$ is derived from the ideal of A in a very natural way.

EXAMPLE 5.6: *Hypersurfaces.* Let $f(X_0, \ldots, X_n)$ be an irreducible homogeneous polynomial. Then the principal ideal $\langle f \rangle$ is prime, so $f = 0$ defines an irreducible algebraic set in $\mathbb{P}^n$ called a **hypersurface** (e.g., plane curve, surface in 3-space, etc.).

6. Morphisms.

Let V be an irreducible algebraic set in $\mathbb{A}^n$. Then the **affine coordinate ring** is the ring

$$K[V] = K[X_1, \dots, X_n]/I(V). \tag{1}$$

Each element $f \in K[V]$ can be considered as a function from V to K: Lift f to a polynomial $f' \in K[X_1, \dots, X_n]$ and set $f(\mathbf{x}) = f'(\mathbf{x})$ for each $\mathbf{x} \in V$. If f'' is another lifting of f, then $f' - f'' \in I(V)$ and therefore $f'(\mathbf{x}) = f''(\mathbf{x})$. Among the the elements of $K[V]$ we find the coordinate functions $x_i = X_i + I(V)$, $i = 1, \dots, n$, and we have $K[V] = K[x_1, \dots, x_n]$. Note that we are using x_i also to denote the ith coordinate of a point $\mathbf{x}$ of V. If $\mathbf{x}$ and $\mathbf{x}'$ are distinct points of V, then $x_i \neq x_i'$ for at least one i between 1 and n. A fortiori there exists $g \in K[V]$ such that $g(\mathbf{x}) \neq g(\mathbf{x}')$.

For each subset A of V let

$$I(A) = \{f \in K[V] |\ f(\mathbf{x}) = 0 \text{ for each } \mathbf{x} \in A\}.$$

For each subset $\mathfrak{a}$ of $K[V]$ let

$$V(\mathfrak{a}) = \{\mathbf{x} \in V |\ f(\mathbf{x}) = 0 \text{ for each } f \in \mathfrak{a}\}$$

As in the case $V = \mathbb{A}^n$ (Section 3) we have:

(2a) $\mathfrak{a} \subseteq \mathfrak{b}$ implies $V(\mathfrak{a}) \supseteq V(\mathfrak{b})$,

(2b) $A \subseteq B$ implies $I(A) \supseteq I(B)$,

(2c) for ideals $\mathfrak{a}_i$, $i \in I$, of $K[V]$, $V(\langle \mathbf{a}_i \rangle_{i \in I}) = \bigcap_{i \in I} V(\mathfrak{a}_i)$,

(2d) for ideals $\mathfrak{a}, \mathfrak{b}$ of $K[V]$, $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$,

(2e) $I(A) \cap I(B) = I(A \cup B)$.

(2f) $V(\mathfrak{a}) = V(\langle \mathfrak{a} \rangle)$.

Also,

(3a) $\mathfrak{a} \subseteq I(V(\mathfrak{a}))$, and

(3b) $A \subseteq V(I(A))$.

It follows that

(4a) $A = V(\mathfrak{a})$ implies $V(I(A)) = A$, and

(4b) $\mathfrak{a} = I(A)$ implies $I(V(\mathfrak{a})) = \mathfrak{a}$.

Denote the canonical epimorphism $K[\mathbf{X}] \to K[V]$ with kernel $I(V)$ by φ. If $\mathfrak{a}$ is an ideal of $K[V]$, then $\mathfrak{a}' = \varphi^{-1}(\mathfrak{a})$ is an ideal of $K[\mathbf{X}]$ which contains $I(V)$. Hence, $V(\mathfrak{a}) = V(\mathfrak{a}')$. In particular $V(\mathfrak{a})$ is a closed subset of $\mathbb{A}^n$. Conversely, if $\mathfrak{b}$ is an ideal of $K[\mathbf{X}]$, then $V(\mathfrak{b}) \cap V = V(\varphi(\mathfrak{b}))$. Thus, the Zariski topology of V defined by the complements of $V(\mathfrak{a})$ in V coincides with the topology induced on V by the Zariski topology of $\mathbb{A}^n$.

A basis for the open sets in the Zariski topology on V is given by the open sets:

$$V_f = \{\mathbf{x} \in V \mid f(\mathbf{x}) \neq 0\}$$

for elements $f \in K[V]$. In fact $V_f = V - V(f)$, hence V_f is open. And if $U = V - V(\mathfrak{a})$ is an arbitrary open set, then

$$U = \bigcup_{f \in \mathfrak{a}} V_f.$$

EXERCISE 6.1: In the above notation prove that $1/f\colon V_f \to K$ is a continuous map.

LEMMA 6.2: *Let V be an irreducible algebraic set in $\mathbb{A}^n$.*

(a) *If A is a subset of V, then $\overline{A} = V(I(V))$ is the closure of A.*

(b) *If $\mathfrak{a}$ is an ideal of $K[V]$, then $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*

Proof of (a): Since $V(I(A))$ is a closed set, $\overline{A} \subseteq V(I(A))$. If $A \subseteq B \subseteq V(I(A))$ is a closed set, then $I(A) \supseteq I(B)$ and therefore $V(I(A)) \subseteq B$ (by (4a)). Hence $B = V(I(A))$ and therefore $\overline{A} = V(I(A))$.

Proof of (b): Let $f \in I(V(\mathfrak{a}))$. Lift f to polynomial f' in $K[\mathbf{X}]$. In the above notation, let $\mathfrak{a}' = \varphi^{-1}(\mathfrak{a})$. Then $f' \in I(V(\mathfrak{a}')) = \sqrt{\mathfrak{a}'}$. Hence $f'^r \in \mathfrak{a}'$ for some positive integer r (Hilbert's Nullstellenzatz). Apply φ to conclude that $f^r \in \mathfrak{a}$ and therefore $f \in \sqrt{\mathfrak{a}}$. ∎

If $\mathfrak{m}$ is a maximal ideal of $K[V]$, then $V(\mathfrak{m}) = \{\mathbf{a}\}$ is a one point subset of V. Conversely, for each $\mathbf{a} \in V$, $\mathfrak{m} = \{h \in K[V] \mid h(\mathbf{a}) = 0\}$ is a maximal ideal of $K[V]$.

One should notice that the Zariski topology is very weak. On $\mathbb{A}^1$ itself, for instance, it is just the topology of cofinite sets, the weakest T_1-topology (since any ideal $\mathfrak{a}$ in $K[X]$ is principle – $\mathfrak{a} = \langle f \rangle$ – therefore $V(\mathfrak{a})$ is just the finite set of roots of f). It follows that any bijection $\alpha\colon K \to K$ is continuous, so not all continuous maps are morphisms. In any case this is a very unclassical type of topological space.

We will certainly want to know when two algebraic sets are to be considered isomorphic. More genrally, we will need to define not just the set of all algebraic sets, but the category of algebraic sets (for simplicity, we will stick to the irreducible ones).

EXAMPLE 6.3: *Parabola.* Look at

(5a) $\mathbb{A}^1$, the affine line,

(5b) $Y = X^2$ in $\mathbb{A}^2$, the parabola. Projection of the parabola onto the X-axis should surely be an isomorphism between these algebraic sets.

More generally, if $V \subseteq \mathbb{A}^n$ is an irreducible algebraic set and if $f \in K[\mathbf{X}]$, then the set

$$W = \{(\mathbf{x}, f(\mathbf{x})) \in \mathbb{A}^{n+1} |\ \mathbf{x} \in V\}$$

is an irreducible algebraic set.

Indeed, rewrite W as $\{(\mathbf{x}, x_{n+1}) |\ \mathbf{x} \in V \text{ and } x_{n+1} - f(\mathbf{x}) = 0\}$ to observe that W is an algebraic set. Secondly, $I(V) \subseteq I(W)$ and $X_{n+1} - f(\mathbf{X}) \in I(W)$. Conversely, for $g \in K[X_1, \ldots, X_{n+1}]$ we have

$$(2) \qquad g(\mathbf{X}, X_{n+1}) \cong g(\mathbf{X}, f(\mathbf{X})) \bmod X_{n+1} - f(\mathbf{X}).$$

If $g \in I(W)$, then $g(\mathbf{x}, f(\mathbf{x})) = 0$ for each $\mathbf{x} \in V$. Hence $g(\mathbf{X}, f(\mathbf{X})) \in I(V)$ and therefore $g(\mathbf{X}, X_{n+1}) = g(\mathbf{X}, f(\mathbf{X})) + \big(g(\mathbf{X}, X_{n+1}) - g(\mathbf{X}, f(\mathbf{X}))\big)$ belongs to $\langle I(V), X_{n+1} - f(\mathbf{X})\rangle$. Conclude that $I(W) = \langle I(V), X_{n+1} - f(\mathbf{X})\rangle$. By (2), the maps $h(\mathbf{X}) \mapsto h(\mathbf{X})$ and $g(\mathbf{X}, X_{n+1}) \mapsto g(\mathbf{X}, f(\mathbf{X}))$ define an isomorphism of rings:

$$K[\mathbf{X}]/I(V) \cong K[\mathbf{X}, X_{n+1}]/I(W).$$

Since the left hand side is an integral domain so is the right hand side and therefore $I(W)$ is a prime ideal. Conclude that W is an irreducible algebraic set.

The projection $(\mathbf{x}, x_{n+1}) \mapsto \mathbf{x}$ should define an isomorphism from W onto V.

EXAMPLE 6.4: *Conic.* A **conic** in $\mathbb{P}^2$ is a curve defined by a homogeneous polynomial of degree 2. An irreducible conic C will turn out to be isomorphic to the projective line $\mathbb{P}^1$ under the following map: Fix a point $\mathbf{p}_0 \in C$ which we may assume to be the origin. Identify $\mathbb{P}^1$ with the set of all lines through $\mathbf{p}_0$ in the classical way. Then define a map

$$\varphi\colon \mathbb{P}^1 \to C$$

by letting $\varphi(L)$ for all lines through $\mathbf{p}_0$ be the second point in which L meets C, besides $\mathbf{p}_0$. Also, if L is the tangent line to C at $\mathbf{p}_0$, define $\varphi(L)$ to be $\mathbf{p}_0$ itself (since $\mathbf{p}_0$ is a "double" intersection of C and this tangent line).

EXAMPLE 6.5: *Twisted cubic.* Define an embedding

$$\varphi\colon \mathbb{P}^1 \to \mathbb{P}^3$$

by $\varphi(x) = 1{:}x{:}x^2{:}x^3$: and $\varphi(\infty) = 0{:}0{:}0{:}1$. Check that the image of φ is the algebraic set

$$C = \{\mathbf{x} \in \mathbb{P}^3 |\ x_0x_2 = x_1^2,\ x_1x_3 = x_2^2,\ x_1x_2 = x_0x_3\}$$

The "affine part" of C, i.e., the open subset $C_0 = C - \{0{:}0{:}0{:}1\}$, is defined in $\mathbb{A}^3$ by the equations $Y = X^2$, $XZ = Y^2$ and $Z = XY$. Its projection on the (X, Z)-plance is the cubic $Z = X^3$. Therefore C is called the **twisted cubic**.

The map $(X, Y, Z) \mapsto (X, X^2, X^3)$ defines an isomrphism

$$K[X, Y, Z]/\langle Y - X^2, XZ - Y^2, Z - XY\rangle \cong K[X, X^2, X^3]$$

whose inverse is induced by the map $X \mapsto X$. Hence, $I(C_0)$ is a prime ideal in $K[X, Y, Z]$ and and C_0 is a closed irreducible algebraic subset of $\mathbb{A}^3$. If we prove that C_0 is dense in C, then we may conclude from Corollary 4.2 that C is irreducible in $\mathbb{P}^3$.

Indeed, consider a homogeneous polynomial $f \in K[X_0, X_1, X_2, X_3]$ of degree d such that $f(0, 0, 0, 1) \neq 0$. We have to prove that there exists t such that $f(1, t, t^2, t^3) \neq 0$. In fact, f can be written as

$$f(\mathbf{X}) = \sum c_{\mathbf{i}} X_0^{i_0} X_1^{i_1} X_2^{i_2} X_3^{i_3} + cX_3^d$$

where $\mathbf{i}$ ranges over all 4-tuples $(i_0, i_1, i_2, i_3) \neq (0, 0, 0, 1)$ of nonnegative integers such that $i_0 + i_1 + i_2 + i_3 = d$ and $c_{\mathbf{i}}, c \in K$, $c \neq 0$. For each such $\mathbf{i}$ we have $i_1 + 2i_2 + 3i_3 < d$. Hence $\deg f(1, T, T^2, T^3)$ is d. In particular, this polynomial is nonzero. Conclude that there exists t such that $f(1, t, t^2, t^3) \neq 0$, as desired.

EXAMPLE 6.6: *A cubic curve in* $\mathbb{P}^2$. Let C be a cubic curve in $\mathbb{P}^2$ (C is defined by a form in X_0, X_1, X_2 of degree 3) and let $\mathbf{p}_0 \in C$. For any point $\mathbf{p} \in C$, let L be the line through $\mathbf{p}$ and $\mathbf{p}_0$ And let $\alpha(\mathbf{p})$ be the third point in which L meets C. Although this may not seem as obvious as the previous examples, α will be an automorphism of C of order 2.

Now turn to the problem of actually defining morphisms, and hence isomorphisms, of irreducible algebraic sets. First consider the case of two irreducible affine algebraic sets.

DEFINITION 6.7: *Morphism.* Let $V \subseteq \mathbb{A}^m$ and $W \subseteq \mathbb{A}^n$ be two irreducible algebraic sets. A map

$$\varphi\colon A \to B$$

is called a **morphism** if there exist n polynomials $f_1, \ldots, f_n \in K[X_1, \ldots, X_m]$ such that

$$\varphi(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})) \tag{6}$$

for all points $\mathbf{x} = (x_1, \ldots, x_m) \in V$.

Note one feature of this definition: it implies that every morphism φ from V to W is the restriction of a morphism φ' from $\mathbb{A}^m$ to $\mathbb{A}^n$. Note also that with this definition the map in Example 6.3 above is an **isomorphism**, i.e., both it and its inverse are morphisms.

To analyze the definition further, set

$$R = K[X_1, \ldots, X_m]/I(V) \text{ and } S = K[X_1, \ldots, X_n]/I(W)$$

Then R (resp., S) is just the ring of K-valued functions on V (resp., W) obtained by restricting the ring of polynomial functions on ambient affine space. Suppose that $g \in S$.

Regarding g as a function on W, the definition of morphism implies that the function $g \circ \varphi$ on V is in R – in fact

$$(g \circ \varphi)(X_1, \ldots, X_m) = g\big(f_1(X_1, \ldots, X_m), \ldots, f_n(X_1, \ldots, X_m)\big).$$

Therefore φ induces a K-homomorphism

$$\varphi^* \colon S \to R,$$

$\varphi^*(g) = g \circ \varphi$. Moreover, note that φ is determined by φ^*. This is so because the polynomials $f_1, \ldots, f_n$ can be recovered – up to an element of $I(V)$ – as $\varphi^*(X_1), \ldots, \varphi^*(X_n)$; and the point $\varphi(\mathbf{x})$, for $\mathbf{x} \in V$, is determined via $f_1 \ldots, f_n$ modulo $I(V)$ by the equation (6).

Even more is true. Suppose you start with an arbitrary K-homomorphism

$$\lambda \colon S \to R.$$

Let f_i be a polynomial in $K[X_1, \ldots, X_m]$ such that $\lambda(X_j + I(W)) = f_j + I(V)$, $j = 1, \ldots, n$. Then define a map

$$\varphi' \colon \mathbb{A}^m \to \mathbb{A}^n$$

by

$$\varphi'(x_1, \ldots, x_m) = \big(f_1(x_1, \ldots, x_m), \ldots, f(x_1, \ldots, x_m)\big).$$

If $\mathbf{x} = (x_1, \ldots, x_m) \in V$, then actually $\varphi'(\mathbf{x})$ is in W: for if $g \in I(W)$, then

$$g(\varphi'(\mathbf{x})) = g(f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})).$$

But

$$\begin{aligned} g(f_1, \ldots, f_n) &\cong g\big(\lambda(X_1), \ldots, \lambda(X_n)\big) \bmod I(V) \\ &\cong \lambda(g) \bmod I(V) \\ &\cong 0 \bmod I(V). \end{aligned}$$

Conclude that $g(\varphi'(\mathbf{x})) = 0$ and $\varphi'(\mathbf{x}) \in W$.

We can summarize this discussion in the following:

Proposition 6.8: *If V and W are two irreducible affine algebraic sets, then the set of morphisms from V to W and the set of K-homomorphism from $K[W]$ to $K[V]$ are canonically isomorphic:*

$$\mathrm{Hom}(V, W) \cong \mathrm{Hom}_K\big(K[W], K[V]\big).$$

Corollary 6.9: *If V is an irreducible affine algebraic set, then $K[V]$ is canonically isomorphic to the set of morphisms from V to $\mathbb{A}^1$.*

Proof: Note that $K[\mathbb{A}^1]$ is just $K[X]$. Also the map $\eta \mapsto \eta(X)$ is a K-isomorphism of

$$\mathrm{Hom}_K(K[X], K[V]) \cong K[V]. \quad \blacksquare$$

Even more than Proposition 6.8 is true:

Proposition 6.10: *The assignment*

$$V \mapsto K[V]$$

extends to a contravariant functor:

$$\left\{\begin{array}{c}\textit{Category of irreducible}\\ \textit{algebraic sets +}\\ \textit{morphisms}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\textit{Category of finitely generated}\\ \textit{integral domains over } K \textit{ +}\\ K\textit{-homomorphisms}\end{array}\right\}$$

which is an equivalence of categories.

Proof: Proposition 6.8 asserts that the assignment is a fully faithful functor. The other fact to check is that every finitely generated integral domain R over K occurs as $K[V]$. But every such domain can be represented as

$$R \cong K[X_1, \ldots, X_n]/\langle f_1, \ldots, f_m\rangle,$$

therefore as $K[V]$ where V is the locus of zeros of $f_1, \ldots, f_m$ in K^n. $\blacksquare$

Note that morphisms are continuous maps in the Zariski topology. Indeed, it suffices to prove the statement for each morphism $\varphi\colon \mathbb{A}^m \to \mathbb{A}^n$, given by (6). But if $B = V(g_1, \ldots, g_k)$ is a closed algebraic subset of $\mathbb{A}^n$, then

$$\varphi^{-1}(B) = \{\mathbf{x} \in \mathbb{A}^m \mid g_j(f_1(\mathbf{x}), \ldots, f_k(\mathbf{x})) = 0,\ j = 1, \ldots, k\}$$

is a closed algebraic subset of $\mathbb{A}^m$.

Our simple definition of morphism for affine algebraic sets does not work for projective algebraic sets. The trouble is that it automatically implied that the morphism will extend to a morphism of the ambient space. There is no analogous fact in the projective case. Consider for example the parabola

$$C: \qquad X_0 X_2 = X_1^2$$

Define a map $\varphi_1\colon C \to \mathbb{P}^1$ by $\varphi_1(x_0{:}x_1{:}x_2) = x_1{:}x_2$. The coordinates of the image of φ_1 are homogeneous polynomials in the coordinates of the domain but φ_1 is not defined at the origin 1:0:0. Consider therefore a second map $\varphi_2\colon C \to \mathbb{P}^1$ defined by $\varphi_2(x_0{:}x_1{:}x_2) = x_0{:}x_1$. This map is defined at the origin but is undefined at 0:0:1. If $\mathbf{x}$ is in the common domain of definition of φ_1 and φ_2, then $x_2, x_1 \neq 0$, $x_0/x_1 = x_1/x_2$ and therefore $\varphi_1(\mathbf{x}) = \varphi_2(\mathbf{x})$. Hence φ_1 and φ_2 together define everywhere a map from C to $\mathbb{P}^1$. On the other hand, it will turn out that there are no epimorphism at all from $\mathbb{P}^2$ to $\mathbb{P}^1$.

Thus defining morphisms between projective sets is more subtle. We find that we must define morphisms locally and patch them together. But the problem arises: on which local pieces? We could use the affine algebraic sets $V - H$ where $H \subseteq \mathbb{P}^n$ is a hyperplane. But in general these will not be small enough. We shall need arbitrarily small open sets in the Zariski-topology.

Finally to define morphisms locally, we will need to attach affine coordinate rings to a lot of the Zariski-open sets U and give a definition of affine morphism in terms of local properties. Clearly, we should begin by constructing the apparatus used for defining things locally.

7. Sheaves.

Definition 7.1: *Let X be a topological space. A* **presheaf** $\mathcal{F}$ *on X is a data consists of*

(a1) *a set $\mathcal{F}(U)$ for each open set $U \subseteq X$ and*

(a2) *a map $\operatorname{res}_{U_2,U_1}\colon \mathcal{F}(U_2) \to \mathcal{F}(U_1)$ (called* **restriction***) for each pair of open set $U_1 \subseteq U_2$, such that the following axioms are satisfied:*

(1a) $\operatorname{res}_{U,U} = \operatorname{id}_{\mathcal{F}(U)}$ *for all U and*

(1b) if $U_1 \subseteq U_2 \subseteq U_3$, then $\operatorname{res}_{U_2,U_1} \circ \operatorname{res}_{U_3,U_2} = \operatorname{res}_{U_3,U_1}$.

Definition 7.2: *If $\mathcal{F}_1, \mathcal{F}_2$ are presheaves on X, a* **map** $\varphi\colon \mathcal{F}_1 \to \mathcal{F}_2$ *is a collection of maps $\varphi(U)\colon \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ for each open U such that if $U \subseteq V$, then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{F}_1(V) & \xrightarrow{\varphi(V)} & \mathcal{F}_2(V) \\ {\scriptstyle \operatorname{res}_{V,U}}\downarrow & & \downarrow{\scriptstyle \operatorname{res}_{V,U}} \\ \mathcal{F}_1(U) & \xrightarrow{\varphi(U)} & \mathcal{F}_2(V) \end{array}$$

Definition 7.3: *A presheaf $\mathcal{F}$ is a* **sheaf** *if for each collection $\{U_i\}$ of open sets in X with $U = \bigcup U_i$, the diagram*

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_I \cap U_j)$$

is exact, i.e.: the map

$$\prod_i \operatorname{res}_{U,U_i}\colon \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i)$$

is injective, and its image is the set on which

$$\prod_{i,j} \operatorname{res}_{U_i,U_i\cap U_j}\colon \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

and

$$\prod_{i,j} \operatorname{res}_{U_j,U_i\cap U_j}\colon \prod_i \mathcal{F}(U_j) \longrightarrow \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

agree.

When we pull this high-flown terminology down to earth, it says this:

(2a) If $f_1, f_2 \in \mathcal{F}(U)$ and for all i we have $res_{U,U_i} f_1 = \mathrm{res}_{U,U_i} f_2$, then $f_1 = f_2$. (That is, elements are uniquely determined by local data.)

(2b) If we have a collection of elements $f_i \in \mathcal{F}(U_i)$ such that $\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$ for all i and j, then there is an $f \in \mathcal{F}(U)$ such that $\mathrm{res}_{U,U_i} f = f_i$ for all i. (That is, if we have local data which are compatible, they actually "patch together" to form something in $\mathcal{F}(U)$.)

EXAMPLE 7.4: *Sheaf of continuous functions.* Let X and Y be topological spaces. For each open set $U \subseteq X$, let $\mathcal{F}(U)$ be the set of continuous maps $U \to Y$. This is a presheaf with the restriction maps given by simply restricting maps to smaller sets. It is a sheaf because a function is continuous on $\bigcup U_i$ if and only if its restrictions to each U_i are continuous.

EXAMPLE 7.5: *Presheaf of continuous functions with bounded images.* Let X and Y be topological spaces. For each open subset U of X let $\mathcal{G}(U)$ be the set of continuous functions $U \to Y$ which have relatively compact images. This is a subpresheaf of $\mathcal{F}$, but clearly need not be a sheaf. For example, the function $f(x) = 1/x$ is bounded on each interval $(\varepsilon, 1]$, with $\varepsilon > 0$, but is not bounded on $[0, 1]$.

EXAMPLE 7.6: *Presheaf of locally constant functions.* Let X be a topological space and for each open subset U of X let $\mathcal{F}(U)$ be the vector space of locally constant real-valued functions on U (i.e., functions which are constant on a compact neighborhood of each point), modulo the constant functions on U. This is clearly a presheaf. But every $s \in \mathcal{F}(U)$ goes to zero in $\prod \mathcal{F}(U_i)$ for some open covering $\{U_i\}$, while if U is not connected, $\mathcal{F}(U) \neq 0$. Therefore it is not a sheaf.

DEFINITION 7.7: *Stalks.* Let $\mathcal{F}$ be a sheaf on X and let $x \in X$. The collection of $\mathcal{F}(U)$, U open containing x, is a directed system and we can form the direct limit

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U),$$

called the **stalk** of $\mathcal{F}$ at x.

EXAMPLE 7.8: *Germs of continuous functions.* For each open subset U of a topological space X let $\mathcal{F}(U)$ be the set of continuous functions $U \to \mathbb{R}$. Then $\mathcal{F}_x$ is the set of **germs of continuous functions** at x. It is $\bigcup_{x \in U} \mathcal{F}(U)$ modulo an equivalence relation: $f_1 \sim f_2$ if f_1 and f_2 agree in a neighborhood of x.

EXERCISE 7.9: *Sheafification of a presheaf.* Let $\mathcal{F}_0$ be a presheaf on X. Show that there is a sheaf $\mathcal{F}$ and a map $f\colon \mathcal{F}_0 \to \mathcal{F}$ such that if $g\colon \mathcal{F}_0 \to \mathcal{F}'$ is any map with $\mathcal{F}'$ a sheaf, then there is a unique map $h\colon \mathcal{F} \to \mathcal{F}'$ such that $h \circ f = g$.

DEFINITION 7.10: *Sections.* We may write $\Gamma(U, \mathcal{F})$ for $\mathcal{F}(U)$ and call it the set of **sections** of $\mathcal{F}$ over U. The set $\Gamma(X, \mathcal{F})$ is the set of **global sections** of $\mathcal{F}$. In other contexts we may denote $\mathcal{F}(X)$ by $H^0(X, \mathcal{F})$ and call it the "the zeorth cohomology group" (In those contexts it will be a group, and there will be higher cohomology groups.)

Suppose that for all U, $\mathcal{F}(U)$ is a group (ring, module, etc.) and that all the restriction maps are group (ring, module, etc.) homomorphisms. Then $\mathcal{F}$ is called a **sheaf of groups** (rings, modules, etc.). In this case $\mathcal{F}_x$ is a group (ring, module, etc.).

EXAMPLE 7.11: *Sheaf of rings.* For any topological space X let $\mathcal{F}_{\text{cont},X}(U)$ be the set of all continuous functions $U \to \mathbb{R}$. Then $\mathcal{F}_{\text{cont},X}(U)$ is a sheaf of rings.

Note that if $\varphi\colon X \to Y$ is a continuous function, the operation $f \mapsto f \circ \varphi$ gives for every open $U \subseteq Y$ a map $\mathcal{F}_{\text{cont},Y}(U) \to \mathcal{F}_{\text{cont},X}(\varphi^{-1}(U))$ such that the diagram

$$\begin{array}{ccc} \mathcal{F}_{\text{cont},Y}(U) & \longrightarrow & \mathcal{F}_{\text{cont},X}(\varphi^{-1}(U)) \\ {\scriptstyle\text{res}}\big\downarrow & & \big\downarrow{\scriptstyle\text{res}} \\ \mathcal{F}_{\text{cont},Y}(V) & \longrightarrow & \mathcal{F}_{\text{cont},X}(\varphi^{-1}(V)) \end{array}$$

commutes for all open sets $V \subseteq U$. This set-up is called a **morphism** of the pair $(X, \mathcal{F}_{\text{cont},X})$ to the pair $(Y, \mathcal{F}_{\text{cont},Y})$.

DEFINITION 7.12: *Subsheaf.* A **subsheaf** of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ such that for every open set $U \subseteq X$, $\mathcal{F}'(U)$ is a subset (subgroup, subring, submodule if $\mathcal{F}$ is a sheaf of groups, rings, modules, respectively) of $\mathcal{F}(U)$, and the restriction maps of the sheaf $\mathcal{F}'$ are induced by those of $\mathcal{F}$. It follows that for any point x, the stalk $\mathcal{F}'_x$ is a subset (resp., subgroup, subring, submodule) of $\mathcal{F}_x$.

Definition 7.13: *Restriction of a sheaf to an open subset.* Suppose that Z is an open subset of a topological space X. Let $\mathcal{F}$ be a sheaf over X. Define the **restriction** $\mathcal{F}\big|_Z$ of $\mathcal{F}$ to Z to be the sheaf such that for each open subset U of Z

$$\mathcal{F}\big|_Z(U) = \mathcal{F}(U).$$

8. Affine varieties.

Let $X \subseteq \mathbb{A}^n$ be an irreducible algebraic set, R its coordinate ring. Since X is irreducible, $I(X)$ is prime and R is an integral domain. Let F be its field of fractions. Recall that R has been identified with the ring of all polynomial functions on X. For $\mathbf{x} \in X$, $\mathfrak{m}_{\mathbf{x}} = \{f \in R|\ f(\mathbf{x}) = 0\}$ is a maximal ideal. The local ring $R_{\mathfrak{m}_{\mathbf{x}}}$ is called the **local ring** of X at $\mathbf{x}$ and is denoted by $\mathcal{O}_{\mathbf{x},X}$ or also by $\mathcal{O}_{\mathbf{x}}$ if X is clear from the context. By definition

$$\mathcal{O}_{\mathbf{x}} = \{f/g|\ f, g \in R,\ g(\mathbf{x}) \neq 0\}$$

and $R \subseteq \mathcal{O}_{\mathbf{x}} \subseteq F$. For U open in X, let

$$\mathcal{O}_X(U) = \bigcap_{\mathbf{x} \in U} \mathcal{O}_{\mathbf{x}}.$$

All the $O_X(U)$ are subrings of F. If $U \subseteq V$, then $\mathcal{O}_X(U) \supseteq \mathcal{O}_X(V)$. If we take the inclusion as the restriction map, this defines a sheaf $\mathcal{O}_X$ called the **structure sheaf** of X. We verify only axiom (2b) of Section 7: Let $U = \bigcup_i U_i$ be open subsets of X. Suppose that for each i, f_i is an element of $\mathcal{O}_X(U_i)$ such that $f_i = f_j$ for each i, j. The element $f = f_i$ belongs to $\mathcal{O}_{\mathbf{x}}$ for each $\mathbf{x} \in U_i$ and for each i. Hence $f \in \mathcal{O}_{\mathbf{x}}$ for each $\mathbf{x} \in U$, that is $f \in \mathcal{O}_X(U)$. We shall therefore denote $\mathcal{O}_X(U)$ also by $\Gamma(U, \mathcal{O}_X)$.

The elements of $\Gamma(U, \mathcal{O}_X)$ can be viewed as functions on U. Say $h \in \Gamma(U, \mathcal{O}_X)$, and $\mathbf{x} \in U$. Then $h \in \mathcal{O}_{\mathbf{x}}$, so we can write $h = f/g$ with $f, g \in R$ and $g(\mathbf{x}) \neq 0$. We then define $h(\mathbf{x}) = f(\mathbf{x})/g(\mathbf{x})$. This definition is independent of f and g. To prove that this identification is faithful suppose that $h(\mathbf{x}) = 0$ for each $\mathbf{x} \in U$. Choose $\mathbf{x} \in U$ and write $h = f/g$ with $g(\mathbf{x}) \neq 0$. Then $f(\mathbf{x}) = 0$. If $\mathbf{x}'$ is another point of U, then $h = f'/g'$ with $g'(\mathbf{x}') \neq 0$ and $f'(\mathbf{x}') = 0$. From $fg' = f'g$ deduce that $f(\mathbf{x}')g'(\mathbf{x}') = f'(\mathbf{x}')g(\mathbf{x}') = 0$. Hence $f(\mathbf{x}') = 0$. Since U is dense in X (Lemma 4.1(c)) and f is continuous f vanishes on X. Conclude that $f = 0$ and therefore that $h = 0$.

PROPOSITION 8.1: *Let X be an irreducible algebraic set and let R be its coordinate ring. For $f \in R$ let $X_f = \{\mathbf{x} \in X|\ f(\mathbf{x}) \neq 0\}$ and let $R_f = \{g/f^n|\ g \in R,\ n \in \mathbb{N}\}$. Then $\Gamma(X_f, \mathcal{O}_X) = R_f$.*

Proof: By definition $\Gamma(X_f, \mathcal{O}_X) = \bigcap_{f(\mathbf{x}) \neq 0} \mathcal{O}_x$. Hence, $R_f \subseteq \Gamma(X_f, \mathcal{O}_X)$.

Conversely, suppose that $u \in \Gamma(X_f, \mathcal{O}_X)$. Then $\mathfrak{b} = \{h \in R \mid hu \in R\}$ is an ideal of R. If we prove that f vanishes on $V(\mathfrak{b})$, then, by the Nullstellensatz, $f_r \in \mathfrak{b}$ for some positive integer r and therefore $u \in R_f$.

Indeed, if $\mathbf{x} \in X$ and $f(\mathbf{x}) \neq 0$, then $u \in \mathcal{O}_{\mathbf{x}}$. Hence, $u = g/h$ with $g, h \in R$ and $h(\mathbf{x}) \neq 0$. In particular $h \in \mathfrak{b}$ and $\mathbf{x}$ is not a zero of $\mathfrak{b}$. ∎

In particular for $f = 1$ we have $X_f = X$ and $R_f = R$.

Corollary 8.2: $\Gamma(X, \mathcal{O}_X) = R$.

Remark 8.3: (a) Assume that $f \in \Gamma(U, \mathcal{O}_X)$ and that f vanishes nowhere on U. Then $1/f \in \Gamma(U, \mathcal{O}_X)$.

Indeed, $f(\mathbf{x}) \neq 0$ implies that $1/f \in \mathcal{O}_{\mathbf{x}}$.

(b) The stalk of $\mathcal{O}_X$ at $\mathbf{x}$ is the local ring $\mathcal{O}_{\mathbf{x}}$.

Indeed, since the sets X_f, $f \in R$, form a basis of the Zariski topology of X and since the restrictions maps are inclusions, the stalk is

$$\varinjlim_{\mathbf{x} \in U} \Gamma(U, \mathcal{O}_X) = \bigcup_{\mathbf{x} \in U} \Gamma(U, \mathcal{O}_X) = \bigcup_{f(\mathbf{x}) \neq 0} \Gamma(X_f, \mathcal{O}_X) = \bigcup_{f(\mathbf{x}) \neq 0} R_f = \mathcal{O}_{\mathbf{x}},$$

by Proposition 8.1.

(c) The field F can also be recovered from the sheaf $\mathcal{O}_X$. Recall that since X is irreducible, the intersection of any two nonempty open sets is nonempty. Hence, the collection of nonempty open sets forms a direct system and therefore we may define a **generic stalk** of any sheaf over X as the corresponding direct limit. In particular the generic stalk of $\mathcal{O}_X$ is the union of all local rings $\mathcal{O}_{\mathbf{x}}$, i.e., the field F itself.

(d) If $h \in \Gamma(U, \mathcal{O}_X)$ for some open $U \subseteq X$, then it need not be true that $h = f/g$, with $f, g \in R$ and g vanishing nowhere on U.

For example, let $V \subseteq \mathbb{A}^4$ be $V(XW - YZ)$, let x, y, z, w be the images of X, Y, Z, W modulo $I(V)$. Consider the open set $U = V_y \cup V_w$. Define a function $h \in \Gamma(U, \mathcal{O}_V)$ in the following way: $h = x/y$ on V_y, and $h = z/w$ on V_w. Using the notion of dimension we will show in Section ? that h is not equal to f/g with $g \neq 0$ on U.

Proposition 8.1 shows that this is true however if U has the form X_g. ∎

Proposition 8.4: *Let $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$ be irreducible algebraic sets, and let $\varphi\colon X \to Y$ be continuous map. The following conditions are equivalent:*

(a) *φ is a morphism,*

(b) *for all $g \in \Gamma(Y, \mathcal{O}_Y)$, $g \circ \varphi \in \Gamma(X, \mathcal{O}_X)$,*

(c) *for all open $U \subseteq Y$ and $g \in \Gamma(U, \mathcal{O}_Y)$ we have $g \circ \varphi \in \Gamma(\varphi^{-1}(U), \mathcal{O}_X)$, and*

(d) *for all $\mathbf{x} \in X$ and $g \in \mathcal{O}_{\varphi(\mathbf{x})}$ we have $g \circ \varphi \in \mathcal{O}_{\mathbf{x}}$.*

Proof: Condition (b) is a special case of condition (c).

To prove that (d) implies (c) let U be an open subset of Y and $g \in \Gamma(U, \mathcal{O}_Y)$. Then $\varphi^{-1}(U)$ is an open subset of X and $g \in \mathcal{O}_{\varphi(\mathbf{x})}$ for each $\mathbf{x} \in \varphi^{-1}(U)$. Hence, by (d), $g \circ \varphi \in \bigcap_{\mathbf{x} \in \varphi^{-1}(U)} \mathcal{O}_{\mathbf{x}} = \Gamma(\varphi^{-1}(U), X)$.

The equivalence of (a) and (b) is essentially proved in Proposition 6.6.

Finally, to prove that (b) implies (d) let $\mathbf{x} \in X$ and $g \in \mathcal{O}_{\varphi(\mathbf{x})}$. Then $g = p/q$ with $p, q \in \Gamma(Y, \mathcal{O}_Y)$ and $(q \circ \varphi)(\mathbf{x}) = q(\varphi(\mathbf{x})) \neq 0$. By (b), $p \circ \varphi$, $q \circ \varphi \in \mathcal{O}_{\mathbf{x}}$. Hence, $g \circ \varphi = p \circ \varphi / q \circ \varphi \in \mathcal{O}_{\mathbf{x}}$. ∎

This shows, among other things, that our sheaf gives us all the information we need for defining morphisms. We are ready, then to cut loose from the ambient spaces.

Definition 8.5: *An* **affine variety** *is a topological space X plus a sheaf of K-valued functions $\mathcal{O}_X$ on X which is isomorphic to the structure sheaf of an irreducible algebraic subset of some $\mathbb{A}^n$.*

In particular $\mathbb{A}^n$ together with its structure sheaf is called the **affine n-space** *and is also denoted by $\mathbb{A}^n$.*

Consider the following statement:

Bijective morphisms are isomorphisms.

This statement is correct, for example, in the category of compact Hausdorff topological spaces and the category of Banach spaces (the open map theorem). On the other hand, it is false for differential manifolds – consider the map $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = x^3$.

The statement is also false in the category of affine varieties: A homeomorphism $\varphi\colon X_1 \to X_2$ may well correspond to an isomorphism of the ring of X_2 with a proper subring of the ring of X_1. Here are three key examples to bear in mind.

EXAMPLE 8.6: *Frobenius map.* Let $\mathrm{char}(K) = p \neq 0$. Define a morphism $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^1$ by $\varphi(t) = t^p$. This is bijective. On the ring level, this corresponds to the inclusion map in the pair or rings: $K[X^p] \hookrightarrow K[X]$. Since these rings are not equal φ is not an isomorphism.

EXAMPLE 8.7: Let K be any algebraically closed field. Define the morphism $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^2$ by $\varphi(t) = (t^2, t^3)$. The image of this morphism is the irreducible closed curve

$$C : \qquad X^3 = Y^2.$$

The morphism φ from $\mathbb{A}^1$ to C is a bijection which corresponds to the inclusion map in the pair of rings: $K[T^2, T^3] \hookrightarrow K[T]$. These rings are not equal, so φ is not an isomorphism.

EXAMPLE 8.8: Define $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^2$ by $x = t^2 - 1$, $y = t(t^2 - 1)$. It is not hard to check that the image of this morphism is the curve

$$D : \qquad Y^2 = X^2(X + 1).$$

(Simply note that one can solve for the coordinate t of the point in $\mathbb{A}^1$ by the equation $t = y/x$.) Also, φ is bijective between $\mathbb{A}^1$ and D except that both the points $t = -1$ and $t = 1$ are mapped to the origin. Let $X_1 = \mathbb{A}^1 - \{1\}$, an affine variety with coordinate ring $K[T, (T - 1)^{-1}]$ (Lemma 8.18 or better Proposition 9.2). Then φ restricts to a bijection φ' from X_1 to D. This morphism corresponds to the inclusion in the pairs of rings

$$K[T^2 - 1, T(T^2 - 1)] \hookrightarrow K[T, (T - 1)^{-1}].$$

Since these rings are unequal, φ' is not an isomorphism.

9. Open and closed sets of affine varieties.

We study in this section the induced variety structure on open and closed subsets of affine varieties.

Let Y be an irreducible closed subset of an affine variety $(X, \mathcal{O}_X)$. (Irreducible, now, in the sense given by the topology on X.) Define an induced sheaf of functions $\mathcal{O}'_Y$ on Y as follows: If V is open in Y, then

$$\mathcal{O}'_Y(V) = \left\{ f\colon V \to K \;\middle|\; \begin{array}{c} \text{For all } \mathbf{x} \in V \text{ there exists a neighborhood } U \text{ of} \\ \mathbf{x} \text{ in } X \text{ and a function } f' \in \Gamma(U, \mathcal{O}_X) \text{ such that} \\ \text{the restrictions of } f \text{ and } f' \text{ to } U \cap V \text{ coincide} \end{array} \right\}$$

We varify only axiom (2b) of Section 7 for $\mathcal{O}'_Y$. Suppose that $V = \bigcup_i V_i$ is a union of open subsets of V and for each i we are given $f_i \Gamma(V, \mathcal{O}'_Y)$ such that f_i and f_j coincide on $V_i \cap V_j$. We show that the unique function $f\colon V \to K$ that coincides with f_i on V_i, for each i, belongs to $\mathcal{O}'_Y(V)$. Indeed, let $x \in V$. Take i such that $x \in V_i$. By assumption x has a neighborhood U_i in X and there exists $f_i \in \Gamma(U_i, \mathcal{O}_X)$ that coincides with f on $U_i \cap V_i$. Also, $V_i = Y \cap U'_i = V \cap U'_i$ for some open subset U'_i of X. Let f' be the restriction of f_i to $U_i \cap U'_i$. Then $f' \in \Gamma(U_i \cap U'_i, \mathcal{O}_X)$ and f' coincides with f on $V \cap U_i \cap U'_i$. Conclude that $f \in \mathcal{O}'_Y(V)$.

Proposition 9.1: *$(Y, \mathcal{O}'_Y)$ is an affine variety.*

Proof: Without loss we may assume that $X \subseteq \mathbb{A}^n$. Then Y is an irreducible closed subset of $\mathbb{A}^n$ and therefore $(Y, \mathcal{O}_Y)$ is an affine variety. We prove, that under this identification $(Y, \mathcal{O}'_Y) = (Y, \mathcal{O}_Y)$. Indeed, for each open set V of Y we identify $\mathcal{O}'_Y(V)$ with a subset of $K(Y)$ and show that under this identification $\mathcal{O}'_Y(V) = \Gamma(V, \mathcal{O}_Y)$.

Let $f \in \mathcal{O}'_Y(V)$. Choose $\mathbf{x} \in V$. By assumption there is an open set $U \subseteq X$ that contains $\mathbf{x}$ and $g \in \Gamma(U, \mathcal{O}_X)$ that coincides with f on $U \cap V$. In particular $g \in \mathcal{O}_{\mathbf{x},X} \subseteq \mathcal{O}_{\mathbf{x},Y}$. If $\mathbf{x}'$ is another point of V, U' is an open neighborhood of $\mathbf{x}'$ in X, and $g' \in \Gamma(U', \mathcal{O}_X)$ is a function that coincides with f on $V \cap U'$, then g and g' coincide on $V \cap U \cap U'$. Since the latter set is nonempty and open in Y the elements $\mathrm{res}_{V\cap U}\, g$ and $\mathrm{res}_{V\cap U'}\, g'$ of $K(Y)$ are the same. We may therefore identify f with $\mathrm{res}_{V\cap U}\, g$. In particular $f \in \mathcal{O}_{\mathbf{x},Y}$ for every $\mathbf{x} \in V$. Conclude that $f \in \Gamma(V, \mathcal{O}_Y)$.

Conversely, if $f \in \Gamma(V, \mathcal{O}_Y)$ and $\mathbf{x} \in V$, then $f \in \mathcal{O}_{\mathbf{x},Y}$. This means that $f = g/h$ where $g, h \in \Gamma(Y, \mathcal{O}_Y)$ and $h(\mathbf{x}) \neq 0$. It follows that there exists polynomial functions g', h' on $\mathbb{A}^n$ whose restrictions to Y is g, h, respectively. In particular $\mathbf{x} \in X_{h'}$. Denote the restriction of g'/h' to $X_{h'}$ by f'. Then $f' \in \Gamma(X_{h'}, \mathcal{O}_X)$ and the restriction of f' to $X_{h'} \cap V$ is f. Conclude that $f \in \mathcal{O}'_Y(V)$. ∎

If $(X, \mathcal{O}_X)$ is an affine variety and U is an open subset of X, then the restriction of $\mathcal{O}_X$ to U is a sheaf over U which we have denoted by $(U, \mathcal{O}_X\big|_U)$. By definition 7.13, the ring of functions associated with an open subset of V of U is $\Gamma(V, \mathcal{O}_X\big|_U) = \Gamma(V, \mathcal{O}_X)$. There is one important case where this sheaf is an affine variety.

Proposition 9.2: *Let $(X, \mathcal{O}_X)$ be an affine variety, and let $f \in \Gamma(X, \mathcal{O}_X)$. Then $(X_f, \mathcal{O}_X\big|_{X_f})$ is an affine variety.*

Proof: Identify $(X, \mathcal{O}_X)$ with $(V, \mathcal{O}_V)$ where V is an irreducible algebraic set in $\mathbb{A}^n$. Then f is the restriction to X of a polynomial $f \in K[\mathbf{X}]$, with $\mathbf{X} = (X_1, \dots, X_n)$. If f vanishes at each point of V, then $f = 0$ in $\Gamma(V, \mathcal{O}_f)$, V_f is empty and the proposition holds. So assume that $f \notin I(V)$.

Let $\mathbf{x} = (x_1, \dots, x_n)$ where $x_i = X_i + I(V)$, $i = 1, \dots, n$. Then $K[\mathbf{x}] = \Gamma(V, \mathcal{O}_V)$ is the coordinate ring of V. Since V is irreducible, $K[\mathbf{x}]$ is an integral domain with quotient field F, and $f(\mathbf{x}) \neq 0$.

Let $\mathfrak{b}$ be the ideal of $K[\mathbf{X}, X_{n+1}]$ generated by $I(V)$ and $X_{n+1}f(\mathbf{X}) - 1$. Consider the algebraic subset $W = V(\mathfrak{b})$ of $\mathbb{A}^{n+1}$. The projection $(\mathbf{a}, a_{n+1}) \mapsto \mathbf{a}$ maps W continuously and bijectively onto V_f. Denote its restriction to W by φ. If $h \in K[\mathbf{X}, X_{n+1}]$ is a polynomial of degree d in X_{n+1}, then $h'(\mathbf{X}) = f(\mathbf{X})^d h(\mathbf{X}, f(\mathbf{X})^{-1}) \in K[\mathbf{X}]$ and $\varphi(W_h) = V_f \cap V'_h$. Hence φ is also an open map and therefore a homeomorphism of W onto V_f. Since V_f is open in V it is irreducible (Lemma 4.1) and therefore W is also irreducible. We prove that

$$(V_f, \mathcal{O}_V\big|_{V_f}) \cong (W, \mathcal{O}_W). \tag{1}$$

and thereby prove that $(V_f, \mathcal{O}_V\big|_{V_f})$ is an affine variety.

First we identify $\Gamma(W, \mathcal{O}_W)$ with $\Gamma(V_f, \mathcal{O}_V)$: The map $(\mathbf{X}, X_{n+1}) \mapsto (\mathbf{x}, f(\mathbf{x})^{-1})$ extends to an epimorphism

$$\alpha\colon K[\mathbf{X}, X_{n+1}] \to K[\mathbf{x}, f(\mathbf{x})^{-1}]$$

whose kernel contains $\mathfrak{b}$. Conversely, suppose that $g \in K[\mathbf{X}, X_{n+1}]$ and $g(\mathbf{x}, f(\mathbf{x})^{-1}) = 0$. Write g as $g(\mathbf{X}, X_{n+1}) = \sum_{i=0}^{d} g_i(\mathbf{X}) X_{n+1}^i$. Then

$$\begin{aligned} g(\mathbf{X}, X_{n+1}) &= g(\mathbf{X}, f(\mathbf{X})^{-1}) + \big[g(\mathbf{X}, X_{n+1}) - g(\mathbf{X}, f(\mathbf{X})^{-1})\big] \\ &= g(\mathbf{X}, f(\mathbf{X})^{-1}) + \sum_{i=0}^{d} g_i(\mathbf{X})(X_{n+1}^i - f(\mathbf{X})^{-i}) \\ &= g(\mathbf{X}, f(\mathbf{X})^{-1}) + (X_{n+1} - f(\mathbf{X})^{-1}) h(\mathbf{X}, X_{n+1}, f(\mathbf{X})^{-1}) \end{aligned}$$

where $h \in K[\mathbf{X}, X_{n+1}, X_{n+2}]$ has degree $d-1$ in X_{n+2}. Multiply this equation by $f(\mathbf{X})^d$:

$$f(\mathbf{X})^d g(\mathbf{X}, X_{n+1}) = g_1(\mathbf{X}) + (X_{n+1} f(\mathbf{X}) - 1) h_1(\mathbf{X}, X_{n+1}),$$

where $g_1(\mathbf{X}) = f(\mathbf{X})^d g(\mathbf{X}, f(\mathbf{X})^{-1})$ vanishes on V and $h_1 \in K[\mathbf{X}]$. Hence $f(\mathbf{X})^d g(\mathbf{X}, X_{n+1}) \cong 0 \bmod \mathfrak{b}$, and therefore $\big(X_{n+1} f(\mathbf{X})\big)^d g(\mathbf{X}, X_{n+1}) \cong 0 \bmod \mathfrak{b}$. But $X_{n+1} f(\mathbf{X}) \cong 1 \bmod \mathfrak{b}$. Hence $g \in \mathfrak{b}$. Conclude that $\mathfrak{b} = \mathrm{Ker}(\alpha)$. Hence $\mathfrak{b}$ is a prime ideal and therefore $\mathfrak{b} = I(W)$ and $\Gamma(W, \mathcal{O}_W) = K[\mathbf{X}, X_{n+1}]/I(W) = K[\mathbf{x}, f(\mathbf{x})^{-1}]$. By Proposition 8.1, this ring is isomorphic to $\Gamma(V_f, \mathcal{O}_V)$.

For each open subset U of V_f let $U' = \{(\mathbf{a}, f(\mathbf{a})^{-1}) |\ \mathbf{a} \in U\}$ be the corresponding open subset of U. We have identified $\Gamma(U, \mathcal{O}_V\big|_{V_f}) = \Gamma(U, \mathcal{O}_V)$ and $\Gamma(U', \mathcal{O}_W)$ with subrings of $K(V)$. To prove (1) it suffices to prove that the two rings are equal.

Indeed, $\Gamma(U, \mathcal{O}_V) = \bigcap_{\mathbf{a} \in U} \mathcal{O}_{\mathbf{a}, V}$ and $\Gamma(U', \mathcal{O}_W) = \bigcap_{\mathbf{a} \in U} \mathcal{O}_{(\mathbf{a}, f(\mathbf{a})^{-1}), W}$. So the equality of the above rings follows from the equality

$$\mathcal{O}_{\mathbf{a}, V} = \mathcal{O}_{(\mathbf{a}, f(\mathbf{a})^{-1}), W}.$$

which obviously holds for each $\mathbf{a} \in V_f$. ∎

What we have done to get X_f is to push the zeros of f out to infinity. For example, suppose $X = \mathbb{A}^1$ and f is the coordinate X_1. Then $\mathfrak{b} = \langle 1 - X_1 X_2 \rangle$ gives the hyperbola. Projection of the hyperbola down to the axis is an isomorphism with X_f.

EXAMPLE 9.3: *An open subset of an affine variety which is not affine.* We prove that the open subset $A = \mathbb{A}^2 - \{(0,0)\}$ of $\mathbb{A}^2$ is not affine.

Indeed, suppose that

$$\varphi\colon (A, \mathcal{O}_{\mathbb{A}^2}\big|_A) \to (W, \mathcal{O}_W),$$

is an isomorphism, where W is an irreducible algebraic subset of $\mathbb{A}^n$, for some positive integer n. Let π_i be the projection of $\mathbb{A}^n$ on the ith coordinate. Then $f_i = \pi_i \circ \varphi$ is an element of $\Gamma(A, \mathcal{O}_A)$, i.e., an element of $K(X,Y)$ which belongs to the local ring of each point of A (We then say that f is defined at each point of A.)

CLAIM: If $f \in K(X,Y)$ is defined at each point of A, then $f \in K[X,Y]$.

Indeed, let $f = g/h$, where $g, h \in K[X,Y]$ are relatively prime. If h is not a constant, then, say $\deg_X h > 0$. Since $K(X)[Y]$ is a Euclidean domain there exist $p, q \in K[X,Y]$ and $r \in K[X]$, $r \neq 0$, such that

$$p(X,Y)g(X,Y) + q(X,Y)h(X,Y) = r(X). \tag{2}$$

If r were not relatively prime to h, then r had a zero x such that $X - x$ divides h. For each $y \neq 0$ there are by assumption $g', h' \in K[X,Y]$ such that $f = g'/h'$ and $h'(x,y) \neq 0$. From $gh' = g'h$ conclude that $g(x,y) = 0$. Hence $X - x$ divides each coefficient of $g(X,Y)$ and therefore g itself. But, as g and h are relativly prime, this is a contradiction. Conclude that r is relatively prime to h.

As $K(Y)[Y]$ is Euclidean there exist $u, v \in K[X,Y]$ and $w \in K[Y]$, $w \neq 0$ such that

$$u(X,Y)r(X) + v(X,Y)h(X,Y) = w(Y). \tag{3}$$

Now choose $y \in K$ such that $\deg_X h(X,y) = \deg_X h(X,Y) > 0$ and $w(y) \neq 0$. Then choose $x \in K$ such that $h(x,y) = 0$. It follows from (3) that $r(x) \neq 0$ and hence, by (2), $g(x,y) \neq 0$.

By assumption $f = g'/h'$ where $g', h' \in K[X,Y]$ and $h'(x,y) \neq 0$. But then $g(x,y)h'(x,y) = g'(x,y)h(x,y)$. As the left hand side is nonzero while the right hand side is zero we get a contradiction, and our claim is true.

It follow that f_i is a polynomial, $i = 1, \ldots, n$. Hence φ extends to a continuous map $\psi\colon \mathbb{A}^2 \to \mathbb{A}^n$. As W is closed and $(0,0) \in \overline{A}$, we have $\psi(0,0) \in W$. Let $(a,b) \in A$ be the unique point such that $\varphi(a,b) = \psi(0,0)$. Assume for example that $b \neq 0$. Consider the open subset $A_0 = \{(x,y) \in \mathbb{A}^2 |\ y \neq 0\}$. Since $\varphi\colon A \to W$ is a homeomorphism $W_0 = \varphi(A_0)$ is an open subset of W. But then $A_1 = A_0 \cup \{(0,0)\} = \psi^{-1}(W_0)$ is an open subset of $\mathbb{A}^2$. It follows that for $\mathbb{A}^1 = \{(x,0)|\ x \in K\}$ the intersection $A_1 \cap \mathbb{A}^1 = \{(0,0)\}$ is a nonempty open set. However such a set must be cofinite, and we have a contradiction. ∎

10. Prevarieties.

Let $\mathcal{O}_X$ be a sheaf of K-valued functions over a topological space X. An open subset U of X is called an **open affine set** if $(U, \mathcal{O}_X\big|_U)$ is an affine variety.

The pair $(X, \mathcal{O}_X)$ is a **prevariety** if

(1a) X is connected, and

(1b) there is a finite covering $\{U_i\}$ of X by affine open sets U_i.

Note that the open affine sets form then a basis of the topology. In fact, we know from Proposition 9.2 that this is true within each U_i and they cover X.

Proposition 10.1: *Every prevariety X is an irreducible topological space. In particular every open set is dense.*

Proof: Let U and V be an open and nonempty in X. We have to prove that $U \cap V \neq \emptyset$.

Denote the union of all open affine sets that meets V by U_1. We claim that $U_1 = X$.

Otherwise the union U_2 of all open affine sets which are disjoint from V is nonempty. As $U_1 \cup U_2 = X$ and X is connected $U_1 \cap U_2 \neq \emptyset$. Let $\mathbf{y} \in U_1 \cap U_2$. Then there are affine open sets W_1, W_2 containing $\mathbf{y}$, such that $W_1 \cap V \neq \emptyset$ and $W_2 \cap V = \emptyset$. But then $W_1 \cap V$ and $W_1 \cap W_2$ are nonempty open sets in the affine set W_1 and therefore they meet. In particular $W_2 \cap V = \emptyset$. This contradiction proves the claim.

Now let $\mathbf{x} \in U$. By the claim, there is an affine open set W containing x and meeting V. Then both $V \cap W$ and $U \cap W$ are nonempty open sets in the affine set W, so $V \cap W \cap U \neq \emptyset$. A fortiori $U \cap V \neq \emptyset$.

Conclude that X is irreducible. ∎

Proposition 10.2: *If X is a prevariety, then X is a Noetherian space. In particular X is compact.*

Proof: Let $Z_1 \supseteq Z_2 \supseteq Z_3 \cdots$ be a descending sequence of closed sets. Since $\{Z_i \cap U\}$ is stationary for each open affine set U (Corollary 4.9) and since X has an open affine covering $\{Z_i\}$ is stationary. Conclude that X is Noetherian.

Let X be a prevariety. The **function field** $K(X)$ is the generic stalk of $\mathcal{O}_X$, i.e.,

$$K(X) = \varinjlim \Gamma(U, \mathcal{O}_X),$$

where U ranges over all nonempty subsets of X. Since each two of them has a nonempty intersection the directed limit makes sense. Moreover, for each open affine set U in X, the open subsets of U are cofinal. Hence, $K(X) = K(U)$. In particular, this shows that $K(X)$ is really a field. The elements of $K(X)$ are called **rational functions on** X, although they are, strictly speaking, only functions on open dense subsets of X.

Every open set U is covered by a finite set of open affine sets U_i. If $f \in \Gamma(U, \mathcal{O}_X)$, then its restriction f_i to U_i is a polynomial function into K (whose "number of variables" depends on i) and it belongs to $K(X)$. Hence f is a function into K and it belongs to $K(X)$.

Another type of directed limit over $\Gamma(U, \mathcal{O}_X)$'s is sometimes very useful. This is intermediate between the limit that lead to $\mathcal{O}_x$ and that that leads to $K(X)$. Let Y be an irreducible subset of X. Set:

$$\mathcal{O}_{Y,X} = \varinjlim \Gamma(U, \mathcal{O}_X),$$

where U ranges over all open sets that intersect Y.

To express more simply the ring which we get in this way, fix one open affine set V which intersects Y. Let R be the coordinate ring of V and $\mathfrak{p} = I(Y \cap V)$ the ideal of all $f \in R$ which vanish on $Y \cap V$. If U is an arbitrary open set that intersects Y, then, since Y is irreducible, $Y \cap U \cap V \neq \emptyset$. Take an open subset of $Y \cap U \cap V$ of the form V_f. If $\mathbf{x} \in V_f$, then $f(\mathbf{x}) \neq 0$. Thus f does not vanish on all $Y \cap V$ and therefore $f \notin \mathfrak{p}$. It follows that

$$\mathcal{O}_{Y,X} = \varinjlim \Gamma(V_f, \mathcal{O}_X), \qquad f \text{ ranges on } R - \mathfrak{p}.$$

By Lemma 8.1, $\Gamma(V_f, \mathcal{O}_X) = \Gamma(V_f, \mathcal{O}_V) = R_f$. Hence

$$\mathcal{O}_{Y,X} = R_\mathfrak{p}.$$

In particular $\mathcal{O}_{Y,X}$ is a local ring with quotient field $K(X)$. Its residue field is $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = K(Y \cap V) = K(Y)$.

If we identify each $\Gamma(U, \mathcal{O}_X)$ with a subring of $K(X)$, and consider $K(X)$ as a field of functions on X, then $\mathcal{O}_{Y,X}$ is the ring of all functions in $K(X)$ which are defined on an open dense subset of Y and the maximal ideal $\mathfrak{m}_{Y,X}$ of $\mathcal{O}_{Y,X}$ is the ideal of all functions in $K(X)$ which are defined and vanishes on an open dense subset of Y.

PROPOSITION 10.3: *An open subset of a prevariety is a prevariety.*

Proof: Let U be an open subset of a variety X. By Corollary 4.3, U is irreducible and therefore connected. Also, U is a union of affine open subsets. Since X is Noetherian, U is a union of finitely many open affine subsets. Thus, U is a prevariety. ∎

Now let Y be a closed irreducible subset of a prevariety X. The sheaf $\mathcal{O}_X$ induces a sheaf $\mathcal{O}_Y$ on Y. For V open in Y:

$$\Gamma(V, \mathcal{O}_Y) = \left\{ f\colon V \to K \;\middle|\; \begin{array}{c} \text{For all } \mathbf{x} \in V \text{ there exists a neighborhood } U \text{ of} \\ \mathbf{x} \text{ in } X \text{ and a function } f' \in \Gamma(U, \mathcal{O}_X) \text{ such that} \\ \text{the restrictions of } f \text{ and } f' \text{ to } U \cap V \text{ coincide} \end{array} \right\}$$

PROPOSITION 10.4: *If Y is a closed irreducible subset of a prevariety X, then $(Y, \mathcal{O}_Y)$ is a prevariety.*

Proof: As an irreducible set Y is connected. Secondly, let $\{U_i\}$ be a finite open affine covering of X. Then, by Proposition 9.1, $\{Y \cap U_i\}$ is an open affine covering of Y. Thus, $(Y, \mathcal{O}_Y)$ is a prevariety. ∎

A **locally closed** subset Z of a topological space X is an intersection of open and closed subsets of X. Alternatively, Z in locally closed in X if, for every point $z \in Z$, there exists an open neighborhood U_z of z in X such that $U_z \cap Z$ is closed in U_z, i.e., $U_z \cap Z = U_z \cap C_z$ for some closed subset C_z in X. Indeed, in this case $Z = \bigcup_{z\in Z} U_z \cap C_z$. Let $C = \bigcap_{z\in Z} C_z \cup (X - U_z)$ and $U = \bigcup_{z\in Z} U_z$. Then C is closed, U is open, and $Z = U \cap C$.

Combining Propositions 10.3 and 10.4, we can now give a prevariety structure to every locally closed irreducible subset of a prevariety X. Each variety obtained in this way is called **sub-prevariety** of X.

EXAMPLE 10.5: *The projective line.* Take two copies U and V of $\mathbb{A}^1$. Let u, v be the coordinates on these two affine lines. Let $U_0 \subseteq U$ (resp. $V_0 \subseteq V$) be defined by

$u \neq 0$ (resp. $v \neq 0$). Then $\Gamma(U, \mathcal{O}_U) = K[u]$, so $\Gamma(U_0, \mathcal{O}_U) = K[u, u^{-1}]$. Similarly $\Gamma(V_0, \mathcal{O}_V) = K[v, v^{-1}]$. Define a map $\varphi\colon U_0 \to V_0$ by $\varphi(a) = a^{-1}$. This gives a map $\varphi^*\colon K[v, v^{-1}] \to K[u, u^{-1}]$ taking v to u^{-1} and v^{-1} to u. As φ^* is an isomorphism of rings, φ is an isomorphism of varieties (φ has an inverse). Now we patch together U and V via φ, i.e., we form $U \cup V$ with U_0 and V_0 identified via φ. This has a sheaf structure on it in the obvious way, and is a prevariety. The space is homeomoprhic to $\mathbb{P}^1$ (say $u \mapsto 1 : u$ and $v \mapsto v : 1$). We call it the **variety** $\mathbb{P}^1$.

We could have patched U and V by the map $a \mapsto a$, for $a \in U_0$. This also gives an isomorphism of U_0 onto V_0. However, we are leaving out the same point each time and the result is $\mathbb{A}^1$ with the zero doubled. This is still a prevariety, but it will not be a "variety" (cf. Section 14).

EXERCISE 10.6: Suppose that $X = \bigcup_{i=1}^n U_i$ is an open covering of a topological space X.

(a) Let Z be a subset of X. Prove that if $Z \cap U_i$ is closed in U_i for $i = 1, \ldots, n$, then Z is closed.

(b) Prove that if each of the sets U_i is connected and $U_i \cap U_j \neq \emptyset$ for all i, j, then X is connected.

11. Projective prevarieties.

In this section we abbreviate $(X_0, \ldots, X_n)$ by $\mathbf{X}$. Let P be a homogeneous prime ideal of $K[\mathbf{X}]$ and let $X = V(P)$ the corresponding irreducible subset of $\mathbb{P}^n$ (Section 5). We want to make X (with the Zariski topology) into a prevariety. We do it by defining a function field, getting local rings, and intersecting them, just as for affine varieties.

The elements of $K[\mathbf{X}]$, even the homogeneous ones, do not give functions on X; but the ratio of any two having the same degree is a function. Consider the integral domain $R = K[\mathbf{X}]/P$. For each nonnegative integer d let $K[\mathbf{X}]_d$ be the additive group of all **forms** (i.e., homogeneous polynomials) $f \in K[\mathbf{X}]$ of degree d (and 0). Then

$$R_d = \{f + P|\ f \in K[\mathbf{X}]_d\}$$

is also an additive group. If $g \in K[\mathbf{X}]_e$ with $e \neq d$ and $f + P = g + P$, then, since P is homogeneous, $f, g \in P$. Hence $R_d \cap R_e = 0$. It follows that $R = \bigoplus_{d=0}^{\infty} R_d$. Moreover, $R_d R_e \subseteq R_{d+e}$. Thus R is a **graded ring**. Let

$$K(X) = \left\{ \frac{f}{g} |\ f, g \in K[\mathbf{X}]_d \text{ for the same } d \text{ and } g \neq 0 \right\}.$$

If $\mathbf{a} \in X$ and $g \in R_d$, it makes sense to say $g(\mathbf{a}) \neq 0$, even though g is not a function on X; for g changes by a nonzero factor as we change the homogeneous coordinates of a. Hence we can define a subring of $K(X)$:

$$\mathcal{O}_{\mathbf{a}} = \left\{ \frac{f}{g} |\ f, g \in R_d \text{ for some } d \text{ and } g(\mathbf{a}) \neq 0. \right\}$$

The set

$$\mathfrak{m}_{\mathbf{a}} = \left\{ \frac{f}{g} |\ f, g \in R_d \text{ for some } d,\ f(\mathbf{a}) = 0 \text{ and } g(\mathbf{a}) \neq 0 \right\}$$

is clearly an ideal in the ring $\mathcal{O}_{\mathbf{a}}$ and any element not in $\mathfrak{m}_{\mathbf{a}}$ is invertable in $\mathcal{O}_{\mathbf{a}}$. Thus $\mathcal{O}_{\mathbf{a}}$ is a local ring.

We now define a sheaf $\mathcal{O}_X$ on X. For open subset U of X set

$$\mathcal{O}_X(U) = \bigcap_{\mathbf{a} \in U} \mathcal{O}_{\mathbf{a}}.$$

We can identify $\mathcal{O}_X$ with a sheaf of K-valued functions. For suppose $\mathbf{a} \in U$ and $h \in \Gamma(U, \mathcal{O}_X)$. Then $h = f/g$ with $f, g \in R_d$, $g(\mathbf{a}) \neq 0$, and we define $h(\mathbf{a}) = f(\mathbf{a})/g(\mathbf{a})$. This means of course that we lift f, g to forms $\tilde{f}, \tilde{g}$ of degree d let $\tilde{\mathbf{a}}$ be a set of homogeneous coordinates of $\mathbf{a}$ and take $\tilde{f}(\mathbf{a})/\tilde{g}(\mathbf{a})$. This value is unchanged if we take a different set of homogeneous coordinates, or we change $\tilde{f}$ and $\tilde{g}$ by a form in $P \cap K[\mathbf{X}]_d$. Thus h is a well defined function.

We still should check that if $h \in \Gamma(U, \mathcal{O}_X)$ and $h(\mathbf{a}) = 0$ for all $\mathbf{a} \in U$, then $h = 0$. But this also comes out of the next step, which consists in checking that $(X, \mathcal{O}_X)$ is locally isomorphic to an affine variety. In fact we claim that for each i between 0 and n, and with $X_i = \{\mathbf{a} \in X |\ a_i \neq 0\}$,

$$(X_i, \mathcal{O}_X\big|_{X_i})$$

is an affine variety. We will check this only for $i = 0$, since the general case goes just the same.

If $a_0 = 0$ for each $\mathbf{a} \in X$, then X_0 is empty and there is nothing to prove. So assume that X_0 is nonempty.

Let $\mathbf{Y} = (Y_1, \dots, Y_n)$. To each polynomial $f \in K[\mathbf{X}]$ we attach the polynomial $f' \in K[\mathbf{Y}]$, $f'(\mathbf{Y}) = f(1, Y_1, \dots, Y_n)$. Conversely, for each polynomial $g \in K[\mathbf{Y}]$ of degree d we attach the form $g^* \in K[\mathbf{X}]$ of degree d:

$$g^*(\mathbf{X}) = X_0^d g\big(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\big).$$

The prime ideal P of $K[\mathbf{X}]$ corresponds to the prime ideal $P' = \{f' \in K[\mathbf{Y}] |\ f \in P\}$. Indeed, let $x_i = X_i + P$, $i = 0, \dots, n$. Then P' is the kernel of the K-epimorphism $K[\mathbf{Y}] \to K[x_1/x_0, \dots, x_n/x_0]$ that maps Y_i onto x_i/x_0, $i = 1, \dots, n$. As the latter ring is an integral domain P' is prime.

Let $X' = V(P')$ be the corresponding irreducible algebraic subset of $\mathbb{A}^n$. The map $\mathbf{a} \mapsto (a_1/a_0, \dots, a_n/a_0)$ is a homeomorphism of X_0 onto X' whose inverse is $(b_1, \dots, b_n) \mapsto 1{:}b_1{:}\cdots{:}b_n$.

For forms $f, g \in K[\mathbf{X}]_d$ such that $g(\mathbf{x}) \neq 0$ and for $\mathbf{a} \in X_0$ the identity

$$\frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{f(\mathbf{x})/x_0^d}{g(\mathbf{x})/x_0^d} = \frac{f'(\mathbf{x}')}{g(\mathbf{x}')}$$

implies that $\mathcal{O}_{\mathbf{a},X} = \mathcal{O}_{\mathbf{a}',X'}$. Since the sheaves were defined by intersecting local rings, the ring of sections over open sets correspond and the claimed isomorphism has been established.

Definition 11.1: *Morphism of prevarieties.* Let X and Y be prevarieties. A map $\varphi\colon X \to Y$ is a **morphism** if φ is continuous and, for all open sets V in Y,

$$g \in \Gamma(V, \mathcal{O}_Y) \text{ implies } g \circ \varphi \in \Gamma(\varphi^{-1}(V), \mathcal{O}_X).$$

Proposition 11.2: *Let $\varphi\colon X \to Y$ be a map of prevarieties. Let $\{V_i\}$ be a collection of open affine subsets covering Y. Suppose that $\{U_i\}$ is an open covering of X such that*

(a1) $\varphi(U_i) \subseteq V_i$ *and*

(a2) φ^* *maps* $\Gamma(V_i, \mathcal{O}_Y)$ *into* $\Gamma(U_i, \mathcal{O}_X)$.

Then φ is a morphism.

Proof: We may assume that the U_i are affine; for if $U \subseteq U_i$ is affine, then $\Gamma(U_i, \mathcal{O}_X) \subseteq \Gamma(U, \mathcal{O}_X)$ and therefore φ^* maps $\Gamma(V_i, \mathcal{O}_Y)$ into $\Gamma(U, \mathcal{O}_X)$. So, we may replace U_i by a collection of affine open sets that cover U_i.

First of all, the restriction φ_i of φ to a map from U_i to V_i is a morphism. In fact, the homomorphism

$$\varphi_i^*\colon \Gamma(V_i, \mathcal{O}_Y) \to \Gamma(U_i, \mathcal{O}_X)$$

is induced by some morphism $\psi_i\colon U_i \to V_i$ (Proposition 6.6). If $\mathbf{x} \in U_i$ and π_j is the jth coordinate function of the ambient affine space of V_i, then

$$\pi_j(\varphi_i(\mathbf{x})) = (\varphi_i^* \circ \pi_j)(\mathbf{x}) = (\psi_i^* \circ \pi_j)(\mathbf{x}) = \pi_j(\psi_i(\mathbf{x})),$$

and therefore $\varphi_i(\mathbf{x}) = \psi_i(\mathbf{x})$. Hence $\varphi_i = \psi_i$ and φ_i is a morphism. In particular φ_i is continuous and therefore φ itself is continuous.

It remains to check that φ^* maps sections of $\mathcal{O}_Y$ to sections of $\mathcal{O}_X$. But if $V \subseteq Y$ is open, and $g \in \Gamma(V, \mathcal{O}_Y)$, then $g \in \Gamma(V \cap V_i, \mathcal{O}_Y)$. Hence $g \circ \varphi = \varphi^*(g) \in \Gamma(\varphi^{-1}(V \cap V_i), \mathcal{O}_X) \subseteq \Gamma(\varphi^{-1}(V) \cap U_i, \mathcal{O}_X)$. Since $\mathcal{O}_X$ is a sheaf, $g \circ \varphi$ actually belongs to $\Gamma(\varphi^{-1}(V), \mathcal{O}_X)$. ∎

11.3: *Morphism is a local property.* Let φ: $X \to Y$ be a map of prevarieties. Suppose that $\{U_i\}_{i=1}^m$ is an open covering of X and $\{V_i\}_{i=1}^m$ is an open covering of Y such that the restriction of φ to U_i is a morphism into Y_i. Prove that φ is a morphism.

EXAMPLE 11.4: *Plane cubic.* To illustrate the meaning of our definitions it seems worthwhile to work out in detail a non-trivial example. Let C be the plane cubic curve defined in homogeneous coordinates by:

$$X_0X_2^2 = X_1(X_1^2 - X_0^2).$$

Look first at C_0, with affine coordinates $X = X_1/X_0$ and $Y = X_2/X_0$. The equation of C_0 is:

$$Y^2 = X(X^2 - 1).$$

For all lines L through the origin we want to interchange the two points in $L \cap C_0$ (other that the origin). Start with a point $(a, b) \in C_0$, $(a, b) \neq (0, 0)$. This is joined to the origin by the line

$$X = at \qquad Y = bt.$$

Intersecting this with the cubic, we get the equation

$$b^2t^2 = at(a^2t^2 - 1)$$

or, using $b^2 = a(a^2 - 1)$,

$$at(t - 1)(a^2t + 1) = 0.$$

Thus the second point of intersection is given by $t = -1/a^2$. In other words, the morphism on C_0 is to be given by:

(1) $$(a, b) \mapsto (-1/a,\ -b/a^2).$$

These are not polynomials, so at any rate they do not define a map from C_0 into itself. This is as it should be, since we want the origin itself to go to the unique point, 0:0:1, at infinity on the cubic.

To describe the subsets on which we will get a morphism, we must throw out the various "bad" points one at at time. We need names for them:

$\mathbf{p}_0$	$= 1{:}0{:}0$	the origin
$\mathbf{p}_\infty$	$= 0{:}0{:}1$	the only point at infinity with respect to C_0
$\mathbf{q}_1$	$= 1{:}1{:}0$	a point on the X-axis
$\mathbf{q}_2$	$= 1{:}-1{:}0$	another point on the X-axis

The morphism - call it φ - should coincide with (1) on $C - \{\mathbf{p}_0, \mathbf{p}_\infty\}$ (in particular it should interchange $\mathbf{q}_0$ and $\mathbf{p}_\infty$, and $\mathbf{q}_1$ with $\mathbf{q}_2$. Define

$$\begin{aligned} C_0 &= C - \{\mathbf{p}_\infty\} \\ C_2 &= C - \{\mathbf{p}_0, \mathbf{q}_1, \mathbf{q}_2\} \\ U_1 &= C - \{\mathbf{p}_0, \mathbf{p}_\infty\} \subseteq C_0 \\ U_2 &= C - \{\mathbf{p}_\infty, \mathbf{q}_1, \mathbf{q}_2\} \subseteq C_0 \\ U_3 &= C - \{\mathbf{p}_0, \mathbf{q}_1, \mathbf{q}_2\} = C_2 \\ U &= C - \{\mathbf{p}_0, \mathbf{p}_\infty, \mathbf{q}_1, \mathbf{q}_2\} \end{aligned}$$

Then

(1a) U_1, U_2, U_3 is an open covering of C,

(1b) C_0, C_2 is an affine open covering of C, and

(1c) if φ is defined set-theoretically as above, then $\varphi(U_1) \subseteq C_0$, $\varphi(U_2) \subseteq C_2$, and $\varphi(U_3) \subseteq C_0$.

(1d) $U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = U$.

Hence, by Proposition 11.2, it suffices to check that

$$\varphi^*(\Gamma(C_0, \mathcal{O}_C)) \subseteq \Gamma(U_1, \mathcal{O}_C) \cap \Gamma(U_3, \mathcal{O}_C), \text{ and } \varphi^*(\Gamma(C_2, \mathcal{O}_C)) \subseteq \Gamma(U_2, \mathcal{O}_C)$$

and then it follows that φ is a morphism.

Let x_0, x_1, x_2 be the residue classes of X_0, X_1, X_2, respectively, modulo the prime ideal of $K[X_0, X_1, X_2]$ generated by $X_0X_2^2 - X_1(X_1^2 - X_0^2)$. Then $x = x_1/x_0$ and $y = x_2/x_0$ are the coordinate functions for C_0. In particular $\Gamma(C_0, \mathcal{O}_C) = K[x, y]$ and

$K(C) = K(x,y)$. Also, $s = x_0/x_2$ and $t = x_1/x_2$ are the coordinate functions for C_2. In particular $\Gamma(C_2, \mathcal{O}_C) = K[s,t]$. These coordinates are related by the formulas:

$$x = \frac{t}{s},\ y = \frac{1}{s} \qquad t = \frac{x}{y},\ s = \frac{1}{y}.$$

The corresponding variables are denoted by the corresponding capitals. The defining equations for C_0 and C_2 are then:

$$C_0:\ Y^2 = X(X^2 - 1) \qquad C_2:\ S = T(T^2 - S^2).$$

As we have already mentioned φ maps U_1 into C_0 by the formula:

$$\varphi(a,b) = \big(-\frac{1}{a}, -\frac{b}{a^2}\big), \qquad (a,b) \in U_1 \tag{2}$$

In particular $\varphi(1,0) = (-1,0)$ and $\varphi(-1,0)$. So φ interchange $\mathbf{q}_1$ and $\mathbf{q}_2$, as desired. As the denominators of the right hand side do not vanish on U_1, we have $x \circ \varphi, y \circ \varphi \in \Gamma(U_1, \mathcal{O}_C)$.

Note that $U_2 \subseteq C_0$. So define φ from U_2 to C_2 by the formula:

$$\varphi(a,b) = \big(\frac{-ab}{a^2-1}, \frac{b}{a^2-1}\big), \qquad (a,b) \in U_2 \tag{3}$$

This means that $\varphi^*(s) = -xy/(x^2-1)$ and $\varphi^*(t) = y/(x^2-1)$. As $a^2 - 1 \neq 0$ for $(a,b) \in U_2$, we have $\varphi^*(s), \varphi^*(t) \in \Gamma(U_2, \mathcal{O}_C)$, as it should be.

We still have to check that the two definitions agree on U. But if $(a,b) \in U$, then $b^2 = a(a^2-1)$ and therefore $s(\varphi(a,b)) = -a^2/b$, $t(\varphi(a,b)) = a/b$. Hence $t(\varphi(a,b))/s(\varphi(a,b)) = -1/a = x(\varphi(a,b))$ and $1/s(\varphi(a,b)) = -b/a^2 = y(\varphi(a,b))$, as it should be. Also, $\varphi(0,0) = (0,0)$. So φ maps $\mathbf{p}_0$ to $\mathbf{p}_\infty$.

To define φ from U_3 to C_0 use the (s,t)-coordinates for U_3 and the (x,y)-coordinates for C_0:

$$\varphi(c,d) = (c^2 - d^2,\ c(d^2 - c^2) - d) \qquad (c,d) \in U_3. \tag{4}$$

In other words $\varphi^*(x) = s^2 - t^2$ and $\varphi^*(y) = s(t^2 - s^2) - t \in \Gamma(U_3, \mathcal{O}_C)$.

Finally we check that if $(c,d) \in U$, then definition (2) agrees with definition (4). Indeed, if (a,b) is the point (c,d) in the coordinates (x,y), then $c = a/b$ and $d = 1/b$. Using the relation $d = c(c^2 - d^2)$ we find:

$$\varphi(c,d) = (c^2 - d^2,\, c(d^2 - c^2) - d) = \big(-\frac{c}{d}, -\frac{c}{d^2}\big) = \big(-\frac{1}{a}, -\frac{b}{a^2}\big) = \varphi(a,b).$$

As $\varphi(0,0) = (0,0)$, φ maps $\mathbf{p}_\infty$ to $\mathbf{p}_0$. The definition of φ is complete. ∎

Problem 11.5: *Veronese map.* For a positive integer d let M be the set of all monomials in $X_0, \cdots, X_n$ of degree d. Prove by induction on n that

$$|M| = \binom{n+d}{n}.$$

For $m = |M| - 1$ define a map $\varphi \mathbb{P}^n \to \mathbb{P}^m$ by

$$\varphi(\mathbf{x}) = (\mu(\mathbf{x}))_{\mu \in M}.$$

Prove that φ is an isomorphism of φ onto the subvariety V of $\mathbb{P}^m$ defined by all the equations

$$Y_\mu Y_{\mu'} = Y_\nu Y_{\nu'},$$

where μ, μ', ν, ν' range over all monomials in M such that $\mu\mu' = \nu\nu'$.

Problem 11.6: *Removing a hypersurface from* $\mathbb{P}^n$*.* Generalize the result by which we have covered $\mathbb{P}^n$ by open affine set as follows: Prove that if $h \in K[X_0, \ldots, X_n]$ is a form of positive degree, then $(\mathbb{P}^n)_h$ is an affine variety. Hint: Prove first for hyperplanes. Then for $d = \deg(h)$ study the image of the hypersurface $h = 0$ under the veronesian map defined in Problem 11.4.

Problem 11.7: *Very few sections.* Prove that $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \cong K$. In particular, unlike for affine varieties, the elements of $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ do not separate points.

Problem 11.8: *Morphism of* $\mathbb{P}^1$ *to* $\mathbb{A}^n$*.* Prove that every morphism $\varphi\colon \mathbb{P}^1 \to \mathbb{A}^n$ maps $\mathbb{P}^1$ onto a point.

12. Tensor product of modules.

Let R be a ring (commutative with 1), and let M, N be R-modules. Denote the free R-module generated by the direct product (of sets) $M \times N$ by (M, N). An element of (M, N) is a formal sum of the form $\sum \gamma_{m,n}(m, n)$ where $\gamma_{m,n} \in \mathbb{Z}$. Let T be the submodule of (M, N) generated by all elements of the form

$$\begin{array}{ll} (m_1 + m_2, n) - (m_1, n) - (m_2, n) & m_1, m_2 \in M;\ n \in N \\ (m, n_1 + n_2) - (m, n_1) - (m, n_2) & m \in M;\ n_1, n_2 \in N \\ (\gamma m, n) - (m, \gamma n) & r \in R;\ m \in M; n \in N \\ \gamma(m, n) - (\gamma m, n) & \end{array}$$

The quotient module $(M, N)/T$ is denoted by $M \otimes_R N$ and is called the **tensor product** of M and N over R. The coset of (m, n) module T is denoted by $m \otimes n$.

The definition implies that the following rules are valid in $M \otimes_R N$:

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$$

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2, \text{ and}$$

$$\gamma m \otimes n = m \otimes \gamma n = \gamma(m \otimes n).$$

Let π: $(M, N) \to M \otimes_R N$ be the canonical map, i.e., $\pi(m, n) = m \otimes n$. Then

$$\pi(m_1 + m_2, n) = \pi(m_1, n) + \pi(m_2, n)$$

$$\pi(m, n_1 + n_2) = \pi(m, n_1) + \pi(m, n_2)$$

$$\pi(\gamma m, n) = \pi(m, \gamma n) = \gamma\, \pi(m, n)$$

A map of (M, N) into an R-module A satisfying these relations is said to be R-**bilinear**.

LEMMA 12.1 (Universality property): *Let A be an R-module and let ψ: $M \times N \to A$ be an R-bilinear map. Then there exists a unique homomorphism θ: $M \otimes N \to A$ such that $\theta \circ \pi = \psi$.*

Proof: The map θ must satisfies $\theta(m \otimes n) = \psi(m, n)$. ∎

COROLLARY 12.2: *Let α: $M \to M'$ and β: $N \to N'$ be R-homomorphisms. Then there exists a unique homomorphism*

$$\alpha \otimes \beta\colon M \otimes N \to M' \otimes N'$$

such that $(\alpha \otimes \beta)(m \otimes n) = \alpha(m) \otimes \beta(n)$.

If M is also a module with respect to another ring S, then $M \otimes_R N$ is not only an R-module but also an S-module. Consider an element $s \in S$ as a homomorphism of M into M, acting by multiplication from the left. Put 1 for the identity map of N. Then, Lemma 12.2 gives an R-homomorphism $s \otimes 1$: $M \otimes_R N \to M \otimes_R N$ such that $(s \otimes 1)(m \otimes n) = sm \otimes n$. We can therefore define $s(m \otimes n) = sm \otimes n$. The same definition applies if N is an S-module.

If M and N are R-modules, then the map $(m, n) \to n \otimes m$, induces a natural homomorphism θ: $M \otimes_R N \to N \otimes_R M$ for which $\theta(m \otimes n) = n \otimes m$. Similarly we have an inverse map θ': $N \otimes_R M \to M \otimes_R N$ for which $\theta'(n \otimes m) = m \otimes n$. Hence θ is an isomorphism.

(1) $$M \otimes_R N \cong N \otimes_R M.$$

Next we have the following natural isomorphism

(2) $$R \otimes_R N \cong N.$$

In this isomorphism the map $R \otimes_R N \to N$ is determined by $\gamma \otimes n \mapsto \gamma n$ and the inverse map is given by $n \mapsto 1 \otimes n$.

If N is also an S-module and P is an S-module, then the map $(m \otimes n) \otimes p \mapsto m \otimes (n \otimes p)$ induces a natural isomorphism

(3) $$(M \otimes_R N) \otimes_S P \cong M \otimes_R (N \otimes_S P).$$

This is the associative law for tensor products.

The tensor product preserves direct sums:

(4) $$M \otimes_R (N_1 \oplus N_2) \cong (M \otimes_R N_1) \oplus (M \otimes_R N_2).$$

The map from the left hand side of (4) is defined by $m \otimes (n_1, n_2) \to (m \otimes n_1, m \otimes n_2)$. The inverse map is the direct sum of the map $M \otimes_R N_1 \to M \otimes_R (N_1 \oplus N_2)$ with the map $M \otimes_R N_2 \to M \otimes_R (N_1 \oplus N_2)$ given by $m \otimes n_1 \to m \otimes (n_1, 0)$ and $m \otimes n_2 \mapsto m \otimes (0, n_2)$, respectively.

Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ be an exact sequence of R-modules. Let M be an R-module. We claim that the sequence

$$M \otimes_R A \overset{1\otimes\alpha}{\longrightarrow} M \otimes_R B \overset{1\otimes\beta}{\longrightarrow} M \otimes_R C \longrightarrow 0 \tag{5}$$

is exact. Obviously $1 \otimes \beta$ is surjective. Thus, in order to prove the exactness of (5) we have to prove that

$$M \otimes_R C \cong (M \otimes_R B)/(1 \otimes \alpha)(M \otimes_R A).$$

First define a map ψ: $M \times C \to (M \otimes_R B)/(1 \otimes \alpha)(M \otimes_R A)$ by $\psi(m, c) = \overline{m \otimes \beta^{-1}(c)}$ where $\beta^{-1}(c)$ is an element of B which is mapped by β on c, and the bar means reduction modulo $(1 \otimes \alpha)(M \otimes_R A)$. It is clear that ψ is well defined and that it is an R-bilinear map. Hence, by Lemma 12.1, it induces a homomorphism $\bar{\psi}$: $M \otimes_R C \to (M \otimes_R B)/(1 \otimes \alpha)(M \otimes_R A)$ for which $\bar{\psi}(m \otimes c) = \overline{m \otimes \beta^{-1}(c)}$. On the other hand, $(1 \otimes \alpha)(M \otimes_R A) \subseteq \mathrm{Ker}(1 \otimes \beta)$. Hence $1 \otimes \beta$ induces a homomorphism

$$\overline{1 \otimes \beta}\colon (M \otimes_R B)/(1 \otimes \alpha)(M \otimes_R A) \longrightarrow M \otimes_R C.$$

The map $\bar{\psi}$ and $\overline{1 \otimes \beta}$ are inverse to each other, hence they are isomorphisms.

In general is is not true that if α: $A \to B$ is injective, then $1 \otimes \alpha$: $M \otimes_R A \to M \otimes_R B$ is also injective. For example, multiplication by 2, is an injection $\mathbb{Z} \to 2\mathbb{Z}$. But $2\mathbb{Z} \otimes (\mathbb{Z}/2\mathbb{Z}) = 0$, since $2z \otimes a = z \otimes 2a = 0$ for each $z \in \mathbb{Z}$ and $a \in \mathbb{Z}/2\mathbb{Z}$. Hence $\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \to 2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$ is not an injection.

In the next section we consider a special case, where R is a field, for which left exactness is preserved.

Exercise 12.3: Let M be an R-module and I and ideal of R. Prove that $R/I \otimes_R M = M/IM$.

13. Tensor product of vector spaces.

A module over a field is a vector space. Tensor products of vector spaces can be explicitly described in terms of bases:

LEMMA 13.1: *Let U and V be vector spaces over a field F with bases $\{u_i\}_{i\in I}$ and $\{v_j\}_{j\in J}$, respectively. Then $U\otimes_F V$ is a vector space over F with base $\{u_i\otimes v_j\}_{(i,j)\in I\times J}$.*

Proof: Let W be the vector space over F with the base $\{(u_i, v_j)\}_{i\in I, j\in J}$. The map $(u_j, v_j) \mapsto u_i \otimes v_j$ defines an F-linear map of W into $U \otimes_F V$. If $u \in U$ and $v \in V$, then $u = \sum_{i\in I} a_i u_i$ and $v = \sum_{j\in J} b_j v_j$ where $a_i \in F$, almost all a_i and almost all b_j are 0. Set $u \otimes v \mapsto \sum_{i,j} a_i b_j (u_i, v_j)$. By Lemma 12.1, this map is a homomorphism of vector spaces. The equality $\sum_{i,j} a_i b_j\, u_i \otimes v_j = u \otimes v$ show that the two maps are inverse to each other. Hence, they are isomorphisms. ∎

LEMMA 13.2: *Let U and V be vector spaces over a field F. If $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_n\}$ are linearly independent sets of elements in U and V, respectively, then $\{u_i \otimes v_j |\ 1 \le i \le m,\ 1 \le j \le n\}$ is a linearly independent set of elements in $U \otimes_F V$.*

Proof: In fact the sets $\{u_i\}$ and $\{v_j\}$ can be completed to bases of U and V, respectively. The result follows therefore from Lemma 13.1. ∎

Here is another immediate corollary of Lemma 13.1:

LEMMA 13.3: *Let U and V be vector spaces over a field F. Then $\dim(U \otimes_F V) = \dim(U)\cdot\dim(V)$ (This formula holds even if the dimensions are not finite, provided that the appropriate product of cardinal numbers is used.)*

LEMMA 13.4: *Let $\alpha\colon V \to W$ be an injective F-linear map of F-vector spaces V and W. Let U be another F-vector space. Then the map $1 \otimes \alpha\colon U \otimes_F V \to U \otimes_F W$ is injective.*

Proof: Let $\{u_i\}_{i\in I}$, $\{v_j\}_{j\in J}$ and $\{w_k\}_{k\in K}$ be bases for U, V and W over F. Then, according to Lemma 13.1, $\{u_i \otimes v_j\}_{i\in I,\, j\in J}$ and $\{u_i \otimes w_k\}_{i\in I,\, k\in K}$ are bases for $U \otimes_F V$ and $U \otimes_F W$ respectively. A typical element of $U \otimes_F V$ has the form $\sum_{i,j} a_{ij} u_i \otimes v_j$

where $a_{ij} \in F$ and almost all of them are 0. Suppose that

$$(1) \qquad (1 \otimes \alpha)(\sum_{i,j} a_{ij}\, u_i \otimes v_j) = 0.$$

We have to shows that $a_{ij} = 0$ for all i, j.

Let $c_{jk} \in F$ be such that

$$(2) \qquad \alpha(v_j) = \sum_k c_{jk} w_k.$$

By (1) and (2),

$$\sum_{i,k} (\sum_j a_{ij} c_{jk}) u_i \otimes w_k = 0$$

Hence, by Lemma 13.2,

$$\sum_j a_{ij} c_{jk} = 0 \text{ for every } i, k.$$

Combine (1) and (3):

$$\alpha(\sum_j a_{ij} v_j) = \sum_k (\sum_j a_{ij} c_{jk}) w_k = 0.$$

Since α is injective $\sum_j a_{ij} v_j = 0$ for all i. Conclude that $a_{ij} = 0$ for all i, j. ∎

Combine Lemma 13.4 with the last result of Section 12:

PROPOSITION 13.5: *If*

$$0 \longrightarrow V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3 \longrightarrow 0$$

is a short exact sequence of F vector spaces and if U is an F-vector space, then the sequence

$$0 \longrightarrow U \otimes V_1 \overset{1\otimes\alpha}{\longrightarrow} U \otimes_F V_2 \overset{1\otimes\beta}{\longrightarrow} U \otimes V_3 \longrightarrow 0$$

is exact.

Let now V be a vector space over a field F and let F' be a field extension of F. Denote the inclusion map of F into F' by j. Then, by Lemma 13.5, $j \otimes 1\colon F \otimes_F V \to F' \otimes_F V$ is an injective map. Since $v \mapsto 1 \otimes v$ is an isomorphism of V into $F \otimes_F V$ it is also an embedding of V into $F' \otimes_F V$. We identify V with its image. Note that $F' \otimes_F V$ is also a vector space over the field F'. It has the following property:

LEMMA 13.6: *If $v_1, \dots, v_n$ are independent elements of V over F, then they are independent over F'.*

Proof: Suppose that $a'_1, \dots, a'_n$ are elements of F' such that $\sum_{j=1}^{n} a'_j \otimes v_j = 0$. Let $\{x_1, \dots, x_m\}$ be a base for the F-vector space generated by $a'_1, \dots, a'_n$. Then $a'_j = \sum_{i=1}^{m} a_{ji} x_i$ with $a_{ji} \in F$. It follows that $\sum_{i,j} a_{ji} x_i \otimes v_j = 0$. But, according to Corollary 13.2, the set $\{x_i \otimes v_j\}_{i,j}$ is linearly independent over F. Hence $a_{ji} = 0$ for all j and i. Conclude that $a_j = 0$ for all j. ∎

The procedure of going from V to $F' \otimes_F V$ is called **an extension of the field of coefficients of** V. Lemma 13.6 justifies this name.

Next we consider tensor products of algebras. Recall that an **algebra** over a field F is an F-vector space A which is also a ring such that

$$\lambda(a_1 a_2) = (\lambda a_1) a_2 = a_1(\lambda a_2), \text{ for } \lambda \in F;\ a_1, a_2 \in A.$$

We always assume that F is contained in A.

If B is another algebra over F, then $A \otimes_F B$ is an F-vector space. We consider the homomorphisms

$$(A \otimes_F B,\, A \otimes_F B) \xrightarrow{\pi} (A \otimes_F B) \otimes_F (A \otimes_F B) \xrightarrow{\tau} (A \otimes_F A) \otimes_F (B \otimes_F B) \xrightarrow{\kappa} A \otimes_F B$$

where π is the canonical projection:

$$\pi(a \otimes b,\, a' \otimes b') = (a \otimes b) \otimes (a' \otimes b'),$$

τ is the isomorphism

$$\tau\big((a \otimes b) \otimes (a' \otimes b')\big) = (a \otimes a') \otimes (b \otimes b'),$$

and κ is the tensor product of the multiplication maps:

$$\kappa\big((a \otimes a') \otimes (b \otimes b')\big) = aa' \otimes bb'.$$

The map

$$(a \otimes b,\, a' \otimes b') \mapsto aa' \otimes bb'$$

defined in this way is a multiplication in $A \otimes_F B$ which makes it an F-algebra.

In particular, if K and L are two field extensions of F, then $K \otimes_F L$ is an F-algebra. In Section ? we will prove the first part of the following result.

Proposition 13.7: *If R and S are integral domains that contain an algebraically closed field K, then $R \otimes_K S$ is an integral domain. Moreover, if E, F, and L are the quotient fields of R, S and $R \otimes_K S$, respectively, then* $\dim_K(L) = \dim_K(E) + \dim_K(F)$.

Exercise 13.8: If E and F are algebraic extensions of a field K, then $E \otimes_K F$ is a field.

Exercise 13.9: let U and V be topological spaces. Let R (resp., S) be an algebra of continued K-valued functions of U (resp. V). Define a map φ of $R \otimes_K S$ into the K-algebra of all K-valued functions of $U \times V$ by $\varphi(\sum f_i \otimes g_j) = \sum f_i g_j$, where

$$(\sum f_i g_j)(x, y) = \sum f_i(x) g_j(y).$$

Prove that φ is well defined and injective.

14. Products.

We want to define the product $X \times Y$ of any two prevarieties X, Y. We will certainly want to have $\mathbb{A}^m \times \mathbb{A}^n \cong \mathbb{A}^{m+n}$. But the product of the Zariski topologies in $\mathbb{A}^m$ and $\mathbb{A}^n$ does not give the Zariski topology in $\mathbb{A}^{m+n}$. In $\mathbb{A}^1 \times \mathbb{A}^1$, for instance, the only closed sets in the product topology are finite unions of horizontal and vertical lines. The only reliable way to finish the correct definition is to use the general category-theoretic definition of product.

Definition 14.1: Let $\mathcal{C}$ be a category, X, Y objects in $\mathcal{C}$. An object Z plus a pair of morphisms (p: $Z \to X$, q: $Z \to Y$) is a **product** if it has the following universal mapping property:

For each pair of morphsims (ρ: $W \to X$, σ: $W \to Y$) there is a unique morphism τ: $W \to Z$ such that $\rho = p \circ \tau$ and $\sigma = q \circ \tau$.

The induced morphism τ in this situation will always be denoted (ρ, σ). We call p and q the **projections** of the product onto its factors. Sometimes they will be denoted by pr_X and pr_Y, respectively. Clearly a product, if it exists, is unique up to a unique isomorphism commuting with the projections.

We prove that products exist in the category of prevarieties over K. Note that we have no choice for the underlying set. For if $X \times Y$ is a product of the prevarieties X and Y, then $X \times Y$ as a point set must be the usual product of the point sets X and Y. To see this, let W be a single point. This is a prevariety ($\mathbb{A}^0$, in fact). The maps of W to any prevariety S clearly correspond to the points of S. Therefore the points of the product of X and Y bijectively correspond to the points of $X \times Y$.

Proposition 14.2: *Let X and Y be affine varieties, with coordinate rings R and S. Then*

(a) *There is a product prevariety $X \times Y$.*
(b) *$X \times Y$ is affine with coordinate ring $R \otimes_K S$. It can be identified with the ring of all functions $\sum f_i g_i$: $X \times Y \to K$ where $f_i \in R$ and $g_i \in S$.*
(c) *A basis of the topology is given by the open sets $\{(\mathbf{x}, \mathbf{y}) | \sum f_i(\mathbf{x}) g_i(\mathbf{y}) \neq 0\}$, with $f_i \in R$ and $g_i \in S$.*

(d) *The local ring* $\mathcal{O}_{(\mathbf{x},\mathbf{y})}$ *is the localization of* $\mathcal{O}_{\mathbf{x}} \otimes_K \mathcal{O}_{\mathbf{y}}$ *at the maximal ideal* $\mathfrak{m}_{\mathbf{x}}\mathcal{O}_{\mathbf{y}} + \mathcal{O}_{\mathbf{x}}\mathfrak{m}_{\mathbf{y}}$.

Proof: Represent $X \subseteq \mathbb{A}^m$ as $V(f_1, \dots, f_k)$ with $I(X) = \langle f_1, \dots, f_k \rangle$ and $Y \subseteq \mathbb{A}^n$ as $V(g_1, \dots, g_l)$ with $I(Y) = \langle g_1, \dots, g_l \rangle$. Let $\mathbf{X} = (X_1, \dots, X_m)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$. Then the set $X \times Y \subseteq \mathbb{A}^{m+n}$ is the locus of zeros of $f_j(\mathbf{X})$,$g_j(\mathbf{Y})$ in $K[\mathbf{X}, \mathbf{Y}]$. Moreover,

$$(1) \qquad K[\mathbf{X}, \mathbf{Y}]/\langle f_i, g_j \rangle_{i,j} \cong K[\mathbf{X}]/\langle f_i \rangle_i \otimes_K K[\mathbf{Y}]/\langle g_j \rangle_j = R \otimes_K S.$$

Indeed, the map $X_i \mapsto \big(X_i + \langle f_j \rangle\big) \otimes 1$, $i = 1, \dots, m$ and $Y_i \mapsto 1 \otimes \big(Y_i + \langle g_j \rangle\big)$ extends to a K-homomorphism of $K[\mathbf{X}, \mathbf{Y}]$ into the right hand side of (1). As f_j, g_j are mapped into zero, this homomorphism induces one from the left hand side into the right hand side. The pair of obvious homomorphisms of $K[\mathbf{X}]/\langle f_j \rangle$ and $K[\mathbf{Y}]/\langle g_j \rangle$ into the left hand side of (1) gives a homomorphism of the right hand side into the left hand side which is inverse to the one that has been defined before. This establishes (1).

By Proposition 13.7, $R \otimes_K S$ is an integral domain. Hence $\langle f_i, g_j \rangle$ is prime, $X \times Y$ is irreducible, and $R \otimes_K S$ is its coordinate ring.

This gives us an affine variety $X \times Y$. The next step is to prove that it is a categorical product. We have natural projections $p, q\colon X \times Y \to X, Y$ with $p(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ and $q(\mathbf{x}, \mathbf{y}) = \mathbf{y}$. Suppose $\rho\colon Z \to X$ and $\sigma\colon Z \to Y$ are morphisms from a prevariety Z. There is just one map of points sets $\tau\colon Z \to X \times Y$ such that $\rho = p \circ \tau$, $\sigma = q \circ \tau$ (since $X \times Y$ as a point set is the product of X and Y). To verify the universal mapping property we need only check that τ is a morphism.

Since $X \times Y$ is affine, it suffices, by Proposition 11.2, to check that

$$g \in \Gamma(X \times Y, \mathcal{O}_{X \times Y}) \text{ implies } g \circ \tau \in \Gamma(Z, \mathcal{O}_Z).$$

But $\Gamma(X \times Y, \mathcal{O}_{X \times Y})$ is generated by the images of $\Gamma(X, \mathcal{O}_X) = R$ and $\Gamma(Y, \mathcal{O}_Y) = S$. Since ρ and s are morphisms each element of these images goes by composition with τ into $\Gamma(Z, \mathcal{O}_Z)$. Hence all of $\Gamma(X \times Y, _{\mathcal{X} \times \mathcal{Y}})$ goes into $\Gamma(Z, \mathcal{O}_Z)$.

We have proved (a) and (b). Statement (c) follows from (b). We prove (d).

Note that $\mathcal{O}_{(\mathbf{x},\mathbf{y})}$ is the localization of $R \otimes_K S$ at the ideal of all functions vanishing at $(\mathbf{x}, \mathbf{y})$. Clearly $R \otimes_K S \subseteq \mathcal{O}_{\mathbf{x}} \otimes_K \mathcal{O}_{\mathbf{y}} \subseteq \mathcal{O}_{\mathbf{x},\mathbf{y}}$. We can therefore get $\mathcal{O}_{(\mathbf{x},\mathbf{y})}$ by localizing

$\mathcal{O}_{\mathbf{x}} \otimes_K \mathcal{O}_{\mathbf{y}}$ at the ideal $\mathfrak{m}$ of all functions in it vanishing at $(\mathbf{x}, \mathbf{y})$. We claim that $\mathfrak{m}$ is precisely $\mathfrak{m}_{\mathbf{x}}\mathcal{O}_{\mathbf{y}} + \mathcal{O}_{\mathbf{x}}\mathfrak{m}_{\mathbf{y}}$. Evidently all functions in the latter ideal vanish at $(\mathbf{x}, \mathbf{y})$. Conversely, if $h = \sum f_i \otimes g_i \in \mathfrak{m}$ with $f_i \in \mathcal{O}_{\mathbf{x}}$, $g_i \in \mathcal{O}_{\mathbf{y}}$ and $f_i(\mathbf{x}) = a_i$, $g_i(\mathbf{y}) = b_i$, then

$$\begin{aligned} h = h - h(\mathbf{x}, \mathbf{y}) &= \sum f_i \otimes g_i - \sum a_i \otimes b_i \\ &= \sum (f_i - a_i) \otimes g_i + \sum a_i \otimes (g_i - b_i) \in \mathfrak{m}_{\mathbf{x}}\mathcal{O}_{\mathbf{y}} + \mathcal{O}_{\mathbf{x}}\mathfrak{m}_{\mathbf{y}}. \quad \blacksquare \end{aligned}$$

We can now "glue together" these affine products to obtain:

THEOREM 14.3: *Every pair X and Y of prevarieties over K has a product.*

Proof: We start, of course with the product set, $X \times Y$. For all open affine $U \subseteq X$, $V \subseteq Y$ and all finite set of elements $f_i \in \Gamma(U, \mathcal{O}_X)$, $g_i \in \Gamma(V, \mathcal{O}_Y)$ form $(U \times V)_{\sum f_i g_i}$. Take these as a basis of the open sets. They do form a basis. Indeed suppose that

$$(\mathbf{x}, \mathbf{y}) \in (U \times V)_{\sum f_i g_i} \cap (U' \times V')_{\sum f'_j g'_j}$$

where U, U' (resp. V, V') are affine open sets in X (resp. Y). Then $\mathbf{x} \in U \cap U'$ and $\mathbf{y} \in V \cap V'$. Hence $\mathbf{x}$ has an open affine neigborhood $U'' \subseteq U \cap U'$ and $\mathbf{y}$ has an affine open neigborhood $V'' \subseteq V \cap V'$. By Proposition 14.2, $U'' \times V''$ is an affine variety and

$$(\mathbf{x}, \mathbf{y}) \in (U'' \times V'')_{\sum f_i f'_j g_i g'_j}.$$

If $\{U_i\}_i$ (resp., $\{V_j\}_j$) is a finite affine open covering of X (resp., Y), then $\{U_i \times V_j\}_{i,j}$ is a finite affine covering of $X \times Y$. Obviously, the intersection of any two sets in this cover is nonempty. Since, by Proposition 14.2, $U_i \times V_j$ is irreducible, hence connected, $X \times Y$ is connected (Exercise 10.6(b)).

Note that on $U \times V$ this induces the topology of their product as defined in Proposition 14.2.

Let F be the quotient field of the integral domain $K(X) \times_K K(Y)$ (Proposition 13.7). By Exercise 13.9, we may consider the elements of $K(X) \otimes_K K(Y)$ as K-valued functions defined at least on open sets of the form $U \times V$, with U open in X and V open in Y. Namely, if $f_i \in K(X)$ are defined on U and $g_i \in K(Y)$ for $i = 1, \dots, n$ and if $(\mathbf{x}, \mathbf{y}) \in U \times V$, then $\sum f_i \otimes g_i$ is defined at $(\mathbf{x}, \mathbf{y})$ as $\sum f_i(\mathbf{x}) g_i(\mathbf{y})$. Obviously these value is unchanged if $(\mathbf{x}, \mathbf{y}) \in U' \times V'$, $f_i \in \Gamma(U', \mathcal{O}_X)$, and $g_i \in \Gamma(V', \mathcal{O}_Y)$. Then the elements of F are K-valued functions defined at least on basic open sets $(U \times V)_{\sum f_i g_i}$. For $\mathbf{x} \in X$, $\mathbf{y} \in Y$ let $\mathcal{O}_{(\mathbf{x},\mathbf{y})} \subseteq F$ be the localization of $\mathcal{O}_\mathbf{x} \otimes_K \mathcal{O}_\mathbf{y}$ at the ideal $\mathfrak{m} = \mathfrak{m}_x \mathcal{O}_\mathbf{y} + \mathcal{O}_\mathbf{x} \mathfrak{m}_\mathbf{y}$. As in the proof of Proposition 14.2, $\mathfrak{m}$ is the maximal ideal of all functions in $\mathcal{O}_\mathbf{x} \otimes_K \mathcal{O}_\mathbf{y}$ which vanish at $(\mathbf{x}, \mathbf{y})$.

For each open subset U of $X \times Y$ set

$$\Gamma(U, \mathcal{O}_{X \times Y}) = \bigcap_{(\mathbf{x},\mathbf{y}) \in U} \mathcal{O}_{(\mathbf{x},\mathbf{y})}.$$

This gives us a sheaf of functions. By Proposition 14.2, it coincides on each $U \times V$ (U, V affine) with the product of the affine varieties. Conclude that $X \times Y$ is a prevariety.

Now suppose ρ: $Z \to X$ and σ: $Z \to Y$ are morphisms from a prevariety Z. Let τ: $Z \to X \times Y$ be the unique set theoretical map composing properly with the projections. We want to check that it is a morphism. For each open affine U in X and V in Y look at $Z_{U,V} = \rho^{-1}(U) \cap \sigma^{-1}(V)$. These are open sets covering Z. Since being a morphism is a local property (Proposition 11.2) it is enough to prove that the restriction of τ to $Z_{U,V}$ into $U \times V$ is a morphism. This follows however from Proposition 14.2. ∎

Remark 14.4: (a) If U is any open subprevariety of X, then $U \times Y$ is an open subprevariety of $X \times Y$.
(b) If Z is a closed subprevariety of X, then $Z \times Y$ is a closed subprevariety of $X \times Y$. Indeed, it is enough to prove that $(Z \cap U) \times V$ is a closed subprevariety of $U \times V$ for U, V affine. This follows from the first paragraph of the proof of Proposition 14.2.

Theorem 14.5: *The product of two projective varieties is a projective variety.*

Proof: Since a closed subvariety of a projective variety is a projective variety, it is enough to show that $\mathbb{P}^m \times \mathbb{P}^n$ is a projective variety. In fact we embed it as a closed subvariety of $\mathbb{P}^{mn+n+m}$.

Take homogeneous coordinates $X_0, \ldots, X_m$ in $\mathbb{P}^m$, $Y_0, \ldots, Y_n$ in $\mathbb{P}^n$, and U_{ij} ($i = 0, \ldots, m$, $j = 0, \ldots, n$) in $\mathbb{P}^{mn+m+n}$. Define

$$\nu\colon \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$$

by

$$\nu(\mathbf{x}, \mathbf{y}) = \mathbf{u}, \text{ where } u_{ij} = x_i y_j \text{ for all } i, j.$$

This definition makes sense; first of all some u_{ij} is nonzero; and multiplying all x_i or all y_j by λ multiplies all u_{ij} by λ.

Let $\mathbb{P}^m_i = (\mathbb{P}^m)_{X_i}$, $\mathbb{P}^n_j = (\mathbb{P}^n)_{Y_j}$, and $\mathbb{P}^{mn+m+n}_{ij} = (\mathbb{P}^{mn+m+n})_{U_{ij}}$. Clearly

$$\nu^{-1}(\mathbb{P}^{mn+m+n}_{ij}) = \mathbb{P}^m_i \times \mathbb{P}^n_j.$$

We claim first that ν is injective. Assume that for some $\lambda \neq 0$, $x_i y_j = \lambda x'_i y'_j$ for all i and j. Then we want to prove that for some $a, b \neq 0$, $x_i = a x'_i$ and $y_j = b y'_j$. We may by symmetry assume $x_0 \neq 0$, $y_0 \neq 0$. Then $\lambda x'_0 y'_0 = x_0 y_0 \neq 0$, so $x'_0 \neq 0$ and $y'_0 \neq 0$. Therefore $x_i = (\lambda y'_0 / y_0) x'_i$ and $y_j = (\lambda x'_0 / x_0) y'_j$, as desired.

Now we claim that ν is an isomorphism of $\mathbb{P}^m_i \times \mathbb{P}^m_j$ onto a closed subvariety of $\mathbb{P}^{mn+m+n}_{ij}$. We may assume $i = 0$, $j = 0$ for simplicity. On $\mathbb{P}^m_0$ we take affine coordinates $S_i = X_i / X_0$, $i = 1, \ldots, m$. On $\mathbb{P}^m_0$ we take affine coordinates $T_j = Y_j / Y_0$, $j = 1, \ldots, m$. On $\mathbb{P}^{mn+m+n}_{0,0}$ we take affine coordinates $R_{ij} = U_{ij} / U_{0,0}$, $(i, j) \neq (0, 0)$.

In these coordinates $\nu(\mathbf{s}, \mathbf{t}) = \mathbf{r}$ where

$$r_{i,0} = s_i, \ r_{0,j} = t_j \text{ and } r_{ij} = s_i t_j \text{ if } i, j \geq 1.$$

Hence the image of $\mathbb{P}^m_0 \times \mathbb{P}^n_0$ is the locus of points satisfying $r_{ij} = r_{i,0} r_{0,j}$ for all $i, j \geq 1$. This is certainly closed. Its affine coordinate ring is $K[R_{ij}]_{ij} / \langle R_{ij} - R_{i,0} R_{0,j} \rangle_{ij}$, which is isomorphic to the polynomial ring $K[R_{i0}, R_{0j}]_{ij}$. Under ν^*, this is mapped isomorphically onto $K[S_i, T_j]_{ij}$, which is the affine coordinate ring of $\mathbb{P}^m_0 \times \mathbb{P}^n_0$. Indeed,

$$(R_{i0} \circ \nu)(\mathbf{s}, \mathbf{t}) = R_{i0}((s_i t_j)_{ij}) = s_i / s_0 = S_i(\mathbf{s}, \mathbf{t}).$$

So, $\nu^*(R_{i0}) = S_i$ and similarly $\nu^*(R_{0j}) = T_j$. Hence, we do have an isomorphism.

Let $Z = \nu(\mathbb{P}^m \times \mathbb{P}^m)$. Since $Z \cap \mathbb{P}^{mn+m+n}_{ij}$ is closed for all i, j, Z is closed. Also, ν is a homeomorphism on each of these affines, so it is a homeomorphism globally, and in particular Z is irreducible. Thus Z is a projective variety. Since ν is an isomorphism on each affine piece, it is an isomorphism globally. Conclude that Z is isomorphic to $\mathbb{P}^m \times \mathbb{P}^n$. ∎

Exercise 14.6: Prove that Z is the locus of the homogeneous ideal generated by all polynomials $U_{ij}U_{kl} - U_{il}U_{jk}$. So Z is an intersection of **quadrics**. In particular $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to the quadric in $\mathbb{P}^3$ defined by $U_0U_1 - U_2U_3$.

15. Varieties.

If X is a topological space, then X satisfies the Hausdorff axiom if and only if the diagonal

$$\Delta(X) = \{(x,x)|\ x \in X\}$$

is a closed subset of $X \times X$ in the product topology. The topology of a prevariety X is usually not Hausdorff, nor is the topology of $X \times X$ the product topology. So $\Delta(X)$ may or may not be closed in $X \times X$.

LEMMA AND DEFINITION 15.1: *A prevariety X is called a* **variety** *if it satisfies the following equivalent conditions:*

(a) *For each prevariety Y and for each pair of morphisms $\varphi_1, \varphi_2\colon Y \to X$ the set $\{\mathbf{y} \in Y|\ \varphi_1(\mathbf{y}) = \varphi_2(\mathbf{y})\}$ is closed in Y.*

(b) *The diagonal $\Delta(X)$ of X is closed in $X \times X$.*

Proof: If (a) is true, then $\Delta(X) = \{\mathbf{y} \in X \times X|\ \mathrm{pr}_1(\mathbf{y}) = \mathrm{pr}_2(\mathbf{x})\}$ is closed in X.

Conversely, in the situation of (a) $\{\mathbf{y} \in Y|\ \varphi_1(\mathbf{y}) = \varphi_2(\mathbf{y})\} = (\varphi_1, \varphi_2)^{-1}(\Delta(X))$. Hence, if (b) is true, then this set is closed in Y. ∎

EXAMPLE 15.2: $\mathbb{A}^1$ *with a double point.* As in the second part of Example 10.5 let U_1, U_2 be two copies of $\mathbb{A}^1$, way with coordinates x_1 and x_2, patched to a prevariety X by $x_1 = x_2$ on the open sets $x_1 \neq 0$ and $x_2 \neq 0$. Consider the isomorphisms i_1, i_2 of $\mathbb{A}^1$ with U_1, U_2, respectively as embeddings into X. Then $\{y \in \mathbb{A}^1|\ i_1(y) = i_2(y)\} = \mathbb{A}^1 - \{0\}$ is not closed in $\mathbb{A}^1$. Hence X is not a variety.

REMARK 15.3: (a) *A subprevariety of a variety is a variety. A product of two varieties is a variety.*

(b) *An affine variety is a variety.* In fact, since the global sections on an affine variety separates points, if X is affine, Y is an arbitrary prevariety, and $\varphi_1, \varphi_2\colon Y \to X$ are morphisms, then $\{\mathbf{y}|\ \varphi_1(\mathbf{y}) = \varphi_2(\mathbf{y})\}$ is the locus of zeroes of the functions $g \circ \varphi_1 - g \circ \varphi_2$ for all $g \in \Gamma(X, \mathcal{O}_X)$ and this set is closed.

(c) *If $\varphi_1, \varphi_2\colon Y \to X$ are morphisms of any prevarieties, then $Z = \{\mathbf{y}|\ \varphi_1(\mathbf{y}) = \varphi_2(\mathbf{y})\}$ is locally closed.* In fact for $\mathbf{z} \in Z$, let V be an affine open neighborhood of $\varphi_1(\mathbf{z})$. Then

$Y_{\mathbf{z}} = \varphi_1^{-1}(V) \cap \varphi_2^{-1}(V)$ is an open neighborhood of $\mathbf{z}$ and, by (b),

$$Z \cap Y_{\mathbf{z}} = \{\mathbf{z}' \in Y_{\mathbf{z}} |\ \varphi_1(\mathbf{z}') = \varphi_2(\mathbf{z}')\}$$

is closed in Y_z.

(d) *The graph of a morphism.* If φ: $X \to Y$ is a morphism of a prevariety X into a variety Y, then the **graph** of φ,

$$\Gamma_\varphi = \{(\mathbf{x}, \varphi(\mathbf{x})) \in X \times Y |\ \mathbf{x} \in X\},$$

is closed in $X \times Y$. Indeed, consider the two morphisms $\varphi \circ \mathrm{pr}_X, \mathrm{pr}_Y$: $X \times Y \to Y$. Then, $\Gamma_\varphi = \{(\mathbf{x}, \mathbf{y}) \in X \times Y |\ \varphi \circ \mathrm{pr}_X(\mathbf{x}, \mathbf{y}) = \mathrm{pr}_Y(\mathbf{x}, \mathbf{y})\}$.

Moreover, the restriction of pr_X to Γ_φ is an isomorphism onto X whose inverse is (id, φ). ∎

Here is a useful criterion for a prevariety to be a variety.

PROPOSITION 15.4: *Let X be a prevariety. If for each $\mathbf{x}, \mathbf{x}' \in X$ there is an open affine set U containing both $\mathbf{x}$ and $\mathbf{x}'$, then X is a variety.*

Proof: Let φ_1, φ_2: $Y \to X$ be morphisms. We prove that each $\mathbf{z}$ that belongs to the closure of $Z = \{\mathbf{y} \in Y |\ \varphi_1(\mathbf{y}) = \varphi_2(\mathbf{y})\}$ belongs to Z.

Indeed, X has an open affine set V that contains both $\varphi_1(\mathbf{z})$ and $\varphi_2(\mathbf{z})$. Then $U = \varphi^{-1}(V) \cap \varphi^{-1}(V)$ is an open neighborhood of $\mathbf{z}$ in Y. The restrictions φ_1' and φ_2' of φ_1 and φ_2 to U are morphisms into the affine variety V. Hence $Z \cap U = \{\mathbf{y} \in U |\ \varphi_1'(\mathbf{y}) = \varphi_2'(\mathbf{y})\}$ is closed in U and therefore contains $\mathbf{z}$. Conclude that Z is closed. ∎

COROLLARY 15.5: *Every projective prevariety X is a variety.*

Proof: By Remark 15.3(a) it suffices to prove that $\mathbb{P}^n$ is a variety. We apply criterion 15.4.

Let $\mathbf{x} = x_0{:}\cdots{:}x_n$ and $\mathbf{x} = x_0'{:}\cdots{:}x_n'$ be two points in $\mathbb{P}^n$. Since K is an infinite field there exist $a_0, \ldots, a_n \in K$ such that

$$(a_0x_0 + \cdots + a_nx_n)(a_0x_0' + \cdots + a_nx_n') \neq 0.$$

Thus the hyperplane $h(\mathbf{X}) = a_0X_0 + \cdots + a_nX_n = 0$ does not contain $\mathbf{x}, \mathbf{x}'$. In other words, $\mathbf{x}$ and $\mathbf{x}'$ belong to the open set $(\mathbb{P}^n)_h$. Apply now a linear automorphism on $\mathbb{P}^n$ that maps $h = 0$ onto the hyperplane $X_0 = 0$ to conclude that $(\mathbb{P}^n)_h$ is isomorphic to the affine open set $\mathbb{P}^n_0$. Hence $(\mathbb{P}^n)_h$ is also affine. ∎

PROPOSITION 15.6: *let X be a variety and let U, V be affine open subsets with coordinate rings R, S. Then $U \cap V$ is an affine open subset with coordinate ring RS (the compositum being formed in $K(X)$).*

Proof: The set $U \times V$ is open affine in $X \times X$ with coordinate ring $R \otimes_K S$ (Proposition 14.2). Each element of $R \otimes_K S$ is viewed here as a function $(\mathbf{x}, \mathbf{y}) \mapsto \sum f_i(\mathbf{x}) g_i(\mathbf{y})$ from $U \times V$ into K, where $f_i \in R$ and $g_i \in S$ (Exercise 13.9). The set $U \cap V$ is open in X and therefore irreducible. Let $Z = \Delta(X)$. The diagonal map Δ: $X \to X \times X$ maps $U \cap V$ isomorphically onto $Z \cap (U \times V)$. Hence $Z \cap (U \times V)$ is irreducible. By definition Z is closed in $X \times X$ and therefore $Z \cap (U \times V)$ is closed in $U \times V$. It follows that $Z \cap (U \times V)$ is an affine variety. Its coordinate ring T is obtained from $R \otimes_K S$ by restriction of the elements of $R \otimes_K S$ to $Z \cap (U \times V)$. Hence $U \cap V$ is also an affine variety. Its coordinate ring is $\Delta^*(R \otimes_K S)$, i.e., RS. ∎

PROBLEM 15.7: Let $X \subseteq \mathbb{P}^n$ be a projective variety defined by the homogeneous prime ideal $P \subseteq K[X_0, \ldots, X_n]$. Consider the affine variety

$$X^* = \{(x_0, \ldots, x_n) \in K^{n+1} |\ f(x_0, \ldots, x_n) = 0 \text{ for all } f \in P\}$$

defined by P (This is the **cone** over X). Show that

$$\{(x_0, \ldots, x_n) \in X^* |\ x_i \neq 0\} \cong (\mathbb{A}^1 - \{0\}) \times \{\mathbf{x} \in X |\ x_i \neq 0\}.$$

16. Noether's normalization theorem.

Let $R \subseteq S$ be rings and $x \in S$. We way that x is **integral** over R if it satisfies an equation of the form

$$(1a) \qquad x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0, \qquad a_i \in R,\ i = 1, \ldots, n.$$

An equivalent condition to (1a) is:

(1b) There exists a finitely generated nonzero R-module M which is contained in S and contains 1 such that $xM \subseteq M$.

Indeed, suppose that (1b) holds and let $v_1, \ldots, v_n$ be generators of M over R. Then there exist $a_{ij} \in R$ such that $xv_i = \sum_{j=1}^n a_{ij}v_j$ for $i = 1, \ldots, n$. Hence, with I the unit matrix of order $n \times n$ and $A = (a_{ij})$ we have $(xI - A)\mathbf{v} = 0$. Multiple the latter equation from the left by the adjoint of $xI - A$ to get $\det(xI - A)v_j = 0$, $j = 1, \ldots, n$. Since $1 \in M$, we have $\det(xI - A) = 0$. This gives a monic equation for x as in (1a).

Conversely, if $(1a)$ is satisfied, take M to be the R-module generated by $1, x, \ldots, x^{n-1}$. ∎

Problem 16.1: Let R be an integral domain, F a field that contains R, and x an element of F. Then x is integral over R if and only if $\varphi(x) \neq \infty$ for each place φ of F which is finite on R. Hint: If x^{-1} is not a unit of $R[x^{-1}]$, then it is contained in a prime ideal.

Proposition 16.2: *If R is an integral domain, F is the quotient field of R, and x is algebraic over F, then there exists $c \neq 0$ in R such that cx is integral over R.*

Proof: The element x satisfies an equation over R of degree, say n with highest coefficient a_n. Then $a_n^{n-1}x$ is integral over R. ∎

Let $R \subseteq S$ be rings. We say that S is **integral** over R if every element of S is integral over R.

Proposition 16.3: *Let $R \subseteq S$ be rings. If S is integral over R and is finitely generated as an R-algebra (i.e., $S = R[x_1, \ldots, x_n]$), then S is finitely generated as an R-module.*

Proof: Each x_i satisfies an integral equation over R of degree, say, r_i. Then $\{x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n} \mid k_i < r_i,\ i = 1, \ldots, n\}$ is a finite set of generators of S as an R-module. ∎

PROPOSITION 16.4: *Suppose that $R \subseteq S \subseteq T$ are rings. If S is integral over R and T is integral over S, then T is integral over R.*

Proof: Each $x \in T$ satisfies an equation of the form

$$x^n + b_{n-1}x^{n-1} + \cdots + b_0 = 0, \qquad b_i \in S.$$

By Proposition 16.2, the ring $S_0 = R[b_0, \ldots, b_{n-1}]$ is a finitely generated R-module. Therefore so is the S_0-module M generated by $1, x, \ldots, x^{n-1}$. As $xM \subseteq M$, x is integral over R. ∎

PROPOSITION 16.5 (Noether's normalization theorem): *Let K be an infinite field and let $F = K(x_1, \ldots, x_n)$ be a finitely generated field extension of K. Then there exist elements*

$$t_i = \sum_{j=1}^{n} a_{ij}x_j, \qquad i = 1, \ldots, n$$

in F with $a_{ij} \in K$ and $\det(a_{ij}) \neq 0$ such that $t_1, \ldots, t_r$ is a transcendence base for F/K, and the ring $K[\mathbf{t}] = K[\mathbf{x}]$ is integral over $K[t_1, \ldots, t_r]$.

Proof: If each x_i is transcendental over $K(x_1, \ldots, x_{i-1})$, take $r = n$ and $t_i = x_i$, $i = 1, \ldots, n$. Otherwise assume that x_n is algebraic over $K(x_1, \ldots, x_{n-1})$. Take an irreducible polynomial $f \in K[X_1, \ldots, X_n]$ such that $f(\mathbf{x}) = 0$ in which X_n occurs. Write f as the sum of its homogeneous parts, $f(\mathbf{X}) = \sum_{j=0}^{d} f_j(\mathbf{X})$, where $f_d(\mathbf{X}) \neq 0$. Then $f_d(X_1, \ldots, X_{n-1}, 1) \neq 0$. Since K is infinite, there exist $c_1, \ldots, c_{n-1}$ in K such that $f_d(c_1, \ldots, c_{n-1}, 1) \neq 0$. Set $t_i = x_i - c_i x_n$, $i = 1, \ldots, n-1$, and $t_n = x_n$. We claim that t_n is integral over $K[t_1, \ldots, t_{n-1}]$. Hence so are $x_1, \ldots, x_n$.

Indeed, if $f_d(\mathbf{X}) = \sum_{\mathbf{k}} b_k X_1^{k_1} \cdots X_n^{k_n}$, where $k_1 + \cdots + k_n = d$ for each $\mathbf{k} = (k_1, \ldots, k_n)$, then the coefficient of T_n^d in the transformed polynomial $f(T_1 + c_1T_n, \ldots, T_{n-1} + c_{n-1}T_n, T_n) = g(T_1, \ldots, T_{n-1}, T_n)$ is

$$\sum b_{\mathbf{k}} c_1^{k_1} \cdots c_{n-1}^{k_{n-1}} = f_d(c_1, \ldots, c_{n-1}, 1) \neq 0$$

and $\deg_{T_n}(g) = d$. Since $(t_1, \ldots t_{n-1}, T_n)$ is a zero of $g(T_1, \ldots, T_{n-1}, T_n)$, t_n is integral over $K[t_1, \ldots, t_{n-1}]$.

Now apply induction to $K[t_1, \dots, t_{n-1}]$ and use transitivity of integral dependence (Proposition 16.4) to conclude the proposition. ∎

Remark 16.6: If K is a finite field, then a weaker version of Noether normalization theorem asserts that F/K has a transcendence base $t_1, \dots, t_r$ such that $K[\mathbf{x}]$ contains $K[t_1, \dots, t_r]$ and is integral over it. In this case we can not assure that the t_i are linear combinations in $x_1, \dots, x_n$ with coefficients in K.

Noether's normalization theorem has a geometric interpretation.

Definition 16.7: *Finite morphisms.* Let $\varphi\colon X \to Y$ be a morphism of affine varieties. Let R and S be the coordinate rings of X an Y, respectively, and let $\varphi^*\colon S \to R$ be the corresponding homomorphism. Then φ is **finite** if R in integral over $\varphi^*(S)$.

Note that the restriction of a finite morphism $\varphi\colon X \to Y$ to a closed subvariety of X is also finite. The morphism φ in each of Examples 8.6, 8.7, and 8.8 is finite but the morphism φ' in Example 8.8 is not. This is because $K[T, (T-1)^{-1}]$ is not integral over $K[T^2-1, T(T^2-1)]$. Indeed, the K-homomorphism of $K[T]$ which maps T onto 1 is finite on the smaller ring but each extension of it to a place maps $(T-1)^{-1}$ to infinity.

Proposition 16.8 (Geometric form of Noether's normalization theorem): *Let X be an affine variety such that* $\mathrm{tr.deg}_K K(X) = r$. *Then there exists a finite epimorphism* $\pi\colon X \to \mathbb{A}^r$.

Lemma 16.9: (a) *If a field F is integral over a ring R, then R is also a field.*

(b) *Let $R \subseteq S$ be rings, $\mathfrak{m}$ an ideal of R, and $\mathfrak{n}$ an ideal of S that lies over $\mathfrak{m}$. If S is integral over R and $\mathfrak{n}$ is maximal, then so is $\mathfrak{m}$.*

Proof: Statement (b) follows from (a), so we prove (a).

Any nonzero element x of R has an inverse x^{-1} in F. By assumption x satisfies an equation

$$x^{-n} + a_{n-1}x^{-(n-1)} + \cdots + a_0 = 0$$

with $a_i \in R$, $i = 0, \dots, n-1$. Therefore $x^{-1} = -a_{n-1} - a_{n-2}x - \cdots - a_0x^{n-1} \in R$. Conclude that R is a field. ∎

A subset S of a ring A is said to be **multiplicative** if it contains 1 and is closed under multiplication. Consider the set of all quotients x/s with $x \in A$ and $s \in S$. Identify x/s with x'/s' if there exists $s'' \in S$ such that $s''(s'x - sx') = 0$. Then define addition and multiplication by the usual formulas to obtain a ring

$$S^{-1}A = \{x/s |\ x \in A,\ s \in S\}$$

called the **quotient ring** of A with respect to S. In particular if A is an integral domain, then $S^{-1}A$ is an integral domain that contains A with the same quotient field. In general the map $x \mapsto x/1$ is a homomorphism φ: $A \to S^{-1}A$ which is not necessarily injective. The map $\mathfrak{a} \mapsto S^{-1}\mathfrak{a}$ maps ideals of A onto ideals of $S^{-1}A$. If $\mathfrak{p}$ is a prime ideal of A, then $S^{-1}\mathfrak{p} \subset S^{-1}A$ if and only if $\mathfrak{p} \cap S = \emptyset$. In this case $S^{-1}\mathfrak{p}$ is a prime ideal of $S^{-1}A$ and $\varphi^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$.

Examples of multiplicative sets are $A - \mathfrak{p}$, for prime ideal $\mathfrak{p}$ of A and $\{1, f, f^2, f^3, \ldots\}$ for nonzero $f \in A$. In the first case we have denoted the ring of quotients by $A_{\mathfrak{p}}$ and in the latter case by A_f (or also by $A[f^{-1}]$ if A is an integral domain).

If $A \subseteq B$ are rings, B is integral over A, and S is a multiplicative subset of A (and therefore also of B), then $S^{-1}B$ is integral over $S^{-1}A$.

Problem 16.10: Let A, B be rings. Find a multiplicative subset S of $A \times B$ such that $A \cong S^{-1}(A \times B)$.

Proposition 16.11 (Going-up theorem of Cohen-Seidenberg): *Let $R \subseteq S$ be rings such that S is integral over R. Suppose that $\mathfrak{a} \subseteq \mathfrak{p}$ are ideals of R with $\mathfrak{p}$ prime. If $\mathfrak{b}$ is an ideal of S such that $\mathfrak{b} \cap R = \mathfrak{a}$, then S has a prime ideal $\mathfrak{q}$ that contains $\mathfrak{b}$ such that $\mathfrak{q} \cap R = \mathfrak{p}$.*

Proof: Replace R and S by $R/\mathfrak{a}$ and $S/\mathfrak{b}$, respectively, if necessary, to assume that $\mathfrak{a}, \mathfrak{b} = 0$. Now take a maximal ideal $\mathfrak{n}$ in $S_{\mathfrak{p}}$. As $S_{\mathfrak{p}}$ is integral over $R_{\mathfrak{p}}$, Lemma 17.9 implies that $\mathfrak{n} \cap R_{\mathfrak{p}}$ is a maximal ideal of $R_{\mathfrak{p}}$. Hence $\mathfrak{n} \cap R_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$. It follows that $\mathfrak{q} = \mathfrak{n} \cap S$ is a prime ideal of S that lies over $\mathfrak{p}$. ∎

The main properties of finite morphisms are the following:

Proposition 16.12: *Let* $\varphi\colon X \to Y$ *be a finite morphism of affine varieties. Then*

(a) φ *is a* **closed map**, *i.e., maps closed sets into closed sets,*

(b) *for each* $\mathbf{y} \in Y$, $\varphi^{-1}(\mathbf{y})$ *is a finite set, and*

(c) φ *is surjective if and only if the corresponding map* φ^* *of coordinate rings is injective.*

Proof of (a): Let R (resp., S) be the coordinate ring of X (resp., Y). Every closed subset of X has the form $V(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal of R. Let $\varphi^*\colon S \to R$ be the homomorphism of rings corresponding to φ. Set $\mathfrak{b} = (\varphi^*)^{-1}(\mathfrak{a})$. We prove that $\varphi(V(\mathfrak{a})) = V(\mathfrak{b})$.

Indeed, let $\mathbf{a} \in V(\mathfrak{a})$ and $\mathbf{b} = \varphi(\mathbf{a})$. If $g \in \mathfrak{b}$, then $g \circ \varphi \in \mathfrak{a}$. Hence $g(\mathbf{b}) = g(\varphi(\mathbf{a})) = 0$. It follows that $\mathbf{b} \in V(\mathfrak{b})$.

Conversely, let $\mathbf{b} \in V(\mathfrak{b})$. The maximal ideal $\mathfrak{n}$ of all functions of S that vanish at $\mathbf{b}$ contains $\mathfrak{b}$. Obviously, $\varphi^*(\mathfrak{n})$ is a maximal ideal of $\varphi^*(S)$ and $\mathfrak{n} = (\varphi^*)^{-1}(\varphi^*(\mathfrak{n}))$. By the going up theorem R has a maximal ideal $\mathfrak{m}$ that contains $\mathfrak{a}$ whose intersection with $\varphi^*(S)$ is $\varphi^*(\mathfrak{n})$. Thus $(\varphi^*)^{-1}(\mathfrak{m}) = \mathfrak{n}$. The point $\mathbf{a}$ of X that corresponds to $\mathfrak{m}$ belongs to $V(\mathfrak{a})$ and $\varphi(\mathbf{a}) = \mathbf{b}$.

Proof of (b): Let $x_1, \ldots, x_m$ be the coordinate functions of X and let $y_1, \ldots, y_n$ be the coordinate functions of Y. Take polynomials $f_1, \ldots, f_n \in R$ that define φ. Then $\varphi^*(S) = K[f_1, \ldots, f_n]$ and if $\varphi(\mathbf{a}) = \mathbf{b}$ for $\mathbf{a} \in X$ and $\mathbf{b} \in Y$, then $b_j = f_j(\mathbf{a})$, $j = 1, \ldots, n$.

The map $x_i \mapsto a_i$, $i = 1, \ldots, m$, extends to a K-homomorphism $\tau\colon R \to K$. Its restriction to $\varphi^*(S)$ maps f_j into $f_j(\mathbf{a}) = b_j$. By assumption x_i satisfies an equation

$$x_i^{k_i} + c_{i,k-1}x_i^{k_i-1} + \cdots + c_{i,0} = 0$$

with $c_{ij} \in \varphi^*(S)$. Apply τ on this equation to find that a_i satisfies a monic equation over K of the same degree with coefficients which are determinded by $\mathbf{b}$. So, there are only finitely many possibilities for a_i. Conclude that $\varphi^{-1}(\mathbf{b})$ is finite.

Proof of (c): Suppose first that $\varphi(X) = Y$. If $g \in S$ and $\varphi^*(g) = 0$, then $g(\varphi(\mathbf{a})) = 0$ for each $\mathbf{a} \in X$. Hence $g(\mathbf{b}) = 0$ for each $\mathbf{b} \in Y$. Conclude that $g = 0$ and φ is injective.

Conversely, suppose that φ^* is injective. Then φ^* is an isomorphism of S onto $\varphi^*(S)$. Hence, in the notation of the proof of (b), for each $\mathbf{b} \in Y$ the K-homomorphism of S which is defined by $\mathbf{y} \mapsto \mathbf{b}$ defines a homomorphism $\tau_0\colon \varphi^*(S) \to K$ such that $\tau_0(f_j) = b_j$. Since R is integral over $\varphi^*(S)$ this homomorphism extends to a homomorphism $\tau\colon R \to K$. It maps $\mathbf{x}$ to a point $\mathbf{a}$ of X such that $\varphi(\mathbf{a}) = \mathbf{b}$. Conclude that φ is surjective. ∎

17. The dimension theorem.

The **dimension** of a variety X is $\dim(X) = \mathrm{tr.deg}_K K(X)$.

If U is an open nonempty set in X, then $K(U) = K(X)$ and therefore $\dim(U) = \dim(X)$. For example $\dim(\mathbb{A}^n) = n$ and $\dim(\mathbb{P}^n) = n$.

The **dimension** of a closed subset A of a variety X is the maximal dimension of its components.

LEMMA 17.1: *The following statement on a variety X are equivalent:*

(a) $\dim(X) = 0$.

(b) $K(X) = K$.

(c) X *is a point.*

Proof of (a) implies (b): K is algebraically closed.

Proof of (b) implies (c): Let U be a nonempty affine subset of X. Then $\Gamma(U, \mathcal{O}_X) \subseteq K$ and therefore $\Gamma(U, \mathcal{O}_X)$ does not separate the points of U. So U consists of one point, say $\mathbf{x}$. If $\mathbf{x}'$ is another point of X, then $\{\mathbf{x}'\}$ is an open set which is disjoint from U. Since X is irreducible, this cannot happen. Conclude that $X = \{\mathbf{x}\}$.

Proof of (c) implies (a): Each element of $K(X)$ is a constant. ∎

EXAMPLE 17.2: *Hypersurface.* Let $f \in K[X_1, \ldots, X_n]$ be an irreducible polynomial and let $H = V(f)$ be the **hypersurface** defined by f in $\mathbb{A}^n$. The coordinate functions $x_1, \ldots, x_n$ on H satisfy the equation $f(\mathbf{x}) = 0$ and therefore $\dim K(\mathbf{x})$ is at most $n - 1$. On the other hand, suppose for example, that X_n appears in f. Then $x_1, \ldots, x_n$ are algebraically independent over K. Otherwise $g(x_1, \ldots, x_{n-1}) = 0$ for some $g \in K[X_1, \ldots, X_{n-1}]$ and therefore f divides g, a contradiction. Conclude that $\dim(X) = \mathrm{trans.deg}_K K(\mathbf{x}) = n - 1$.

LEMMA 17.3: *Let R be an integral domain containing K with quotient field F. Let $\mathfrak{p}$ be a prime of R and denote the quotient field of $R/\mathfrak{p}$ by $\overline{F}$. Then $\mathrm{tr.deg}_K F \geq \mathrm{tr.deg}_K \overline{F}$. Equality holds if and only if $\mathfrak{p} = 0$ or both sides are infinity.*

Proof: For each $z \in R$ let $\bar{z} = z + \mathfrak{p}$ be its residue class modulo $\mathfrak{p}$. Suppose that $\mathrm{tr.deg}_K \overline{F} = r < \infty$. Choose a transcendence base $\bar{x}_1, \ldots, \bar{x}_r \in R/\mathfrak{p}$ for $\overline{F}/K$. Lift $\bar{x}_i$

to an element x_i of R. If $f(\mathbf{x}) = 0$ for some $f \in K[X_1, \ldots, f_r]$, then $f(\bar{\mathbf{x}}) = 0$ and therefore $f = 0$. It follows that $x_1, \ldots, x_r$ are algebraically independent over K and therefore $\mathrm{tr.deg}_K F \geq r$.

Now suppose that $\mathrm{tr.deg}_K F = r$. Then $x_1, \ldots, x_r$ is a trancsendence base for F/K. We prove that $\mathfrak{p} = 0$. Indeed, let $z \in \mathfrak{p}$. Take an equation

$$a_n(\mathbf{x})z^n + \cdots + a_0(\mathbf{x}) = 0 \tag{1}$$

with $a_i \in K[X_1, \ldots, X_r]$ and with smallest degree n. Reduce it modulo $\mathfrak{p}$ to get $a_0(\bar{\mathbf{x}}) = 0$. Therefore $a_0(\mathbf{x}) = 0$ and we can divide (1) by z unless $z = 0$. ∎

Proposition 17.4: *Let Y be a proper closed subvariety of a variety X. Then* $\dim(Y) < \dim(X)$.

Proof: Choose an affine open set U that intersects Y. Let R be the coordinate ring of U, $\mathfrak{p}$ the prime ideal corresponding to the closed set $U \cap Y$. Then $\mathfrak{p} \neq 0$. If $U \cap Y = U$, then $U \subseteq Y$. Hence, Y being both closed and dense in X is equal to X. This contradiction to the assumption implies that $U \cap Y \subset U$ and therefore $\mathfrak{p} \neq 0$. Note that R is finitely generated over K. Hence $K(X)$, which is the quotient field of R, has a finite transcendence degree over K. Also, $K(Y)$ is the quotient field of $R/\mathfrak{p}$. Conclude from Lemma 17.3 that $\dim(Y) < \dim(X)$. ∎

In the situation of this Proposition $\mathrm{codim}(Y, X) = \dim(X) - \dim(Y)$ is called the **codimension** of Y in X. This is half of what we want so that our definition gives a good dimension function. The other half is that is does not go down too much.

Theorem 17.4 (The dimension theorem): *Let X be a variety, $U \subseteq X$ open, $g \in \Gamma(U, \mathcal{O}_X)$, Z an irreducible component of $H = \{\mathbf{x} \in U |\ g(\mathbf{x}) = 0\}$. If $g \neq 0$, then* $\dim(Z) = \dim(X) - 1$.

Proof: In each of the following parts of the proof we reduce the Theorem to a simpler case. Eventually we reduce it to Example 17.2.

Part A: *Reduction to the affine case.* Let $U_0 \subseteq U$ be an open affine such that $U_0 \cap Z \neq \emptyset$. Then g does not vanish on U_0 (otherwise it would vanish on U). Hence, the restriction g_0 of g to U_0 is a nonzero element of $\Gamma(U_0, \mathcal{O}_{U_0})$. Moreover, $U_0 \cap Z$ is an irreducible component of $U_0 \cap H = \{\mathbf{x} \in U_0 |\ g_0(\mathbf{x}) = 0\}$. Indeed, if V_0 is a closed subset $U_0 \cap H$ that contains $U_0 \cap Z$, then $Z \subseteq V$ and therefore the closure V of V_0 in H is an irreducible closed subset of H that contains Z. Since Z is an irreducible component of H, $V = Z$. Hence $V_0 = U_0 \cap Z$.

Finally observe that as $U_0 \cap Z$ is open in Z and nonempty it has the same dimension as Z. Also, $\dim(U_0) = \dim(X)$. Hence, replacing X by U_0, U by U_0, g by g_0, H by $U_0 \cap H$, and Z by $U_0 \cap Z$, if necessary, we may assume that $X = U$ is affine. Let R be its coordinate ring.

Part B: *Reduction to the case where H is irreducible.* Let $H = Z \cup Z_1 \cup \cdots \cup Z_k$ be the decomposition of H into irreducible components. In particular $Z \not\subseteq Z_1 \cup \cdots \cup Z_k$ (Lemma 4.5). Therefore, there exists $h \in R$ which vanish on each Z_i but not on Z. Thus $X_h \cap Z_i = \emptyset$ for $i = 1, \ldots, k$ but $X_h \cap Z \neq \emptyset$.

Let g_1 be the restriction of g to X_h. Then $X_h \cap H = \{\mathbf{x} \in X_h |\ g_1(\mathbf{x}) = 0\} = X_h \cap Z$ is irreducible. So, replacing X by X_h, g by g_1, H by $X_h \cap H$, and Z by $H_h \cap Z$, we may further assume that $H = Z$ is irreducible.

Part C: *Reduction to the case where $X = \mathbb{A}^r$.* By Noether's normalization theorem $K(X)/K$ has a transcendence base $t_1, \ldots, t_r$ such that R is integral over $R_0 = K[\mathbf{t}]$. Let $F = K(X)$, $F_0 = K(\mathbf{t})$, and $f = \mathrm{Norm}_{F/F_0}(g)$. Then f, as a product of conjugates of g over F_0, is integral over R_0. Since the latter ring is a unique factorization domain it is integrally closed. Hence $f \in R_0$.

We claim that $f \in Rg$. Indeed, let $g^k + a_{k-1}g^{k-1} + \cdots + a_0 = 0$ be an irreducible equation for g over F_0 with $a_i \in R$. Then $a_0 = (-1)^k \mathrm{Norm}_{F_0(g)/F_0} g$ and $a_0^{kl} = (-1)^{kl} \mathrm{Norm}_{F/F_0} g = \pm f$, where $l = [F : F_0(g)]$. As $a_0 \in Rg$, we have $f \in Rg$. Hence, $\sqrt{R_0 f} \subseteq \sqrt{Rg}$.

Conversely, if $h \in \sqrt{Rg} \cap R_0$, then $h^s \in Rg$. Taking norms we find that $h^{s[F:F_0]} \in R_0 f$. Whence $h \in \sqrt{R_0 f}$. Conclude that $\sqrt{Rg} \cap R_0 = \sqrt{R_0 f}$.

But $\sqrt{Rg} = I(H)$ is a prime ideal of R. Hence, $\sqrt{R_0 f}$ is a prime ideal of R_0. Thus, $H_0 = \{\mathbf{a} \in \mathbb{A}^r |\ f(\mathbf{a}) = 0\}$ is irreducible.

In particular $R/\sqrt{Rg}$ is the coordinate ring of H and $R_0/\sqrt{R_0 f}$ is the coordinate ring of H_0. Since the former ring is integral over the latter, the quotient field of the former ring is algebraic over the quotient ring of the latter. In particular they have the same transcendence degree over K. Thus $\dim(H) = \dim(H_0)$. Also, $\dim(X) = \dim(\mathbb{A}^r)$. So, without loss, we may assume that $X = \mathbb{A}^r$ and $H = V(f)$ is irreducible.

Part D: *Conclusion of the proof.* Although f need not be irreducible it is a power of an irreducible polynomial f_0. Replacing f by f_0 does not change H. Hence, we may assume that f is irreducible. But then it follows form Example 17.2 that $\dim(H) = r-1$.

Conclude that our original Z satisfies $\dim(Z) = \dim(X) - 1$. ∎

Exercise 17.6 (Krull's principal ideal theorem): *Let R be finitely generated integral domain over K with quotient field F, $f \in R$, and $\mathfrak{p}$ a minimal ideal among the prime ideals that contain Rf. Denote the quotient field of $R/\mathfrak{p}$ by $\overline{F}$. If $f \neq 0$, then* $\mathrm{tr.deg}_K \overline{F} = \mathrm{tr.deg}_K F - 1$. Hint: Translate algebraic terms to geometric ones and use the dimension theorem.

18. Applications of the dimension theorem.

Let Z be a closed subset of a variety X. Then Z has **pure dimension** r is each of its components has dimension r (similarly for **pure codimension** r).

The conclusion of the dimension theorem may be stated: $V(g)$ has pure codimension 1, for any non-zero $g \in \Gamma(X, \mathcal{O}_X)$.

The theorem has an obvious converse: Suppose Z is an irreducible closed subset of a variety X of codimension 1. Then for each open set U such that $Z \cap U \neq \emptyset$ and for each nonzero $f \in \Gamma(U, \mathcal{O}_X)$ vanishing on Z, $Z \cap U$ is a component of $V(f)$. Indeed, let W be a component of $V(f)$ containing $Z \cap U$. Then, by the dimension theorem,

$$\dim(X) - 1 = \dim(Z \cap U) \leq \dim(W) = \dim(X) - 1.$$

Hence $\dim(Z \cap U) = \dim(W)$. Since both $Z \cap U$ and W are irreducible Proposition 17.4 implies that $Z \cap U = W$. Thus $Z \cap U$ is a component of $V(f)$.

PROPOSITION 18.1: *Let Z be a maximal proper closed irreducible subset of a variety X. Then* $\dim(Z) = \dim(X) - 1$.

Proof: Choose an open affine subset $U \subseteq X$ that intersects Z. Then $U \cap Z$ is a maximal proper closed irreducible subset of U. Indeed, if V is a proper closed irreducible subset of U that contains V, then its closure $\overline{V}$ in X is closed, irreducible and contains Z. Also, $U \cap \overline{V} = V$. Hence $\overline{V} \subset X$ and therefore $\overline{V} = Z$ and $V = U \cap Z$.

Take now nonzero $f \in \Gamma(U, \mathcal{O}_X)$ which vanishes on $U \cap Z$. Then $U \cap Z$ is a component of $V(f)$ and therefore, by the dimension theorem, $\dim(Z) = \dim(Z \cap U) = \dim(X) - 1$. ∎

PROPOSITION 18.2 (Topological characterization of dimension): *Suppose $\emptyset \subset Z_0 \subset Z_1 \subset \cdots \subset Z_r = Z$ is a maximal chain of closed irreducible subsets of X. Then $\dim(X) = r$.*

Proof: Use induction on r and Proposition 18.1. ∎

PROPOSITION 18.3: *Let X be a variety and let Z be a component of $V(f_1, \ldots, f_r)$, where $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. Then* $\dim(Z) \geq \dim(X) - r$.

Proof: Use induction on r. Note that Z is an irreducible subset of $V(f_1,\ldots,f_{r-1})$. So, it is contained in some component Z' of $V(f_1,\ldots,f_{r-1})$. By induction, $\dim(Z') \geq \dim(X)-(r-1)$. Since $Z \subseteq Z' \cap V(f_r) \subseteq V(f_1,\ldots,f_r)$ and Z is a component of $V(f_1,\ldots,f_r)$, Z is also a component of $Z' \cap V(f_r)$. If f_r vanishes on Z', then $Z \subseteq Z' \subseteq V(f_1,\ldots,f_r)$ and therefore $Z' = Z$. In this case $\dim(Z) = \dim(Z') > \dim(X) - r$. Otherwise, by the dimension theorem, $\dim(Z) = \dim(Z') - 1 \geq \dim(X) - r$. ∎

Of course, equality need not hold in the above result: e.g, take $f_1,\ldots,f_r$, $r > 1$. Hence, by the dimension theorem, $\dim(Z)$ component of $V(f)$.

LEMMA 18.4: *Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_m$ be prime ideals and let $\mathfrak{a}$ be an ideal of a ring R. If $\mathfrak{a} \not\subseteq \mathfrak{p}_i$ for $i = 1,\ldots,m$, then $\mathfrak{a} \not\subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_m$.*

Proof: Omitting those $\mathfrak{p}_i$ which are contained in some other $\mathfrak{p}_j$, we may suppose that there are no inclusion relations between the $\mathfrak{p}_i$'s. We use induction on m.

Since $\mathfrak{p}_m$ is a prime and does not contain $\mathfrak{a}$ nor $\mathfrak{p}_i$ for $i = 1,\ldots,m-1$, it does not contain $\mathfrak{a}\mathfrak{p}_1\cdots\mathfrak{p}_{m-1}$. Choose $x \in \mathfrak{a}\mathfrak{p}_1\cdots\mathfrak{p}_{m-1} - \mathfrak{p}_m$. By induction hypothesis there exists an element

$$s \in S = \mathfrak{a} - (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{m-1}).$$

If $s \notin \mathfrak{p}_m$, we are done. So assume $s \in \mathfrak{p}_m$. But then $s + x \in \mathfrak{a}$, $s + x \cong s \not\cong 0 \bmod \mathfrak{p}_i$ for $i = 1,\ldots,m-1$, and $s + x \cong x \not\cong 0 \bmod \mathfrak{p}_m$. So $x + s \in \mathfrak{a} - (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_m)$, ∎

We rephrase the Lemma in geometric terms:

COROLLARY 18.5: *Let X be an affine variety with coordinate ring R. Suppose that $Y_1,\ldots,Y_m \subseteq X$ are irreducible closed sets and $Z \subseteq X$ is a closed set. If $Y_i \not\subseteq Z$ for $i = 1,\ldots,m$, then there exists $f \in R$ which does not vanish identically on Y_i, $i = 1,\ldots,m$ but does vanish on Z.*

Proof: By assumption $I(Y_i)$ is a prime ideal of R which does not contain $I(Z)$, $i = 1,\ldots,m$. Hence, by Lemma 18.4 there exists $f \in I(Z)$ which belongs to none of the $I(Y_i)$. ∎

PROPOSITION 18.6: *Let X be an affine variety, Z a closed irreducible subset of codimension r. Then there exist $f_1,\ldots,f_r$ in $R = \Gamma(X,\mathcal{O}_X)$ such that Z is a component of*

$V(f_1, \ldots, f_r)$ and $V(f_1, \ldots, f_r)$ is of pure codimension r.

Proof: By Proposition 18.2 there exists a chain $Z = Z_r \subset \cdots \subset Z_2 \subset Z_1 \subset X$, of irreducible sets with $\operatorname{codim}(Z_s) = s$, $s = 1, \ldots, r$.

If $r = 1$, take $f_1 \in R$, $f_1 \neq 0$ which vanish on Z_1. By the dimension theorem and its converse each component of $V(f_1)$ has codimension 1, and Z_1 is a component of $V(f_1)$.

Now say $f_1, \ldots, f_{r-1}$ have been chosen such that each of the components, $Y_1, \ldots, Y_l$, of $V(f_1, \ldots, f_{r-1})$ have codimension $r - 1$ and Z_{r-1} is one of them, say, $Z_{r-1} = Y_1$. In particular the dimension of Y_i is greater than that of Z_r and therefore $Y_i \not\subseteq Z_r$ for $i = 1, \ldots, l$. By Corollary 18.5, there exists $f_r \in R$ that vanishes on X_r but not on any of the Y_i's.

Let Y be a component of $V(f_1, \ldots, f_r)$. From $V(f_1, \ldots, f_r) = (Y_1 \cap V(f_r)) \cup \cdots \cup (Y_l \cap V(f_r))$ deduce that Y is a component of $Y_i \cap V(f_r)$ for some i between 1 and l. Since f_r is not zero on Y_i, we have $\dim(Y) = \dim(Y_i) - 1$ and therefore $\operatorname{codim}(Y) = r$. Also, $Z_r \subseteq V(f_1, \ldots, f_r)$. Since Z_s is irreducible it is contained in one of the components of $V(f_1, \ldots, f_r)$. Both Z and this component have the same dimension. Conclude from Proposition 17.4 that they coincide. ■

The **Krull dimension** of a ring R is the the maximal number r for which R has an ascending chain of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r$$

(If such a number does not exist, then $\dim(R) = \infty$). For example, a field has dimension 0, $\mathbb{Z}$ has dimension 1, and $K[X_1, \ldots, X_n]$ has dimension n.

Recall that in Section 10 we have attached a local ring $\mathcal{O}_{Z,X}$ to every irreducible closed subset Z of every variety.

PROPOSITION 18.7: *Let Z be an irreducible closed subset of a variety X. Then the Krull dimension of Z is* $\operatorname{codim}(Z) = \dim(X) - \dim(Z)$.

Proof: Let $r = \operatorname{codim}(Z)$. Assume without loss that X is affine. Let R be the coordinate ring of X and let $\mathfrak{p} = I(Z)$ be the prime ideal corresponding to Z. By Proposition 18.2

each maximal descending chain of irreducible subsets of X ending in Z has length r:

$$X = Z_0 \supset Z_1 \supset \cdots \supset Z_r = Z.$$

So, for each maximal ascending chain of prime ideals of R ending in $\mathfrak{p}$ has length r. Hence, each maximal ascending chain of $\mathcal{O}_{Z,X} = R_{\mathfrak{p}}$, is of length r. Conclude that the Krull dimension of this local ring is r. ∎

Suppose $Z \subseteq X$ is irreducible and of codimension 1. A natural question to ask is whether, for all $\mathbf{z} \in Z$, there is some neighborhood U of $\mathbf{z}$ in X and some function $f \in \Gamma(U, \mathcal{O}_X)$ such that $Z \cap U$ is not just a component of $V(f)$, but actually equal to $V(f)$. More generally, if $Z \subseteq X$ is a closed subset of pure codimension r, one may ask whether, for all $\mathbf{z} \in Z$, there is a neighborhood U of $\mathbf{z}$ and functions $f_1, \ldots, f_r \in \Gamma(U, \mathcal{O}_X)$ such that

$$Z \cap U = V(f_1, \ldots, f_r).$$

This is unfortunately not always true even in the special case where Z is irreducible of codimension 1. A closed set Z with this property is often refereed to as a **local set-theoretic complete intersection**, and it has many other special properties.

There is one case where we can say something however:

Proposition 18.8: *Let X be an affine variety with coordinate ring R. Suppose that R is a unique factorization domain (e.g. $X = \mathbb{A}^n$). Then every closed subset $Z \subseteq X$ of pure codimension 1 equals $V(f)$ for some $f \in R$.*

Proof: Note first that every minimal prime ideal $\mathfrak{p}$ of R is principal. Indeed, let $f \in \mathfrak{p}$, $f \neq 0$. Since $\mathfrak{p}$ is prime it contains one of the irreducible factors p of f. But then $Rf \subseteq \mathfrak{p}$ and by minimality $\mathfrak{p} = Rf$.

Now let $Z = Z_1 \cup \cdots \cup Z_m$ be the factorization of Z into irreducible components. Then $I(Z) = I(Z_1) \cap \cdots \cap I(Z_m)$ is the intersection of minimal prime ideals. By the first paragraph of the proof, there exists $f_i \in R$ such that $I(Z_i) = Rf_i$, $i = 1, \ldots, m$. Then $I(Z) = Rf$, with $f = f_1 \cdots f_m$ and therefore $Z = V(f)$. ∎

Proposition 18.9: $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Proof: Replace X and Y by affine open sets, if necessary, to assume that X and Y are affine with coordinate rings R and S, respectively. The proposition follows then from Proposition 13.7. ∎

19. Application of the dimension theorem to projective varieties.

The results and methods of the last two sections all have projective formulations which give some global as well as some local information:

THEOREM 19.1 (The dimension theorem for projective varieties): *Let* $X \subseteq \mathbb{P}^n$ *be a projective variety of positive dimension and let* $P = I(X) \subseteq K[X_0, \ldots, X_n]$ *be the corresponding prime ideal. If* $f \in K[X_0, \ldots, X_n]$ *is homogeneous and* $f \notin P$*, then* $X \cap V(f)$ *is nonempty and of pure codimension 1.*

Proof: The restriction of f to X defines a nonzero element of $\Gamma(X, \mathcal{O}_X)$. So, by the dimension theorem every component of $X \cap V(f)$ has codimension 1. So, we have only to prove that $X \cap V(f)$ is nonempty.

Let X^* be the cone over X. By Problem 15.7, it has an open subset which is isomorpic to the direct product of $\mathbb{A}^1 - \{0\}$ and an open subset of X. Hence, by Proposition 18.9, $\dim(X^*) = \dim(X) + 1 \geq 2$.

Let $V^*(f)$ be the locus of $f = 0$ in $\mathbb{A}^{n+1}$. Since $(0, 0, \ldots, 0) \in X^* \cap V^*(f)$, the set $X^* \cap V^*(f)$ is nonempty. By the dimension theorem it has a component of codimension 1, hence of dimension at least 1. Choose a nonzero point $(x_0, x_1, \ldots, x_n)$ in this component. It gives a point of $X \cap V(f)$. ∎

COROLLARY 19.2: *Let* $X \subseteq \mathbb{P}^n$ *be a projective variety of positive dimension. If* $f_1, \ldots, f_r \in K[X_0, \ldots, X_n]$ *and nonzero homogeneous polynomials, then all components of* $X \cap V(f_1, \ldots, f_r)$ *have codimension at most* r *in* X*. If* $\dim(X) \geq r$*, then* $X \cap V(f_1, \ldots, f_r)$ *is nonempty.*

COROLLARY 19.3: *A system of homogeneous equations*

$$f_i(X_0, \ldots, X_n) = 0, \qquad i = 1, \ldots, n,$$

always has a nontrivial solution in K.

COROLLARY 19.4: *Suppose that* Y *is a closed subvariety of a variety* $X \subseteq \mathbb{P}^n$ *of codimension* r*. Then there exist homogeneous nonzero polynomials* $f_1, \ldots, f_r \in K[X_0, \ldots, X_n]$ ∎ *such that* $X \cap V(f_1, \ldots, f_r)$ *is of pure codimension* r *and has* Y *as a component.*

If $s = \dim(X)$, *then there exist homomogeneous nonzero polynomials* $f_1, \ldots, f_{s+1} \in K[X_0, \ldots, X_n]$ *such that* $X \cap V(f_1, \ldots, f_{s+1}) = \emptyset$.

Proof: To prove the first part of the Corollary follow exactly the inductive proof of Proposition 18.6, using $K[X_0, \ldots, X_n]$ instead of the affine coordinate ring.

For the second part, we already have $f_1, \ldots, f_s$ such that $X \cap V(f_1, \ldots, f_s)$ is of pure dimesnion zero, i.e., is a finite set of points. Then let f_{s+1} be a homogeneous polynomial that vanish at none of these points. ■

An interesting Corollary of these results is the following global theorem:

PROPOSITION 19.5: *Let* $\varphi\colon \mathbb{P}^m \to \mathbb{P}^n$ *be a morphism such that* $W = \varphi(\mathbb{P}^m)$ *is closed (actually this is always the case as we will see in Section 21). Then either* W *is a single point or* $\dim(W) = m$.

Proof: Let $r = \dim(W)$ and assume that $1 \le r \le m-1$. By Proposition 19.3, there exist nonzero homogeneous polynomials

$$f_1, \ldots, f_{r+1} \in K[X_0, \ldots, X_n]$$

such that $W \cap V(f_1, \ldots, f_{r+1}) = \emptyset$. As $r \ge 1$, Corollary 19.2 implies that $W \cap V(f_i) \neq \emptyset$. Hence $Z_i = \varphi^{-1}(V(f_i)) \neq \emptyset$. Then Z_i is locally defined by the vanishing of one function (since this is true on sufficiently small affine open subsets of $\mathbb{P}^m$). Hence, by the dimension theorem, either $Z_i = \mathbb{P}^m$ or Z_i is of pure codimension 1. In latter case Z_i is a component of $V(g)$ for some homogeneous polynomial $g \in K[X_0, \ldots, X_m]$ (Proposition 19.3), hence $Z_i = V(g_i)$, for some irreducible factor of g. It follows that $V(g_1, \ldots, g_{r+1}) = \emptyset$. As $r + 1 \le m$, this contradicts Proposition 19.2. Conclude that either $r = 0$ or $r = m$. ■

20. The fibres of a morphism.

Let $\varphi\colon X \to Y$ be a morphism of varieties. The purpose of this section is to study the **fibres** of φ, i.e., those closed subsets of X of the form $\varphi^{-1}(\mathbf{y})$, $\mathbf{y} \in Y$.

We say that a morphism $\varphi\colon X \to Y$ is dominating if its image is dense in Y, i.e., $Y = \overline{\varphi(X)}$.

PROPOSITION 20.1: *For a morphism $\varphi\colon X \to Y$ of varieties let $Z = \overline{\varphi(X)}$. Then Z is irreducible, the restricted morphism $\varphi_0\colon X \to Z$ is dominating and φ_0^* induces a K-embedding*

$$(1) \qquad \varphi_0^*\colon K(Z) \to K(X).$$

In particular $\dim(Z) \le \dim(X)$.

Proof: Suppose $Z = W_1 \cup W_2$, where W_1 and W_2 are closed subsets. Then $X = \varphi^{-1}(W_1) \cup \varphi^{-1}(W_2)$. Since X is irreducible $X = \varphi^{-1}(W_1)$ or $X = \varphi^{-1}(W_1)$. Hence $\varphi(X) \subseteq W_1$ or $\varphi(X) \subseteq W_2$. Since W_1 and W_2 are closed this implies $Z \subseteq W_1$ or $Z \subseteq W_2$. It follows that either $Z = W_1$ or $Z = W_2$. Conclude that Z is irreducible.

By definition, φ_0 is dominating. If $U \subseteq Z$ is open and nonempty, then $U \cap \varphi(X) \neq \emptyset$ and therefore $\varphi^{-1}(U)$ is nonempty and open in X. The induced homomorphism φ_0^* maps $\Gamma(U, \mathcal{O}_Z)$ injectively into $\Gamma(\varphi^{-1}(U), \mathcal{O}_X)$ and therefore into $K(X)$. This maps is compatible with restriction and therefore induces an embedding

$$\varphi_0^*\colon \varinjlim \Gamma(U, \mathcal{O}_Z) \to K(X)$$

where U ranges over all open subsets of Z and V ranges over all open subsets of X. As the direct limit is $K(Z)$, this gives the embedding (1). ∎

This reduces the study of fibres of an arbitrary morphism to the case of dominating morphisms. Since a finite morphism is a closed map (Proposition 16.12) it is dominating if and only if it is surjective.

THEOREM 20.2: *Let $\varphi\colon X \to Y$ be a dominating morphism of varieties and let $r = \dim(X) - \dim(Y)$. Consider a closed irreducible subset W of Y and let Z be a component* *of* $\varphi^{-1}(W)$ *that dominates* W. *Then*

$$\dim(Z) \geq \dim(W) + r,$$

i.e.,

$$\dim(X) - \dim(Z) \leq \dim(Y) - \dim(W). \tag{2}$$

Proof: Take an affine open subset U of Y that intersects W. Then replace Y by U, X by $\varphi^{-1}(U)$, W by $W \cap U$ and Z by $Z \cap \varphi^{-1}(U)$, if necessary, to assume that Y is affine. Let $s = \dim(Y) - \dim(W)$. By Proposition 18.6, there are $g_1, \ldots, g_s \in \Gamma(Y, \mathcal{O}_Y)$ such that W is a component of $V(g_1, \ldots, g_s)$. Consider $f_i = \varphi^*(g_i) = g_i \circ \varphi \in \Gamma(X, \mathcal{O}_X)$. Then $Z \subseteq \varphi^{-1}(W) \subseteq \varphi^{-1}(V(g_1, \ldots, g_s)) = V(f_1, \ldots, f_s)$.

We prove that Z is actually a component of $V(f_1, \ldots, f_s)$. Indeed, take a component Z' of $V(f_1, \ldots, f_s)$ that contains Z. Then $W = \overline{\varphi(Z)} \subseteq \overline{\varphi(Z')} \subseteq \varphi(V(f_1, \ldots, f_s)) \subseteq V(g_1, \ldots, g_s)$ and $\overline{\varphi(Z')}$ is irreducible (Proposition 20.1). As W is also a component of $V(g_1, \ldots, g_s)$ we have $W = \overline{\varphi(Z')} \supseteq \varphi(Z')$. Hence $Z \subseteq Z' \subseteq \varphi^{-1}(W)$. Since Z is a component of $\varphi^{-1}(W)$ we have $Z = Z'$. By Proposition 18.3, $\dim(Z) \geq \dim(X) - s$. This is inequality (2). ∎

Corollary 20.3: *If* Z *is a component of* $\varphi^{-1}(\mathbf{y})$ *for some* $\mathbf{y} \in Y$, *then* $\dim(Z) \geq r$.

The next theorem shows that equality in (2) holds "almost everywhere".

Theorem 20.4: *Let* $\varphi\colon X \to Y$ *be a dominating morphism of varieties and let* $r = \dim(X) - \dim(Y)$. *Then there exists a nonempty open set* $U \subseteq Y$ *such that*

(a) $U \subseteq \varphi(X)$ *and*

(b) *for each irreducible closed subset* $W \subseteq Y$ *such that* $W \cap U \neq \emptyset$, *and for each component* Z *of* $\varphi^{-1}(W)$ *that dominates* W

$$\dim(Z) = \dim(W) + r. \tag{3}$$

Proof: As in Theorem 20.3, we may as well replace Y by a nonempty open affine subset; therefore, assume that Y is affine. Moreover, we can also reduce the proof to the case where X is affine. In fact, cover X by affine open sets X_i and let φ_i: $X_i \to Y$ be the restriction of φ. Then $\varphi(\overline{X}_i) \subseteq \overline{\varphi(X_i)}$. Hence, $Y = \overline{\varphi(X)} = \overline{\varphi(\bigcup \overline{X}_i)} = \overline{\bigcup \varphi(X_i)} \subseteq Y$, and therefore $\overline{\varphi(X_i)} = Y$, which means that φ_i is dominating. Let $U_i \cap Y$ satisfy (a) and (b) of the theorem for φ_i. Let $U = \bigcap U_i$. If Z is as above, then for each i $Z \cap X_i$ is a component of $\varphi^{-1}(W) \cap U_i$ of the same dimension as Z and for at least one i, $Z \cap X_i$ dominates W. Hence, $\dim(Z) = \dim(Z \cap X_i) \geq \dim(W) + r$. So with this U, (a) and (b) are correct for φ itself.

Now assume X and Y are affine, and let R and S be their coordinate rings. By Proposition 20.1, φ^* embeds $E = K(Y)$ in $F = K(X)$ and S into R. So, identify E with $\varphi^*(E)$ if necessary to assume that $E \subseteq F$ and $S \subseteq R$. Observe that RE is finitely generated over E and F is its quotient field. By Noether's normalization theorem F/E has a transcendence base $t_1, \ldots, t_r$ such that $t_i \in RE$ and RE is integral over $E[\mathbf{t}]$. Since E is the quotient field of S and $S \subseteq R$ each element of RE is a product of an element of R and and an element of E. Replace each t_i with the corresponding factor in R to assume that $t_i \in R$, $i = 1, \ldots, r$.

Now consider the two rings $S[\mathbf{t}] \subseteq R$. Each element x of R satisfies a monic equation with coefficients in $E[\mathbf{t}]$. These coefficients are themselves polynomials in $\mathbf{t}$ with coefficients in E. Write each of the later coefficients as a quotient of two elements of S. Let g be a common denominator of these coefficients. Then x is integral over $S[g^{-1}]$. Apply this reasoning to a finite set of generators of R as an S-algebra to assume that this is the case for all $x \in R$. Thus $R[g^{-1}]$ is integral over $S[g^{-1}, \mathbf{t}]$.

Let $U = Y_g = \{\mathbf{y} \in Y |\ g(\mathbf{y}) \neq 0\}$. Then $\Gamma(U, \mathcal{O}_Y) = S[g^{-1}]$ and $S[g^{-1}, \mathbf{t}]$ is the coordinate ring of the affine variety $U \times \mathbb{A}^r$. Also $\varphi^{-1}(U) = X_g$ and $\Gamma(\varphi^{-1}(U), O_{\mathcal{X}}) = R[g^{-1}]$. The following diagram of ring inclusions

$$\begin{array}{ccc} R & \hookrightarrow & R[g^{-1}] \\ \big\downarrow & & \big\downarrow \\ S[\mathbf{t}] & \hookrightarrow & S[g^{-1}, \mathbf{t}] \\ \big\downarrow & & \big\downarrow \\ S & \hookrightarrow & S[g^{-1}] \end{array}$$

induces a commutative diagram of morphisms of varieties:

$$\begin{array}{ccc} \varphi^{-1}(U) & \longrightarrow & X \\ {\scriptstyle \pi}\downarrow & & \\ U \times \mathbb{A}^r & & \downarrow{\scriptstyle \varphi} \\ {\scriptstyle \mathrm{pr}_U}\downarrow & & \\ U & \longrightarrow & Y, \end{array}$$

where the horizontal maps are inclusions, and π is finite and surjective. In particular $U \subseteq \varphi(X)$ which is (a).

To show (b), let W be an irreducible closed subset that intersects U, and let Z be a component of $\varphi^{-1}(W)$ that dominates W. By Theorem 20.2, $\dim(Z) \geq \dim(W) + r$. So, we have to prove the other inequality.

As $\overline{\varphi(Z)} = W$, we have $\varphi(Z) \cap U \neq \emptyset$ and hence $Z_0 = Z \cap \varphi^{-1}(U) \neq \emptyset$. Let $W_0 = W \cap U$. Then $\varphi(Z_0) \subseteq W_0$, hence $\pi(Z_0) \subseteq W_0 \times \mathbb{A}^r$ and therefore $\overline{\pi(Z_0)} \subseteq W_0 \times \mathbb{A}^r$. Also, the restriction $\pi'\colon Z_0 \to \overline{\pi(Z_0)}$ of π to Z_0 is still finite and dominating. Hence it induces an inclusion of $K(\overline{\pi(Z_0)})$ in $K(Z_0)$ such that the latter field is algebraic over the former one. It follows that

$$\dim(Z) = \dim(Z_0) = \dim(\overline{\pi(Z_0)}) \leq \dim(W_0 \times \mathbb{A}^r) = \dim(W) + r.$$

This concludes the proof of the theorem. ∎

Remark 20.5: There is a stronger version of Theorem 20.4 in which the condition on Z in (b) to diminate W is relaxed to $Z \cap \varphi^{-1}(U) \neq \emptyset$. In this version U should be chosen to be **normal**, i.e., the local ring of each point of U must be integrally closed. Then $S[g^{-1}, t]$ is an integrally closed domain and the ring extension $R/S[g^{-1}, t]$ satisfies the, so called, "going down theorem". It follows that $Z \cap \varphi^{-1}(U)$ dominates $W \cap U$ if the former set is nonempty. Then one proceeds as before.

Corollary 20.6: *Let $\varphi\colon X \to Y$ be a dominating morphism of varieties and let $r = \dim(X) - \dim(Y)$. Then there exists a nonempty open set $U \subseteq Y$ such that $\varphi^{-1}(\mathbf{y})$ is nonempty and each component of this set has dimension r.*

Theorems 20.2 and 20.4 give a good qualitative picture of the structure of a morphism. We can work this out a bit by some simple inductions.

Definition 20.7: *Constructible sets.* Let X be a variety. A subset A of X is **constructible** if it is a finite union of locally closed subsets of X:

$$A = \bigcup_{i=1}^{m} (U_i \cap C_i), \quad U_i \text{ open, } C_i \text{ closed.}$$

Use the distributive law to find U'_j open and C'_j closed such that

$$A = \bigcap_j (U'_i \cup C'_j).$$

Hence $X - A$ is constructible. Also, the union and intersection of finitely many constructible sets are constructible.

Thus the family of constructible subsets of X form a Boolean algebra. In fact, they are the smallest Boolean algebra containing all open sets.

Exercise 20.8: Prove that the constructible set $\big(\mathbb{A}^2 - V(Y)\big) \cup \{(0,0)\}$ is not locally closed.

Corollary 20.9 (Chevalley): *Let φ: $X \to Y$ be a morphism of varieties. Then the image of φ is a constructible set in Y. More generally, φ maps constructible sets in X to constructible sets in Y.*

Proof: The second statement follows immediately from the first. To prove the first, use induction on $\dim(Y)$. There are two cases.

If φ is not dominating, let $Z = \overline{\varphi(X)}$. Then $\varphi(X) \subseteq Z$ and $\dim(Z) < \dim(Y)$. By induction hypotheses, $\varphi(X)$ is constructible in Z and therefore also in Y.

If φ is dominating Y has a nonempty open subset U that contains $\varphi(X)$. Let $Y - U = W_1 \cup \cdots \cup W_m$ be the decomposition of $Y - U$ into irreducible components. For each i let $Z_{i1}, \ldots, Z_{i,k_i}$ be the components of $\varphi^{-1}(Z_i)$. Then $\varphi(X) = U \cup (\varphi(X) - U) = U \cup (\varphi(\varphi^{-1}(Y - U))) = U \cup (\varphi(\varphi^{-1}(Z_1 \cup \cdots \cup Z_k))) = U \cup (\varphi(\bigcup_{ij} Z_{ij})) = U \cup \bigcup_{ij} \varphi(Z_{ij})$. For each i and j let φ_{ij}: $Z_{ij} \to W_i$ be the restriction of φ. Since $\dim(W_i) < \dim(Y)$ induction shows that $\varphi(Z_{ij})$ is constructible in Y. Hence, so is $\varphi(X)$. ∎

Corollary 20.10 (Upper simicontinuity of dimension): *Let φ: $X \to Y$ be a morphism of varieties. For each $x \in X$ let*

$$e(x) = \{\max(\dim(Z)) \mid Z \textit{ is a component of } \varphi^{-1}(\varphi(x)) \textit{ containing } \mathbf{x}\}.$$

Then e is upper semicontinuous, i.e., for all integers

$$\begin{aligned} S_n(\varphi) &= \{\mathbf{x} \in X \mid e(\mathbf{x}) \geq n\} \\ &= \{\mathbf{x} \in X \mid \varphi^{-1}(\varphi(\mathbf{x})) \textit{ has a component } Z \textit{ containing } \mathbf{x} \textit{ of dimension} \geq n\} \end{aligned}$$

is closed

Proof: Again, make an induction on $\dim(Y)$ and so assume that φ is dominating. Let $r = \dim(X) - \dim(Y)$. Choose, by Corollary 20.6, a nonempty open subset U of Y which is contained in $\varphi(X)$ such that if $\mathbf{x} \in X$ and $\varphi(x) \in U$, then the dimension of each component of $\varphi^{-1}(\varphi(x))$ is r.

First of all, if $n \leq r$, then, by Corollary 20.3, for each $\mathbf{x} \in X$ the dimension of each component of $\varphi^{-1}(\varphi(\mathbf{x}))$ is at least n. So, $S_n(\varphi) = X$ is closed.

Secondly, if $n > r$, then, by the choice of U, $S_n(\varphi) \subseteq X - \varphi^{-1}(U)$. Let $W_1, \ldots, W_m$ be the components of $Y - U$. For each i let $Z_{i1}, \ldots, Z_{i,k_i}$ be the components of $\varphi^{-1}(W_i)$, and let $\varphi_{ij}\colon Z_{ij} \to W_i$ be the restriction of φ to Z_{ij}. Since $\dim(W_i) < \dim(Y)$ the set

$$S_n(\varphi_{ij}) = \{\mathbf{x} \in Z_{ij} \mid \varphi^{-1}(\varphi(x)) \cap Z_{ij} \text{ has a component of dimension} \geq n \text{ that contains } x\}$$

is closed. To conclude the proof it suffice therefore to prove that

$$S_n(\varphi) = \bigcup_{ij} S_n(\varphi_{ij}).$$

Obviously, the right hand side is contained in the left hand side. So let $\mathbf{x} \in S_n(\varphi)$. Then $\varphi^{-1}(\varphi(\mathbf{x}))$ has a component Z that contains x of dimension at least n. We claim that $\varphi(Z) \subseteq Y - U$. Indeed, if $\mathbf{z} \in Z$ and $\varphi(\mathbf{z}) \in U$, then $\varphi(z) = \varphi(x)$ and Z is a component of $\varphi^{-1}(\varphi(z))$, and therefore $r < n \leq \dim(Z) = r$, a contradiction. Hence,

$$Z \subseteq \varphi^{-1}(\varphi(Z)) \subseteq \varphi^{-1}(Y - U) = \varphi^{-1}(W_1) \cup \cdots \cup \varphi^{-1}(W_m) = \bigcup Z_{ij}.$$

As Z is irreducible there are i, j such that $Z \subseteq Z_{ij}$. Let Z'' be a component of $\varphi^{-1}(\varphi(\mathbf{x})) \cap Z_{ij}$ that contains Z. This component contains x and of dimension at least n. Conclude that $\mathbf{x} \in S_n(\varphi_{ij})$. ∎

Definition 20.11: *Birational equivalence.* A morphism $\varphi\colon X \to Y$ of varieties is **birational** if it is dominating and the induced map

$$\varphi^*\colon K(Y) \to K(X)$$

is an isomorphism.

Theorem 20.12: *Let $\varphi\colon X \to Y$ be a birational morphism of varieties. Then Y has a nonempty open subset U such that φ restricts to an isomorphism from $\varphi^{-1}(U)$ to U.*

Proof: We first reduce the Theorem to the case where both X and Y are affine.

Choose by Theorem 20.4 an open subset V of Y which is contained in $\varphi(X)$. It is the union of affine open sets. So assume that V is affine. Now replace X by $\varphi^{-1}(V)$ and Y by V, if necessary, to assume that Y is affine with coordinate ring, say, S.

Next let $U \subseteq X$ be an open affine set with coordinate ring R. The dimension of $X - U$ is less than $\dim(X)$. Hence, the dimension of $W = \overline{\varphi(X - U)}$ is less than $\dim(Y)$ (which is equal to $\dim(X)$). In particular $W \subset Y$. Choose nonzero $g \in S$ that vanish on W. Then $\varphi^{-1}(Y_g) \subseteq U$. Indeed, if $\mathbf{x} \in \varphi^{-1}(Y_g)$, then $g(\varphi(\mathbf{x})) \neq 0$. Hence $\varphi(x) \notin W$, therefore $\varphi(\mathbf{x}) \notin \varphi(X - U)$. Conclude that $\mathbf{x} \in U$. In fact, for $f = g \circ \varphi \in R$ we have $\varphi^{-1}(Y_g) = U_f$. So, replace X by U_f and Y by Y_g to assume that both X and Y are affine.

In this case $\varphi^*\colon S \to R$ is an embedding (since $\varphi^*\colon K(Y) \to K(X)$ is) which we identify with inclusion. Let $x_1, \ldots, x_n$ be a set of generators of R over K. Write $x_i = y_i/h$ with $y_1, \ldots, y_n, h \in S$. Then $S[h^{-1}] = R[h^{-1}]$. Therefore $\varphi\colon X_h \to Y_h$ is an isomorphism. ∎

Exercise 20.13: Two varieties X and Y are said to be **birationally equivalent** if $K(X) = K(Y)$. Use the primitive element theorem to prove that every variety is birationally equivalent to a hypersurface in some affine space $\mathbb{A}^r$.

The theory developed in this section cries out for examples. Theorem 20.4 and its corollaries are illustrated in the following:

Example 20.14: Consider the morphism $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2$ defined by $\varphi(x, y) = (xy, y)$.

(a) The image of φ is $U \cap \{(0,0)\}$ where $U = (\mathbb{A}^2)_y = \mathbb{A}^2 - \{(x,0)|\ x \in \mathbb{A}^1\}$. This set is not locally closed (Exercise 20.8). It is an affine variety with coordinate ring $K[X, Y, Y^{-1}]$.

(b) The map φ is birational. Its restriction to U is an isomorphism onto U whose inverse $\varphi'\colon U \to U$ is defined by $\varphi'(u, v) = (uv^{-1}, v)$. In particular for each point $(x, y) \in U$, the fibre $\varphi^{-1}(\varphi(x, y))$ consists of one point.

(c) On the other hand for each $x \in \mathbb{A}^1$ the fibre $\varphi^{-1}(\varphi(x, 0)) = \varphi^{-1}(0, 0)$ is the whole line of points $(x', 0)$.

(d) Thus, in the notation of 20.10, $S_0(\varphi) = \mathbb{A}^2$, $S_1(\varphi) = \{(x,0)|\ x \in \mathbb{A}^1\}$, and $S_2(\varphi) = \emptyset$.

EXAMPLE 20.15: To illustrate Theorem 20.12, reconsider the finite birational morphism

$$\varphi\colon \mathbb{A}^1 \to C$$

defined in Example 8.7 by $\varphi(t) = (t^2, t^3)$ where C is the plane $X^3 = Y^2$. The open subset $U = C - \{(0,0)\} = C_{T^2}$ of C is the an affine variety with coordinate ring $K[T^2, T^3, T^{-2}]$. Also, $\varphi^{-1}(U) = \mathbb{A}^1 - \{0\}$ is an affine variety with coordinate ring $K[T, T^{-1}]$. As the two ring coincide the restriction of φ to $\varphi^{-1}(U)$ is an isomorphism onto U whose inverse is given by $\varphi(x, y) = y/x$. The restriction of φ to this open subset is an isomorphism onto U

Next reconsider the finite birational morphism

$$\varphi\colon \mathbb{A}^1 \to D$$

defined in Example 8.8 by $\varphi(t) = (t^2 - 1, t(t^2 - 1))$. Here D is the elliptic curve $Y^2 = X^2(X + 1)$ with coordinate ring $K[x, y] = K[X, Y]/\langle Y^2 - X^2(X + 1)\rangle$. Consider the open subset $D_x = D - \{(0,0)\}$ of D. We have $\varphi^{-1}(D_x) = \mathbb{A}^1 - \{1, -1\}$ and φ maps $\mathbb{A}^1 - \{1, -1\}$ bijectively onto D_x. To prove that the restriction of φ to $\mathbb{A}^{-1} - \{1, -1\}$ is an isomorphism define a morphism ψ from D_x to $\mathbb{A}^1 - \{1, -1\}$ by $\psi(x, y) = yx^{-1}$. It is the inverse morphism to the restriction of φ to $\mathbb{A}^1 - \{1, -1\}$.

21. Complete varieties.

An affine variety can be embedded as an open dense set in a projective variety by a birational morphism. Can a projective variety be embedded birationally in anything even bigger? The answer is no; there is a type of variety, called "complete", which in our algebraic theory plays the same role as compact Hausdorff spaces do in the theory of topological spaces. Namely, if a subset X of a Hausdorff topological space Y is a compact, then X is closed in Y and therefore if it X is dense in Y, then $X = Y$. Thus complete varieties are "maximal" and projective varieties turn out to be complete.

DEFINITION 21.1: A variety X is **Complete** if for each variety Y, the projection morphism

$$\mathrm{pr}_Y \colon X \times Y \to Y$$

is a closed map.

The analogous property in the category of separable topological spaces characterizes compact spaces X.

Indeed, suppose that X is a compact, Y is arbitrary, and Z is a closed subset of $X \times Y$. Let $\pi = \mathrm{pr}_Y$. Suppose that $y \in \overline{\pi(Z)}$. Then y is the limit of a sequence $\{y_i\}$ of elements of $\pi(Z)$. For each i there exist $x_i \in X$ such that $(x_i, y_i) \in Z$. Replace $\{x_i\}$ by a subsequence, if necessary, to assume that it converges to an element x of X. Then $\lim_{i\to\infty} (x_i, y_i) = (x, y)$. Since Z is closed $(x, y) \in Z$ and therefore $y \in \pi(Z)$. Conclude that $\pi(Z)$ is closed.

Conversely, suppose that a separable topological space X satisfies the above condition. To prove that X is compact we have to show that every sequence $\{x_n\}$ of elements of X contains a converging subsequence. To this end, let Y be the topological space $\{1, 2, 3, \ldots, \infty\}$, whose open sets are either finite subsets of $\mathbb{N}$ or cofinite sets that contain ∞. Denote the closure of $\{(x_n, n) |\ n \in \mathbb{N}\}$ by Z. By assumption $\pi(Z)$ is closed in Y. Since $\pi(Z)$ contains $\mathbb{N}$ it contains ∞. Thus, there exists $x \in X$ such that $(x, \infty) \in Z$. It follows that $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ such that $\{(x_{n_i}, n_i)\}$ converges to (x, ∞). Conclude that $\{x_{n_i}\}$ converges to x.

The definition of complete variety is very nice from a category-theoretic point of view. It gives the elementary properties of completeness very easily:

PROPOSITION 21.2: (a) *Let* $\varphi\colon X \to Y$ *be a morphism of varieties with* X *complete. Then* $\varphi(X)$ *is closed in* Y *and is complete.*

(b) *If* X *and* Y *are complete, then* $X \times Y$ *is complete.*

(c) *If* X *is complete and* $Y \subseteq X$ *is a closed subvariety, then* Y *is complete.*

(d) *An affine variety* X *is complete only if* $\dim(X) = 0$, *i.e.,* X *consists of a a single point.*

Proof of (a): As Y is a variety, the **graph** of φ is a closed subset of $X \times Y$ (Remark 15.3(d)). Since $\varphi(X)$ is the projection of the graph on Y it is closed.

Let now Z be a variety and W a closed subset of $\varphi(X) \times Z$. To show that $\mathrm{pr}_Z(W)$ is closed decompose $\mathrm{pr}_Z\colon X \times Z \to Z$ as $\mathrm{pr}_Z \circ \psi$, where $\psi = (\varphi, \mathrm{id}_Z)\colon X \times Z \to Y \times Z$. Then $V = \psi^{-1}(W)$ is a closed subset of $X \times Z$ and $\mathrm{pr}_Z(W) = \mathrm{pr}_Z(V)$ is therefore closed in Z.

Proof of (b): Let Z be an arbitrary variety and let W be closed subset of $X \times Y \times Z$. Then the projection of W onto Z is the projection onto Z of the projection onto $Y \times Z$ of W. Conclude that all these projections are closed sets.

Proof of (c): Clear.

Proof of (d): Suppose that $X = V(f_1, \ldots, f_m) \subseteq \mathbb{A}^n$ is an affine variety of positive dimension defined by $f_i \in K[X_1, \ldots, X_n]$. For each i let $g_i(X_0, \ldots, X_n) = X_0^{\deg(f_i)} f_i(X_1/X_0, \ldots, X_n/X_0)$. Then $\overline{X} = V(g_1, \ldots, g_m) \subseteq \mathbb{P}^n$. We claim that $X_0 \notin \langle g_1, \ldots, g_m \rangle$. Otherwise there exist polynomials h_i such that $X_0 = h_1 g_1 + \ldots + h_m g_m$. Substitute $X_0 = 1$ to obtain $1 = h_1' f_1 + \ldots + h_m' f_m$ for some polynomials h_i' in $K[X_1, \ldots, X_m]$. Hence X is empty, a contradiction.

It follows that $\overline{X} \not\subseteq V(X_0)$. As $\dim(\overline{X}) > 0$, the dimension theorem for projective varieties implies that $V(X_0) \cap \overline{X}$ is nonempty. This means that the image of X in $\mathbb{P}^n$ under the embedding $\mathbb{A}^n \to \mathbb{P}^n$ is not closed. Conclude from (a) that X is not complete. ∎

LEMMA 21.3 (Nakayama's Lemma): *Let R be a ring and M a finitely generated module. Suppose that $\mathfrak{a}$ is an ideal of R such that $\mathfrak{a}M = M$. Then there is $f \in 1 + \mathfrak{a}$ which annihilates M.*

Proof: Let $m_1, \ldots, m_n$ be generators of M over R. By assumption

$$m_i = \sum_{j=1}^{n} a_{ij} m_j$$

for suitable $a_{ij} \in \mathfrak{a}$. Denote the $n \times n$ unit matrix by I and let $A = (a_{ij})$. Then $(I - A)\mathbf{m} = 0$. Multiply this equation from the left by the adjoint matrix of $(I - A)$ to find that $\det(I - A)m_j = 0$ for each j. Thus $f = \det(I - A)$ annihilates M and $f - 1 \in \mathfrak{a}$. ∎

EXERCISE 21.4: Let R be a local ring with a maximal ideal $\mathfrak{m}$. Suppose that M is a finitely generated R-modul. Prove that if $\mathfrak{m}M = M$, then $M = 0$. Prove that if M_0 is a submodule of M such that $M = M_0 + \mathfrak{m}M$, then $M = M_0$.

THEOREM 21.5: *$\mathbb{P}^n$ is complete.*

Proof (Grothendieck): We have to prove that for each variety the projection $\mathrm{pr}_Y\colon \mathbb{P}^n \times Y \to Y$ is a closed map. Let $\{Y_i\}$ be an affine open covering of Y. If we prove that the restriction of pr_Y to each $\mathbb{P}^n \times Y_i$ is closed, then pr_Y will be closed. So, we may assume that Y is affine with coordinate ring R. We break the rest of the proof into parts.

PART A: *A graded ring.* The variety $\mathbb{P}^n \times Y$ is covered by the open affine sets $U_i = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{P}^n \times Y \mid x_i \neq 0\}$ whose coordinate rings are $R[X_0/X_i, \ldots, X_n/X_i]$.

Consider the graded ring $S = R[X_0, \ldots, X_n]$. For each k let S_k be the set of all homogeneous polynomials in S of degree k. Let

$$A_k = \{f \in S_k \mid f(\mathbf{X}/X_i) \in I(Z \cap U_i),\ i = 0, \ldots, n\}.$$

As S is Noetherian, $A_k \subseteq S_k$ are finitely generated R-module. Moreover

$$A_k S_l \subseteq A_{k+l} \text{ and } S_k S_l \subseteq S_{k+l}.$$

Claim 1: For each i and each $g \in I(Z \cap U_i)$ there is r_0 such that for all $r \geq r_0$ we have $X_i^r g \in A_r$.

Indeed, if r is large enough, $f = X_i^r g$ is a homogeneous polynomial in R of degree r. To check that $f \in A_r$ let j be an integer between 1 and n. Then $f(\mathbf{X}/X_j)$ vanishes on $Z \cap U_i \cap U_j$. and therefore on $Z \cap U_j$ (since the former set is dense in the latter). Thus $f \in A_r$.

Part B: $Y - \mathrm{pr}_Y(Z)$ *is an open set.* Let $\mathbf{y} \in Y - \mathrm{pr}_Y(Z)$. We have to show that $\mathbf{y}$ has an open neighborhood which is contained in $Y - \mathrm{pr}_Y(Z)$.

Indeed, let $\mathfrak{m} = \{g \in R |\ g(\mathbf{y}) = 0\}$ be the maximal ideal of R corresponding to $\mathbf{y}$. Then, for each i, $Z \cap U_i$ and $(\mathbb{P}^n)_{X_i} \times \{\mathbf{y}\}$ are closed and disjoint subsets of U_i. By Hilbert's Nullstellensatz

$$I(Z \cap U_i) + \mathfrak{m}R[\mathbf{X}/X_i] = R[\mathbf{X}/X_i].$$

In particular there exist $a_i \in I(Z \cap U_i)$, $m_{ij} \in \mathfrak{m}$, and $g_{ij} \in R[\mathbf{X}/X_i]$ such that

$$1 = a_i + \sum_j m_{ij} g_{ij}.$$

If we multiply this equation through by X_i^r for big r and use the Claim we get

$$X_i^r = a_i' + \sum_j m_{ij} g_{ij}'$$

where $a_i' \in A_r$ and $g_{ij} \in S_r$. In fact choose r large enough such that this works for all i simultanouesly. In other words $X_i^r \in A_r + \mathfrak{m}S_r$ for $i = 0, \ldots, n$.

Claim 2: For $q = r(n+1)$ we have $S_q = A_q + \mathfrak{m}S_q$.

Indeed, if $\mu = X_0^{r_0} X_1^{r_1} \cdots X_n^{r_i}$ is a monomial of degree q, then there exist i such that $r_i \geq r$. Hence $X_i^{r_i} \in A_{r_i}$ and therefore $\mu \in A_q + \mathfrak{m}S_q$, by (1).

By Nakayama's Lemma, applied to the R-module S_q/R_q there exists $h \in 1+\mathfrak{m}$ (in particular $h(\mathbf{y}) \neq 0$) such that $hS_q \subseteq A_q$. Hence, for each i, $h \cdot X_i^1 \in A_q$, which implies $h \in I(Z \cap U_i)$ (recall that $h \in R$). If $\mathbf{y}' \in \mathrm{pr}_Y(Z)$ there exists $(\mathbf{x}', \mathbf{y}') \in Z$. Choose i such that $(\mathbf{x}', \mathbf{y}') \in Z \cap U_i$. Then h vanish at this point, i.e., $h(\mathbf{y}') = 0$. So, Y_h is an open neighborhood of $\mathbf{y}$ which is contained in $Y - \mathrm{pr}_Y(Z)$.

Conclude that $\mathrm{pr}_Y(Z)$ is closed and therefore $\mathbb{P}^n$ is complete. ∎

COROLLARY 21.6: *Every projective variety is complete.*

Here is another Corollary of the completeness of $\mathbb{P}^n$.

THEOREM 21.7 (The main theorem of elimination theory): *Given r polynomials with coefficients in K:*

$$f_1(X_0, \cdots, X_n; Y_1, \ldots, Y_m)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$f_r(X_1, \ldots, x_n; Y_1, \ldots, y_m),$$

all of which are homogeneous in the variables $X_0, \ldots, X_n$ there is a second set of polynomials with coefficients in K:

$$g_1(Y_1, \ldots, Y_m)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$g_s(Y_1, \ldots, Y_m)$$

such that for all $\mathbf{a} \in \mathbb{A}^m$, $g_j(\mathbf{a}) = 0$ for $j = 1, \ldots, s$ if and only if there is a nonzero $\mathbf{b} \in \mathbb{A}^{n+1}$ such that $f_i(\mathbf{b}; \mathbf{a}) = 0$ for $i = 1, \ldots, r$.

Proof: The equations $f_1 = \cdots = f_r = 0$ define a closed subset Z of $\mathbb{P}^n \times \mathbb{A}^m$. By Theorem 21.5, the projection of Z on $\mathbb{A}^m$ is a closed set which may be defined by equations $g_1 = \cdots = g_s = 0$. ∎

EXERCISE 21.8: *If Z is a closed subset of $\mathbb{P}^n \times \mathbb{A}^m$, then Z is the zero set of polynomials $f(\mathbf{X}, \mathbf{Y})$ which are homogeneous in $\mathbf{X} = (X_0, \ldots, X_n)$ and where $\mathbf{Y} = (Y_1, \ldots, Y_m)$.*

22. Chow's Lemma.

For some years people were not sure whether or not all complete varieties might not actually be projective varieties. In this section we will see that even if a complete variety is not projective, it can still be dominated by a projective variety with the same function field. Thus the problem is a "birational" one, i.e., concerned with the comparison of the collection of all varieties with a common function field.

LEMMA 22.1: *Let S and T be varieties, with isomorphic open subsets $V_S \subseteq S$, $V_T \subseteq T$. For simplicity identify $V = V_S$ with V_T and look at the diagonal morphism $\Delta\colon V \to V \times V$. If $\overline{V}$ is the closure of $\Delta(V)$ in $S \times T$, then*

$$\overline{V} \cap (S \times V) = \Delta(V) = \overline{V} \cap (V \times T).$$

Proof: Note that $\overline{V} \cap (V \times T)$ is the closure of $\Delta(V)$ in $V \times T$. On the other hand, $\Delta(V)$ is the graph of the inclusion morphism $V \to T$, which is closed by Remark 15.3(d). Conclude that $\Delta(V)$ coincides with $\overline{V} \cap (V \times T)$. By symmetry, $\Delta(V) = \overline{V} \cap (S \times V)$. ∎

THEOREM 22.2 (Chow's Lemma): *Let X be a complete variety. Then there is a projective variety Y of $\mathbb{P}^n$ and a birational epimorphism $\pi\colon Y \to X$.*

Proof: Break the proof into several parts.

PART A: *Construction of Y.* Cover X by open affine subsets U_i with coordinate rings R_i, $i = 1, \ldots, m$, Embed all the U_i's as closed subvarieties of $\mathbb{A}^n$ for some n and let $V = U_1 \cap \cdots \cap U_m$. With respect to the composite inclusion

$$U_i \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$$

U_i is a locally closed subvariety of $\mathbb{P}^n$. Let $\overline{U}_i$ be its closure in $\mathbb{P}^n$. Then $\overline{U}_1 \times \cdots \times \overline{U}_m$ is isomorphic to a closed subvariety of $\mathbb{P}^q$ for some q (Theorem 14.5).

Consider the map

$$\alpha\colon V \to \overline{U}_1 \times \cdots \times \overline{U}_n$$

defined by $\alpha(\mathbf{v}) = (\mathbf{v}, \ldots, \mathbf{v})$. It is an isomorphism of V onto a locally closed subset $V_0 = \{(\mathbf{v}, \ldots, \mathbf{v}) |\ \mathbf{v} \in V\}$ (V_0 is a closed subset of the open subset $U_1 \times \cdots \times U_n$.) Denote the closure of V_0 in $\overline{U}_1 \times \cdots \times \overline{U}_n$ by Y. Then Y is a closed subvariety of $\mathbb{P}^q$ and so it is a projective variety. In particular Y is complete. Also $K(X) \cong_K K(V) \cong_K K(V_0) \cong_K K(Y)$. We will construct an epimorphism $\pi\colon Y \to X$.

Part B: *Construction of* π. Let $\delta\colon V \to X \times Y$ be the morphism defined by $\delta(\mathbf{v}) = (\mathbf{v}, \alpha(\mathbf{v}))$. Then δ maps V isomorphically onto its image $V' = \{(\mathbf{v}, \mathbf{v}, \ldots, \mathbf{v}) |\ \mathbf{v} \in V\}$. Denote the closure of V' in $X \times Y$ by $\widetilde{Y}$.

Let β be the restriction of pr_X to $\widetilde{Y}$. It maps V' isomorphically onto V. As both X and Y are complete, so is $X \times Y$. Hence $\beta(\widetilde{Y})$ is closed in X and contains the dense subset V. Conclude that $\beta(\widetilde{Y}) = X$.

Similarly let γ be the restriction of pr_Y to $\widetilde{Y}$. Again, it maps V' isomorphically onto V_0 and therefore $\widetilde{Y}$ onto Y.

$$\begin{array}{ccccc}
X & \xleftarrow{\mathrm{pr}_X} & X \times Y & \xrightarrow{\mathrm{pr}_Y} & Y \\
\| & & | & & \| \\
X & \xleftarrow{\beta} & \widetilde{Y} & \xrightarrow{\gamma} & Y \\
| & & | & & | \\
V & \longleftarrow & V' & \longrightarrow & V_0
\end{array}$$

If we prove that γ is an isomorphism, then $\pi = \beta \circ \gamma^{-1}\colon Y \to X$ will be the desired birational epimorphism.

Part C: γ *is an isomorphism.* Let i be an integer between 1 and m. Denote the projection of $X \times \overline{U}_1 \times \cdots \times \overline{U}_m$ onto $X \times \overline{U}_i$ by ρ_i. It maps V' isomorphically onto $V_i = \{(\mathbf{v}, \mathbf{v}) |\ \mathbf{v} \in V\}$. Since $\widetilde{Y}$ is the closure of V' and $\rho_i(\widetilde{Y})$ is closed, $\rho_i(\widetilde{Y})$ is the closure of V_i in $X \times \overline{U}_i$. Also, the diagonal map, $\overline{U}_i \to X \times \overline{U}_i$ maps U_i isomorphically onto $U_i' = \{(\mathbf{u}, \mathbf{u}) |\ \mathbf{u} \in U_i\}$ and V onto V_i. In particular $V_i \subseteq U_i' \subseteq \rho_i(\widetilde{Y})$ and so $\rho_i(\widetilde{Y})$

is the closure of U_i' in $X \times \overline{U}_i$.

$$
\begin{array}{ccc}
X \times \overline{U}_1 \times \cdots \times \overline{U}_m & \xrightarrow{\rho_i} & X \times \overline{U}_i \\
| & & | \\
X \times Y & & | \\
| & & | \\
\widetilde{Y} & \xrightarrow{\rho_i} & \rho_i(\widetilde{Y}) \\
| & & | \\
| & & U_i' \\
| & & | \\
V' & \longrightarrow & V_i
\end{array}
$$

Apply Lemma 22.1 on X, $\overline{U}_i$ and U_i instead of S, T, and V, respectively:

$$\rho_i(\widetilde{Y}) \cap (X \times U_i) = U_i' = \rho_i(\widetilde{Y}) \cap (U_i \times \overline{U}_i).$$

Hence

$$
\begin{aligned}
\widetilde{Y}_i &= \widetilde{Y} \cap (X \times \overline{U}_1 \times \cdots \times U_i \times \cdots \times \overline{U}_m) \\
&= \{(\mathbf{u}_i, \mathbf{u}_1, \ldots, \mathbf{u}_m) \in \widetilde{Y} \mid \mathbf{u}_i \in U_i \text{ and } \mathbf{u}_j \in \overline{U}_j \text{ for } j \neq i\} \\
&= \widetilde{Y} \cap (U_i \times \overline{U}_1 \times \cdots \times \overline{U}_m).
\end{aligned}
$$

Let

$$Y_i = Y \cap (\overline{U}_1 \times \cdots \times U_i \times \cdots \times \overline{U}_m).$$

From the third form of $\widetilde{Y}_i$ it follows that $\{\widetilde{Y}_i\}$ is an open covering of $\widetilde{Y}$. From the first form it follows that $\widetilde{Y}_i = \gamma^{-1}(Y_i)$. Hence $\{Y_i\}$ is an open covering of Y. So define a morphism $\sigma_i\colon Y_i \to \widetilde{Y}_i$ by

$$\sigma_i(u_1, \ldots, u_m) = (u_i, u_1, \ldots, u_m).$$

Then σ_i is the inverse of the restriction of γ to $\widetilde{Y}_i$:

$$
\begin{array}{c}
\gamma(\sigma_i(u_1, \ldots, u_m)) = \gamma(u_i, u_1, \ldots, u_m) = (u_1, \ldots, u_m) \\
\sigma_i(\gamma(u_i, u_1, \ldots, u_m)) = \sigma_i(u_1, \ldots, u_m) = \sigma_i(u_i, u_1, \ldots, u_m)
\end{array}.
$$

Note that for all i, j the restrictions of σ_i and σ_j to $Y_i \cap Y_j$ coincide. Conclude that γ is an isomorphism (Problem 11.3). ∎

Corollary 22.3: *Let X be a complete variety. Then for every valuation ring R of $K(X)$ whose quotient field is $K(X)$ there exists a point $\mathbf{x} \in X$ such that $O_{\mathbf{x}} \subseteq R$.*

Proof: Suppose first that X is projective. Then X is the zero set in $\mathbb{P}^n$ of some homogeneous ideal I of $K[X_0, \ldots, X_n]$. Let $x_i = X_i + I$, $i = 0, \ldots, n$ be the "homogeneous coordinates" of X. Then $K(X) = K(\mathbf{x}/x_i)$ for each i such that $x_i \neq 0$. Denote the place that corresponds to R by φ. By Exercise 1.2 there is i such that $b_j = \varphi(x_j/x_i)$ is finite for $j = 0, \ldots, n$. By Hilbert's Nullstellensatz there exists a K-homomorphism $\psi\colon K[\mathbf{b}] \to K$. Then $\mathbf{a} = \psi(\mathbf{b})$ is a point of X. If $g \in K[X_0, \ldots, X_n]$ and $g(\mathbf{a}) \neq 0$, then $\varphi(g(\mathbf{x}/x_i) = g(\mathbf{b}) \neq 0$. Hence $g(\mathbf{x}/x_i)^{-1} \in R$. Conclude that $O_{\mathbf{a}} \subseteq R$.

When X is an arbitrary complete variety there is a birational epimorphism $\pi\colon Y \to X$ from a projective variety. In particular π defines an K-isomorphism of $K(X)$ and $K(Y)$. Identifying the two fields we find that R is a valuation ring of $K(Y)$ whose quotient ring is $K(Y)$. By the first paragraph, Y has a point $\mathbf{y}$ such that $\mathcal{O}_{y,Y} \subseteq R$. Let $\mathbf{x} = \pi(\mathbf{y})$. Since π is surjective π induces an embedding of the local ring $\mathcal{O}_{\mathbf{x},X}$ into $\mathcal{O}_{\mathbf{y},Y}$. ∎

23. Complex varieties.

Suppose that our algebraically closed ground field K is given a topology making it into a topological field. The most interesting case of this is when $K = \mathbb{C}$, the complex numbers. Another case is when K is the algebraic closure of $\mathbb{Q}_p$, the field of p-adic numbers. However, we can make at least the first definition in complete generality. Namely, we show that when K is a topological field, there is a unique way to endow all varieties X over K with a new topology, which we will call the **strong topology**, such that the following properties hold:

(1a) The strong topology is stronger than the Zariski-topology, i.e., a closed (resp., open) subset $X \subseteq X$ is strongly closed (resp., strongly open).

(1b) All morphisms are strongly continuous.

(1c) If $Z \subseteq X$ is a locally closed subvariety, then the strong topology on Z is the one induced by the strong topology on X.

(1d) The strong topology on $X \times Y$ is the product of the strong topologies on X and on Y.

(1e) The strong topology on $\mathbb{A}^1$ is exactly the given topology on K.

The uniqueness of the strong topology is obvious. To prove the existence of the strong topology, equip first $\mathbb{A}^n$ with the product topology of K^n and induce this topology on each locally closed subset of $\mathbb{A}^n$. If φ: $(V, \mathcal{O}_V) \to (V', \mathcal{O}'_V)$ is a morphism of the structure sheaves of irreducible algebraic sets $V \subseteq \mathbb{A}^n$ and $V' \subseteq \mathbb{A}^{n'}$, then φ is defined by polynomials and therefore induces a strongly continuous map from V to V'. In particular, if $(X, \mathcal{O}_X)$ is an affine variety, then the strong topology induced on it by an embedding into an affine space is independent of that embedding. Also, since Zariski-open subsets of $\mathbb{A}^n$ are defined by polynomials inequalities each of them is strongly open. If $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ are closed irreducible sets, then the topology induced on $X \times Y$ by that of $\mathbb{A}^{m+n}$ is the product topology.

For an arbitrary variety X choose an open finite affine covering $\{X_i\}$. Then define a subset $U \subseteq X$ to be strongly open if $U \cap X_i$ is open in X_i for each i. This definition is independent of the covering. The union of arbitrary strongly open sets and the intersection of finitely strongly open sets is again a strongly open set. So are X and the empty set. Conclude that the collection of strongly open sets indeed gives a topology on X.

If $U \subseteq X$ is Zariski-open, then $U \cap X_i$ is Zariski-open and therefore also strongly open in X_i. Hence U is strongly open. In particular, this implies that the strong topology on X does not depend on the open affine covering $\{X_i\}$. As morphisms between varieties are locally defined on affine pieces, they are strongly continuous.

Finally, if X and Y are varieties with finite open affine covers $\{X_i\}$ and $\{Y_j\}$, then $\{X_i \times Y_j\}$ is a finite open affine covering of $X \times Y$. The topology of each $X_i \times Y_j$ is the product topology. Hence, so is the topology of $X \times Y$.

Note that every variety X is a Hausdorff space in its strong topology. In fact, if $\Delta\colon X \to X \times X$ is the diagonal map, then $\Delta(X)$ is Zariski-closed and therefore strongly closed in $X \times X$. Since $X \times X$ has the product strong topology by (1d), this means exactly that X is a Hausdorff space.

From now on suppose that $K = \mathbb{C}$ with its usual topology. The first nontrivial comparison theorem relating the two topologies states that the strong topology is not "too strong".

THEOREM 23.1: *Let X be a variety, and U a nonempty open subvariety. Then U is strongly dense in X.*

Proof (Stolzenberg): Let $\{X_i\}$ be an affine cover of X. Then the strong closure of U is the union of the strong cover of $U \cap X_i$. So, we may assume that X is affine. Break the rest of the proof into three parts.

PART A: *Comparison with $\mathbb{A}^r$.* Let $r = \dim(X)$. By the geometric form of Noether's normalization theorem (Proposition 16.9), there exists a finite epimorphism

$$\pi\colon X \to \mathbb{A}^r.$$

Let $Z = X - U$. Then $\pi(Z)$ is a Zariski closed subset of $\mathbb{A}^r$ (Proposition 16.12). Since all components of Z have dimension $< r$, so do all components of $\pi(Z)$ (Proposition 20.1). Hence $\pi(Z)$ is even a proper closed subset of $\mathbb{A}^r$. In particular, there is a nonzero polynomial $g \in K[Y_1, \ldots, Y_r]$ such that $\pi(Z) \subseteq V(g)$.

Now choose a point $\mathbf{x} \in Z$. Our goal is to prove that $\mathbf{x}$ is a limit of a sequence of points of U. Let us first represent $\mathbf{y} = \pi(\mathbf{x})$ as a limit of points $\mathbf{y} \in \mathbb{A}^r - V(g)$.

To do this, choose any point $\mathbf{y}_1 \in \mathbb{A}^r$ such that $g(\mathbf{y}_1) \neq 0$, and let (in vector notation)

$$h(T) = g((1-T)\mathbf{y} + T\mathbf{y}_1).$$

Then $h \neq 0$, since $h(1) \neq 0$. Hence, the polynomial h has only finitely many zeros, and we can choose a sequence of numbers $t_i \in \mathbb{C}$ such that $t_i \to 0$ strongly, and $h(t_i) \neq 0$. Then let $\mathbf{y}_i = (1-t_i)\mathbf{y} + t_i\mathbf{y}_1$, $i = 2, 3, \ldots$. Obviously $\mathbf{y}_i \to \mathbf{y}$ strongly, and $g(\mathbf{y}_i) = h(t_i) \neq 0$.

The problem now is to lift each $\mathbf{y}_i$ to a point $\mathbf{x}_i \in X$ such that $\mathbf{x}_i \to \mathbf{x}$ strongly. Since $\mathbf{y}_i \notin \pi(Z)$, all the points $\mathbf{x}_i$ must be in U. Hence, it will follow that x is in the strong closure of U.

PART B: *A hypersurface in* $\mathbb{A}^{r+1}$. As π is a finite morphism $\pi^{-1}(\mathbf{y}) = \{\mathbf{x}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(m)}\}$ is a finite set. Choose $q \in K[X]$ such that $q(\mathbf{x}) = 0$ but $q(\mathbf{x}^{(j)}) \neq 0$, $j = 2, \ldots, m$. Abbreviate $(Y_1, \ldots, Y_r)$ by $\mathbf{Y}$. The element q is integral over $K[\mathbf{Y}]$. Let

$$f(\mathbf{Y}, Z) = Z^d + a_1(\mathbf{Y})Z^{d-1} + \ldots + a_d(\mathbf{Y}), \qquad a_1, \ldots, a_d \in K[\mathbf{Y}]$$

be the irreducible polynomial of q over $K(\mathbf{Y})$. Thus $f(\mathbf{Y}, q) = 0$. Hence $a_d(\mathbf{y}) = f(\mathbf{y}, q(\mathbf{x})) = 0$. It follows that $a_d(\mathbf{y}_i) \to 0$ strongly as $i \to \infty$. On the other hand $a_d(\mathbf{y}_i)$ is a the product of the roots of the equation $f(\mathbf{y}_i, Z) = 0$. Hence, we may choose for each i a root z_i of this equation such that $z_i \to 0$ strongly. Thus

(1) $$\lim_{i\to\infty} (\mathbf{y}_i, z_i) = (\mathbf{y}, 0), \text{ strongly, and } f(\mathbf{y}_i, z_i) = 0, \qquad i = 1, 2, 3, \ldots$$

PART C: *Lifting* $(\mathbf{y}_i, z_i)$ *to* $\mathbf{x}_i \in X$. Now view X as an irreducible algebraic subset of some $\mathbb{A}^n$ and let $h_1, \ldots, h_n$ be the coordinate functions on X. Thus $h_i(\mathbf{x}') = \mathbf{x}'_i$ is the ith coordinate of $\mathbf{x}' \in X$, $i = 1, \ldots, n$. Then $K[h_1, \ldots, h_n]$ is the coordinate ring of X and $K[\mathbf{Y}, q]$ is the coordinate ring of $V(f)$ (= the hypersurface in $\mathbb{A}^{r+1}$ defined by f). The inclusion of coordinate rings $K[\mathbf{Y}] \subseteq K[\mathbf{Y}, q] \subseteq K[h_1, \ldots, h_n]$ induces finite morphisms

$$X \xrightarrow{\pi_1} V(f) \xrightarrow{\pi_2} \mathbb{A}^r$$

such that $\pi_2 \circ \pi_1 = \pi$. Thus, $\pi_1(\mathbf{x}') = (\mathbf{y}', q(\mathbf{x}'))$ with $\mathbf{y}' = \pi(\mathbf{x}')$ for each $\mathbf{x}' \in X$.

In particular, as π_1 is surjective (Proposition 16.12(c)) and $(\mathbf{y}_j, z_j) \in V(f)$, there are $\mathbf{x}_j \in X$ such that $\pi_1(\mathbf{x}_j) = (\mathbf{y}_j, z_j)$ $z_j = q(\mathbf{x}_j)$, and therefore $\pi(\mathbf{x}_j) = \mathbf{y}_j$. So, all we have to do is to prove that $\mathbf{x}_j$ strongly converges to $\mathbf{x}$.

Indeed, each h_i satisfies a monic equation

$$h_i^k + b_{i1}h_i^{k-1} + \cdots + b_{ik} = 0 \tag{2}$$

with coefficients $b_{ij} \in K[\mathbf{Y}]$. Since the sequence $\{(\mathbf{y}_j, z_j)\}_{j=1}^{\infty}$ strongly converges it is bounded. If the sequence $\{\mathbf{x}_j\}$ were not bounded, then we may apply (2) on $\mathbf{x}_j$ and multiply it by x_{ji}^{-k}:

$$1 + b_{i1}(\mathbf{y}_j)x_{ji}^{-1} + \cdots + b_{ik}(\mathbf{y}_j)x_{ji}^{-k} = 0.$$

Then let j ranges on an appropriate sequence of positive integers we obtain a contradiction $1 = 0$. Conclude that $\{\mathbf{x}_j\}$ is a bounded sequence.

If $\mathbf{x}_j$ does not strongly converge to $\mathbf{x}$, then there exist $\varepsilon > 0$ such that $|\mathbf{x}_j - \mathbf{x}| > \varepsilon$ for infinitely many j. From the boundness it follows that $\{\mathbf{x}_j\}$ has a subsequence $\{\mathbf{x}_{j(l)}\}$ that converges to $\mathbf{x}' \in X$, $\mathbf{x}' \neq \mathbf{x}$. Apply π to find that $\mathbf{y}_{j(l)}$ converges to $\pi(\mathbf{x}')$. On the other hand $\lim \mathbf{y}_{j(l)} = \mathbf{y}$. Hence $\mathbf{x}' = \mathbf{x}^{(i)}$ for some i between 2 and m. It follows that

$$\lim_{l\to\infty} z_{j(l)} = \lim_{l\to\infty} q(x_{j(l)}) = q(x^{(i)}) \neq 0.$$

This contradiction to (1) proves that $\lim \mathbf{x}_j = \mathbf{x}$, and the proof is complete. ∎

COROLLARY 23.2: *If $Z \subseteq X$ is a constructible subset of a variety, then the Zariski closure and the strong closure of Z are the same. In particular, if Z is constructible and strongly closed, then Z is Zariski-closed.*

The main result of this section is:

THEOREM 23.3: *Let X be a variety over $\mathbb{C}$. Then X is complete if and only if X is compact in its strong topology.*

Proof: Suppose first that X is strongly compact. Let Y be another variety and let $Z \subseteq X \times Y$ be a closed subvariety. Since X is compact and Hausdorff, pr_Y is a closed map in its strong topology (Section 21). Hence $\mathrm{pr}_Y(Z)$ is strongly closed. Since it is also Zariski constructible (by Chevalley's theorem, Corollary 20.8), is is Zariski closed (Corollary 23.2).

Conversely, we must show that complete varieties are strongly compact. First of all, note that $\mathbb{P}^n(\mathbb{C})$ is a strongly continuous image of the n-dimensional sphere

$$S_n = \{(z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} | \sum_{i=0}^{n} |z_i|^2 = 1\}.$$

by the map $(z_0, z_1, \dots, z_n) \mapsto z_0{:}z_1{:}\cdots{:}z_n$. Indeed, if $\mathbf{z} = z_0{:}z_1{:}\cdots{:}z_n$ is an arbitrary point of $\mathbb{P}^n$, then

$$|\mathbf{z}| = \sqrt{\sum_{i=0}^{n} |z_i|^2} \neq 0$$

and $\mathbf{z}$ is the image of the following point of S_n: $(z_0/|\mathbf{z}|^{-1}, z_1/|\mathbf{z}|^{-1}, \dots, z_n/|\mathbf{z}|^{-1})$. Since S_n is compact and $\mathbb{P}^n$ is Hausdorff, $\mathbb{P}^n$ is compact. It follows that every projective variety is strongly compact.

An arbitrary complete variety X is by Chow's Lemma (Theorem 22.2) an image of a projective variety. Conclude that X is strongly compact. ∎

24. Linear disjointness of fields.

Let E and F be two field extensions of a field K and suppose that both are contained in a field Ω. We denote the subring of Ω generated by all the elements xy, with $x \in E$ and $y \in F$ by $E[F]$ and also by $F[E]$. This ring is obviously a K-algebra. Consider the K-linear map

$$\alpha\colon E \otimes_K F \to E[F]$$

defined by $\alpha\colon x \otimes y \mapsto xy$. This map is surjective. We say that E **and** F **are linearly disjoint over** K if α is an isomorphism.

Lemma 24.1: *A necessary and sufficient condition that E and F are linearly disjoint over K is that every finites set of elements of F which is linearly independent over K remains linearly independent over E.*

Proof: The necessity follows from Lemma 13.6. We therefore prove the sufficiency. Let $\{y_j\}_{j\in J}$ be a base for F over K. Then it is a base for $E[F]$ over E. Also, according to Lemma 13.6 $\{1 \otimes y_j\}_{j\in J}$ is a base for $E \otimes_K F$ over E. Hence α is an isomorphism. ∎

Suppose that E is the quotient field of a ring R. Then, to prove that E and F are linearly disjoint over K it suffices to show that every K-linearly independent set of elements of R is still linearly independent over F. Using this remark we prove:

Lemma 24.2 (Tower Lemma): *Let K, E, F, L be fields such that $K \subseteq L \subseteq E$ and $K \subseteq F$. Then E and F are linearly disjoint over K if and only if L, F are linearly disjoint over K and E, LF are linearly disjoint over L.*

$$\begin{array}{ccc} E & \text{——} & EF \\ | & & | \\ L & \text{——} & LF \\ | & & | \\ K & \text{——} & F \end{array}$$

Proof: Suppose first that L, F are linearly disjoint over K and E, LF are linearly disjoint over L. Use the associativity of tensor products:

$$E \otimes_K F \cong (E \otimes_L L) \otimes_K F \cong E \otimes_L (L \otimes_K F) \cong E \otimes_L L[F] \cong E[L[F]] = E[F].$$

Hence E, F are linearly disjoint over K.

Conversely, suppose that E, F are linearly disjoint over K. By Lemma 24.1, L, F are linearly disjoint over K. Hence

$$E \otimes_L L[F] \cong E \otimes_L L \otimes_K F \cong E \otimes_K F \cong E[F].$$

But the quotient field of $E[F]$ is EF. Hence E, LF are linearly disjoint over E.

REMARK 24.3: (a) If E, F are linearly disjoint over K, then $E \otimes_K F = E[F]$ and thus $E \otimes_K F$ is an integral domain. If F is an algebraic extension of K, then $E[F] = EF$ is a field and hence $E \otimes_K F$ is a field which is algebraic over E. If in addition $[F : K] < \infty$, then

$$[F : K] = [EF : E]. \tag{1}$$

It is clear that (1) also suffices for E, F to be linearly disjoint over K.

(b) If E has also a finite degree over K, then this implies that E, F are linearly disjoint over K if and only if

$$[EF : K] = [E : K][F : K].$$

(c) Again, if E, F are arbitrary field extensions of K which are linearly disjoint over K, then $E \cap F = K$. Hence, if in addition F is Galois over K, then the natural restriction embedding of $\mathcal{G}(EF/E)$ into $\mathcal{G}(F/K)$ is an isomorphism. Conversely, if F/K is a finite Galois extension, then $E \cap F = K$ implies that $\mathcal{G}(EF/E) \cong \mathcal{G}(F/K)$, hence $[EF : E] = [F : K]$, and therefore K, L are linearly disjoint over K. A fortiori, this is also true for arbitrary Galois extension F/K.

(d) Note also that if E/K is a separable algebraic extension and F/K is a purely inseparable extension, then E, F are linearly disjoint over K.

25. Algebraic independence.

Let E, F be extensions of a field K. We say that E **is free from** F **over** K if every finite set of elements of E which is algebraically independent over K remains algebraically independent over E.

The relation thus defined is symmetric. Indeed suppose that E is free from F over K. Let $y_1, \ldots, y_n$ be elements of F which are algebraically independent over K. Assume that they become dependent over E. Then there exists $f \in E[Y_1, \ldots, Y_n]$, $f \neq 0$, such that $f(y_1, \ldots, y_n) = 0$. Find a finitely generated extension E_0 of K which contains the coefficients of f. Let $r = \dim_K(E_0)$. Then $\dim_{K(\mathbf{y})}(E_0(\mathbf{y})) = r$. Hence $\dim_K(E_0(\mathbf{y}) = \dim_K(K(\mathbf{y})) + \dim_{K(\mathbf{y})}(E_0(\mathbf{y}) = n + r$. On the other hand $\dim_K(E_0(\mathbf{y})) = \dim_K(E_0) + \dim(E_0(\mathbf{y})) < r + n$, a contradiction. Conclude that F is also free over K from E.

We therefore say, in this case, that E, F are free over K.

Lemma 25.1: *If E and F are linearly disjoint over K, then they are also free over K.*

Proof: Let $x_1, \ldots, x_n$ be elements of E algebraically independent over K. Assume they become algebraically dependent over L. We get a relation $y_i M_x(\mathbf{x}) = 0$ between monomials $M_i(\mathbf{x})$ with coefficients in F. But the $M_i(\mathbf{x})$ are linearly independent over K, since $x_1, \ldots, x_n$ are algebraically independent over K. Hence the $M_i(\mathbf{x})$ are also linearly independent over F, which is a contradiction. ∎

On the other hand, if E is algebraic over K, then it is always the case that E, F are free over K. This shows that the property of linear disjointness is stronger than that of being free.

Lemma 25.2: *Let F be an extension of K. Let $u_1, \ldots, u_r$ be algebraically independent elements over a field F. Then $K(\mathbf{u}), F$ are linearly disjoint over K.*

Proof: It suffices to prove that the elements of a basis for the ring $K[\mathbf{u}]$ over K remain linearly independent over F. In fact the monomials $M(\mathbf{u})$ for a basis for $K[\mathbf{u}]$ over K. They must remain linearly independent over F, because as we have seen, a linear relation between the $M(\mathbf{u})$ gives an algebraic relation between the u_i's. This proves the lemma. ∎

As for linear disjointness we have a tower lemma for freeness:

LEMMA 25.3: *Let L, F be extensions of a field K and let E be an extension of L. Then E, F are free over K if and only if L, F are free over K and E, LF are free over L.* ∎

Proof: *Suppose first that E, F are free over L. Obviously L, F are free over K. We prove that E, LF are free over L.*

Without loss assume that L is finitely generated over K. In particular $\dim_K L = r < \infty$. Hence $\dim_F LF = r$. Now let $x_1, \ldots, x_n$ be elements of E, algebraically independent over L. Then $\dim_K LF(\mathbf{x}) = \dim_K L(\mathbf{x}) = r + n$. Hence $\dim_{LF}(LF(\mathbf{x})) = r$. Conclude that E, LF are free over L.

Conversely, suppose that E, F are free over K and E, LF are free over L. Let $x_1, \ldots, x_n$ be elements of F, algebraically independent over K. Then they are algebraically independent over L and therefore also over E. Conclude that E, F are free over K.

26. Separable extensions.

Let $F = K(\mathbf{x})$ *be a finitely generated extension of a field* K. *We say that* F/K *is* **separably generated** *if it has a transcendence base* $t_1, \ldots, t_r$ *such that* F *is separably algebraic over* $K(\mathbf{t})$. *Such a transcendence base is said to be a* **separating transcendence base for** F **over** K**.**

We always denote by p *the characteristic if it is not 0. The field obtained from* K *by adjoining all the* p^m*th roots of all the elements of* K *is denoted by* K^{1/p^m}. *The compositum of all such fields for* $m = 1, 2, \ldots$ *is denoted by* K^{1/p^∞}.

LEMMA - DEFINITION 26.1: *The following conditions on a field extension* F/K *are equivalent:*

(a) F *is linearly disjoint from* K^{1/p^∞}.

(b) F *is linearly disjoint from* $K^{1/p}$.

(c) *Every subfield of* F *containing* K *and finitely generated over* K *is separably generated.*

If F/K *satisfifs these condition, we say that it is a* **separable extension**.

Proof: It is obvious that (a) implies (b). In order to prove that (b) implies (c), we may clearly assume that F is finitely generated over K, say $F = K(x_1, \ldots, x_n)$. Let the transcendence degree of this extension be r. If $r - n$, the proof is complete. Otherwise, say $x_1, \ldots, x_r$ is a transcendence base. Then x_{r+1} is algebraic over $K(x_1, \ldots, x_r)$. Let $f(X_1, \ldots, X_{r+1})$ be a polynomial of lowest degree such that $f(x_1, \ldots, x_{r+1}) = 0$. Then f is irreducible. We contend that not all X_i ($i = 1, \ldots, r+1$) appear to the pth power throughout. If they did, we could write $f(\mathbf{X}) = \sum c_i M_i(\mathbf{X})^p$ where $M_i(\mathbf{X})$ are monomials in $X_1, \ldots, X_{r+1}$ and $c_i \in K$. This would imply that the $M_i(\mathbf{x})$ are linearly dependent over $K^{1/p}$ (taking the pth root of the equation $\sum c_i M_i(\mathbf{x})^p = 0$). However, the $M_i(\mathbf{x})$ are linearly independent over K (otherwise we would get an equation for $x_1, \ldots, x_{r+1}$ of lower degree) and we thus get a contradiction to the linear disjointness of $K(\mathbf{x})$ and $K^{1/p}$. Say X_1 does not appear to the pth power throughout, but actually appears in $f(\mathbf{X})$. We know that $f(\mathbf{X})$ is irreducible equation for x_1 over $K(x_2, \ldots, x_{r+1})$. Since X_1 does not appear to the pth power throughout, this equation is separable equation for x_1 over $K(x_2, \ldots, x_{r+1})$, in other words, x_1 is separable algebraic over $K(x_2, \ldots, x_{r+1})$. From this it follows that it is separable algebraic over $K(x_2, \ldots, x_n)$.

If $x_2, \ldots, x_n$ is a transcendence base, the proof is complete. If not, say that x_2 is separable over $K(x_3, \ldots, x_n)$. Then $K(\mathbf{x})$ is separable over $K(x_3, \ldots, x_n)$. Proceeding inductively, we see that the procedure can be continued until we get down to a transcendence base. This prove that (b) implies (c). It also prove that a separating transcendence base for $K(\mathbf{x})$ over K can be selected from the given set of generators $x_1, \ldots, x_n$.

To prove that (c) implies (a) we may assume that F is finitely generated over K. Let $\mathbf{u}$ be a separating transcendence base for F/K. Then F is separably algebraic over $K(\mathbf{u})$. By Lemma 25.2, $K(\mathbf{u})$ and $L = K^{1/p^\infty}$ are linearly disjoint. Then $L(\mathbf{u})$ is purely inseparable over $K(\mathbf{u})$, and hence is linearly disjoint from F over $K(\mathbf{u})$. Conclude from the tower lemma that F, L is linearly disjoint over K, thereby proving our theorem. ∎

Note that the definition of "separable" is compatible with the use of the word for algebraic extensions. The first condition of the lemma is known as MacLane's criterion. It has the following immediate corollaries.

Corollary 26.2: *If F/K is a separable extension and E is a subfield of F containing K, then E is separable over K.*

Corollary 26.3: *Let E be a separable extension of K and F a separable extension of E. Then F is a separable extension of K.*

Corollary 26.4: *If K is perfect, every extension of K is separable.*

Proof: In this case $K = K^{1/p}$. ∎

Corollary 26.5: *Let E be a separable extension of K which is free from an extension F of K. Then EF is a separable extension of F.*

Proof: An element of EF has an expression in terms of a finite number of elements of E and F. Hence any finitely generated subfield of EF containing F is contained in a composite field E_0F, where E_0 is a subfield of E which is finitely generated over K. By Corollary 26.2, we may therefore assume that E is finitely generated over K.

Let $\mathbf{t}$ be a separating transcendence base of E/K. So E is separable algebraic over $K(\mathbf{t})$. Hence EF is separably algebraic over $F(\mathbf{t})$. As $F(\mathbf{t})/F$ is a purely transcendental extension, it is separable. Hence, EF is separable over F (Lemma 26.3). ∎

Corollary 26.6: *Let E and F be separable extensions of K, free from each other over K. Then EF is separable over K.*

Proof: Combine Corollaries 26.5 and 26.3. ∎

Corollary 26.7: *Let E, F be linearly disjoint extensions of a field K. Then E/K is separable if and only if EF/F is separable.*

Proof: If E is not separable over K, it is not linearly disjoint from $K^{1/p}$ over K, and hence, a fortiori, it is not linearly disjoint from $K^{1/p}F$ over K. By the tower lemma, this implies that EF is not linearly disjoint from $K^{1/p}F$ over F and hence that EF is not separable over F.

The converse is a special case of Corollary 26.5, taking into account that linearly disjoint fields are free. ∎

27. Regular extensions.

LEMMA 27.1: *Suppose that a field K is algebraically closed in an extension F. If a is algebraic over K, then $K(a)$ and F are linearly disjoint over K and $[K(a) : K] = [F(a) : F]$.*

Proof: Let $f(X) = \mathrm{irr}(a, K)$. The coefficients of each monic factor of $f(X)$ are polynomials in the roots of $f(X)$ with coefficients in K. As K is algebraically closed in F this implies that $f(X)$ remains irreducible over F. Conclude that $[F(a) : F] = [K(a) : K]$ and therefore $F, K(a)$ are linearly disjoint over K. ∎

LEMMA – DEFINITION 27.2: *The following two conditions on a field extension F/K are equivalent:*

(a) *K is algebraically closed in F and F/K is separable.*

(b) *K is linearly disjoint from F over $\widetilde{K}$.*

If this conditions are satisfied, then F/K is said to be **regular**.

Proof: If condition (b) holds, then F is linearly disjoint from $K^{1/p}$ over K. Hence F/K is separable (Lemma 26.1). Obviously K is algebraically closed in F. Thus condition (a) is satisfied.

To prove (b) from (a) we may assume without loss that F is finitely generated over K. It suffices to prove that F is linearly disjoint from an arbitrary finite algebraic extension L of K.

If L is separable algebraic over K, then it can be generated by one element, and we can apply Lemma 27.1.

More generally, Let E be the maximal separable subfield of L containing K. Using the tower lemma we see that it suffices to prove that EF, L are linearly disjoint over E. Let $\mathbf{t}$ be a separating transcendence base for F/K. Then $F/K(\mathbf{t})$ is separably algebraic. Furthermore, $\mathbf{t}$ is also a separating transcendence base for EF/E and $EF/E(\mathbf{t})$ is separably algebraic. Thus EF is separable over E, and by definition it is linearly disjoint from L over E because L is purely inseparable over E. This proves the lemma.
∎

COROLLARY 27.3: *Let F be a regular extension of K and let E be a subfield of F that contains K. Then E is regular over K.*

COROLLARY 27.4: *If E is a regular extension of K and F is a regular extension of E, then F is a regular extension of K.*

Proof: Apply the tower lemma. ∎

COROLLARY 27.5: *If K is algebraically closed, then every extension of K is regular.*

LEMMA 27.6: *Let F/K be a regular extension. Let x be an element which is either algebraic over K, or transcendental over F. Then F is linearly disjoint from $K(x)$ over K and $F(x)$ is a regular extension of $K(x)$.*

Proof: Suppose first that x is algebraic over K. As F is linearly disjoint from $\widetilde{K}$ over K, the tower lemma asserts that F is linearly disjoint from $K(x)$ over K and $F(x)$ is linearly disjoint from $\widetilde{K}$ over $K(x)$. Hence $F(x)$ is a regular extension of $K(x)$.

So assume that x is transcendental over F. By Lemma 25.2, $K(x)$ is linearly disjoint from F over K. Furthermore, by Corollary 26.5, $F(x)/K(x)$ is separable. Hence, in order to prove the second statement it suffices to prove that $K(x)$ is algebraically closed in $F(x)$. Let $y_1 \in F(x)$ and suppose that y is algebraic over $K(x)$. Write $y_1 = f_1/g_1$, where f_1, g_1 are elements of $F[x]$ which are relatively prime. Let c (resp., d) the leading coefficient of f_1 (resp., g_1). Put $y = y_1 d/c$, $f = f_1/c$, and $g = g_1/d$. Then f and g are monic, and $y = f/g$ satisfies and equation

$$a_n(x)(f/g)^n + a_{n-1}(x)(f/g)^{n-1} + \cdots + a_0(x) = 0$$

with $a_i(x) \in K[x]$ and $a_0(x), a_n(x) \neq 0$. Multiply this equation by g^n to conclude that $f(x)|a_0(x)$ and $g(x)|a_n(x)$. It follows that all the roots of $f(x)$ and $g(x)$ belong to $\widetilde{K}$. Since f and g are monic, also their coefficients belong to $\widetilde{K}$. As K is algebraically closed in F, this implies that $f, g \in K[x]$.

It follows that $u = d/c$ is an element of F which is algebraic over $K(x)$. Thus u satisfies an equation

$$b_m(x)u^m + b_{m-1}(x)u^{m-1} + \cdots + b_0(x) = 0$$

with $b_i(x) \in K[x]$ and $b_0(x) \neq 0$. Divide the coefficients by there greatest common divisor in $F[x]$ to assume that x does not divide all of them. Then substitute $x = 0$ in it to conlcude that b is algebraic over K. Thus $b \in K$ and therefore $y_1 \in K(x)$, as desired. ∎

PROPOSITION 27.7: *Let F/K be a regular extension. If L/K is an extension which is free from F/K, then L, F is linearly disjoint over K and LF/L is regular.*

Proof: We can assume that L is finitely generated over K. Then we can describe L as $L = K(t_1, \ldots, t_r, x_1, \ldots, x_m)$ where $t_1, \ldots, t_r$ is a transcendence base for L/K and $x_1, \ldots, x_m$ are algebraic over $K(\mathbf{t})$. Then we use Lemma 27.6 to prove the lemma in $r + m$ steps. ∎

COROLLARY 27.8: *Let E, F be two regular extensions of a field K, free from each other over K. Then EF is a regular extension of K.*

LEMMA 27.9: *Let Ω be an algebraically closed field which has an infinite transcendence degree over a field K. Let $R = K[x_1, \ldots, x_n]$ be an integral domain. Suppose that F is a finitely generated extension of K which is contained in Ω. Then R is K-isomorphic to a subring R' of Ω whose quotient field E' is free from F over K.*

Proof: Denote the quotient field of R by E. Suppose without loss that $x_1, \ldots, x_r$ is a transcendence base for E/K. Choose elements $x'_1, \ldots, x'_r \in \Omega$ which are algebraically independent over F. The algebraic closure Ω_0 of $K(x'_1, \ldots, x'_r)$ is free from F over K. Assume by induction that we have already constructed an embedding φ of $K[x_1, \ldots, x_{n-1}]$ into Ω_0 such that $\varphi(x_i) = x'_i$ for $i = 1, \ldots, r$. Put $\varphi(x_i) = x'_i$ also for $i = r+1, \ldots, n-1$. Note that x_n is algebraic over $K[x_1, \ldots, x_{n-1}]$. Let therefore $f \in K[x_1, \ldots, x_{n-1}, X_n]$ be an irreducible polynomial over $K(x_1, \ldots, x_{n-1})$ such that $f(x_1, \ldots, x_n) = 0$. Then $f(x'_1, \ldots, x'_{n-1}, X_n)$ is irreducible over $K(x'_1, \ldots, x'_{n-1})$. Take a root x'_n of this polynomial in Ω_0. We may extend φ to a K-embedding of R onto $K[x_1, \ldots, x_n]$ that maps x_n onto x'_n. The field $K(x'_1, \ldots, x'_n)$ is obviously free from F over K. ∎

We are now in a position to prove Proposition 13.7:

PROPOSITION 27.10: *If R and S are integral domains that contain an algebraically closed field K, then $R \otimes_K S$ is an integral domain. Moreover, if E, F, and L are the quotient fields of R, S, and $R \otimes_K S$, respectively, then* $\dim_K(L) = \dim_K(E) + \dim_K(F)$.

Proof: Suppose without loss that $R = K[x_1, \ldots, x_m]$ and $S = K[y_1, \ldots, y_n]$ are finitely generated over K. Let Ω be an algebraically closed field of infinite transcendence degree over F. By Lemma 27.9, there exists a K-isomorphism θ of R onto a subring R' of Ω whose quotient field E' is free from F over K. As K is algebraically closed E', F are regular extensions of K (Lemma 27.5). Hence by Lemma 27.7, they are linearly disjoint over K. It follows that the composed map $R \otimes_K S \to R' \otimes_K S \to R[S]$ is an isomorphism. Since $R'[S]$ is a subring of Ω, it is an integral domain. Hence, so is $R \otimes_K S$.

Also, $L \cong_K E'F$. Conclude that $\dim_K(L) = \dim_K(E) + \dim_K(F') = \dim_K(E) + \dim_K(F)$, as asserted. ∎

28. K-algebraic sets.

We consider in this section a field K and a field extension Ω of K which is algebraically closed and of infinite transcendence degree over K, which we call a **universal domain**. Among the algebraic sets defined over Ω we consider now only those which are defined by polynomials with coefficients in K and call them K-algebraic sets. Thus a K-algebraic set is a subset V of Ω^n for which there exists a subset $\mathfrak{a}$ of $K[X_1, \ldots, X_n]$ such that $V = V(\mathfrak{a}) = \{\mathbf{x} \in \Omega^n |\ f(\mathbf{x}) = 0 \text{ for each } f \in \mathfrak{a}\}$. Obviously $V(\mathfrak{a})$ does not change if we replace $\mathfrak{a}$ by the ideal of $K[X_1, \ldots, X_n]$ generated by $\mathfrak{a}$.

To each subset A of Ω^n we associate the ideal $I(V) = \{f \in K[X_1, \ldots, X_n] |\ f(\mathbf{x}) = 0 \text{ for each } \mathbf{x} \in A\}$ of $K[X_1, \ldots, X_n]$. The correspondence between ideals and K-algebraic sets satisfies Lemma 3.1, provided we replace K^n by Ω^n.

The weak Nullstellensatz does not hold any more. For example $X^2 + Y^2 + Z^2$ does not have a $\mathbb{Q}$-rational zero. However, Corollary 3.1, slightly modified, is still true:

LEMMA 28.1: *If $f_1, \ldots, f_m \in K[\mathbf{X}]$ have no common zero in Ω, then there exist $g_1 \ldots, g_m \in K[\mathbf{X}]$ such that $g_1 f_1 + \ldots + g_m f_m = 1$.*

Proof: By Corollary 3.3 there exist $g_1, \ldots, g_m \in \Omega[\mathbf{X}]$ such that $\sum g_i f_i = 1$. Take a base $\{w_\alpha\}_{\alpha \in A}$ for the K-linear space Ω which contains 1. Then $g_i = \sum_{\alpha \in A} g_{i\alpha} w_\alpha$ with $g_{i\alpha} \in K[\mathbf{X}]$. Conclude from $\sum_{\alpha \in A} \big(\sum_{i=1}^m g_{i\alpha} f_i\big) w_\alpha = 1$ that $\sum_{i=1}^m g_{i1} f_i = 1$. This proves our assertion. ∎

Now we may repeat the proof of the strong Nullstellensatz except that we have to use an Ω-rational zero instead of K-rational zero. We obtain a statement on the radical of an ideal in $K[\mathbf{X}]$:

PROPOSITION 28.2 (Strong Nullstellensatz): *Let $\mathfrak{a}$ be an ideal of $K[\mathbf{X}]$. If a polynomial $f \in K[\mathbf{X}]$ vanishes on $V(\mathfrak{a})$, then $f \in \sqrt{\mathfrak{a}}$.*

In particular, it follows for a prime ideal $\mathfrak{p}$ of $K[\mathbf{X}]$ that $I(V(\mathfrak{p})) = \mathfrak{p}$.

Now we may consider the K-topology on Ω^n whose open sets are the complements of the K-algebraic sets. It is weaker than the Zariski topology. By Hilbert's basis theorem Ω^n is Noetherian under the K-topology. In particular, every K-algebraic set has a unique presentation as a union of K-irreducible algebraic sets, none of them is contained in the other. If V is a K-irreducible algebraic set, then $I(V)$ is a prime ideal of $K[\mathbf{X}]$. Conversely, if $\mathfrak{p}$ is a prime ideal, then $V(\mathfrak{p})$ is K-irreducible.

The new aspect we obtain by distinguishing between K and Ω is the possibility to recognize the dimension of the points of V over K and especially the existence of generic points:

Let $\mathfrak{p}$ be a prime ideal of $K[\mathbf{X}]$ and $V = V(\mathfrak{p})$ the corresponding K-irreducible algebraic set. Denote the residue class of X_i modulo $\mathfrak{p}$ by x_i. Then $K[\mathbf{x}] = K[\mathbf{X}]/\mathfrak{p}$ is an integral domain. By Lemma 27.9 we may identify $K[\mathbf{x}]$ with a subring of Ω. As $f(\mathbf{x}) = 0$ for each $f \in \mathfrak{p}$, $\mathbf{x}$ is a point of V which we call **generic**. It is characterized by the following Lemma which is a straight forward application of the second isomorphism theorem for rings:

LEMMA 28.3: *Let $\mathbf{x}' \in \Omega^n$. Then $\mathbf{x}' \in V$ if and only if for each $f \in K[\mathbf{X}]$, $f(\mathbf{x})$ implies $f(\mathbf{x}') = 0$.*

The existence of a generic point characterizes an irreducible K-algebraic set:

LEMMA 28.4: *Let A be a K-algebraic set. Then A is irreducible if and only if it has a generic point.*

Proof: We have only to prove that the condition is sufficient. Indeed, let $\mathbf{x}$ be a generic point of A. Suppose that $A = V(\mathfrak{a}) \cup V(\mathfrak{b})$, where $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of $K[\mathbf{X}]$. Then $\mathbf{x}$ belongs, say to $V(\mathbf{a})$. If $\mathbf{x}'$ is an arbitrary point in A, then $f(\mathbf{x}) = 0$ and hence $f(\mathbf{x}') = 0$ for each $f \in \mathfrak{a}$. Hence $A = V(\mathfrak{a})$. So, A is irreducible. ∎

Consider two points $\mathbf{x}, \mathbf{x}'$ in Ω^n. We say that $\mathbf{x}'$ is a **specialization** of $\mathbf{x}$ (over K, if the reference to K is not obvious) and we write $\mathbf{x} \to \mathbf{x}'$ if $f(\mathbf{x}) = 0$ implies $f(\mathbf{x}') = 0$ for each $f \in K[\mathbf{X}]$. Note that the condition is equivalent to the map $\mathbf{x} \to \mathbf{x}'$ extends to a K-homomorphism $K[\mathbf{x}] \to K[\mathbf{x}']$.

In particular if $\mathbf{x}$ is a generic point of a K-irreducible algebraic set V, then $\mathbf{x} \to \mathbf{x}'$ for every $\mathbf{x}' \in V$.

In the following reinterpretation of Lemma 17.3 we use $\dim_K(\mathbf{x})$ to denote the transcendence degree of $K(\mathbf{x})$ over K.

LEMMA 28.5: *Let* $\mathbf{x}'$ *be a specialization of* $\mathbf{x}$ *over* K. *Then* $\dim_K(\mathbf{x}') \leq \dim_K(\mathbf{x})$ *and equality holds if and only if* $\mathbf{x}' \to \mathbf{x}$.

Thus, if $\mathbf{x}$ is a generic point of a K-irreducible algebraic set V, then it has the maximal dimension over K among all the points of V. The field $K(\mathbf{x})$ is the **function field** of V over K. If $\mathbf{x}'$ is another generic point of V, then $K[\mathbf{x}] \cong_K K[\mathbf{x}']$ and $K(\mathbf{x}) \cong_K K(\mathbf{x}')$. So, the function field of K is unique up to K-isomorphism. So is its transcendence degree over K, which is by definition, the dimension of V.

29. Absolutely irreducible varieties.

We examine in this section what happens to a K-irreducible sets when K grows up to its algebraic closure. We continue to keep K and Ω and consider only intermediate fields L such that Ω is a universal domain over L..

LEMMA 29.1: *Let* $\mathbf{x}$ *be a point in* Ω^n, *and* L *be an extension of* K. *Consider the prime ideals* $\mathfrak{p} = \{f \in K[\mathbf{X}] \mid f(\mathbf{x}) = 0\}$ *and* $\mathfrak{P} = \{f \in L[\mathbf{X}] \mid f(\mathbf{x}) = 0\}$ *determined by* $\mathbf{x}$ *in* $K[\mathbf{X}]$ *and* $L[\mathbf{X}]$, *respectively. Then* $\mathfrak{P} = \mathfrak{p}L[\mathbf{X}]$ *if and only if* $K(\mathbf{x})$ *and* L *are linearly disjoint over* K.

Proof: Suppose that $K(\mathbf{x})$ and L are linearly disjoint over K. Let $f \in \mathfrak{P}$. Write $f(\mathbf{X}) = \sum a_j f_j(\mathbf{X})$ with each f_j in $K[\mathbf{X}]$ and a_j in L are linearly independent over K. Now, $f(\mathbf{x}) = 0$ implies $\sum a_j f_j(\mathbf{x}) = 0$ and by linear disjointness, $f_j(\mathbf{x}) = 0$ for all j. Hence each $f_j(\mathbf{X})$ is in $\mathfrak{P}$. It is also in $K[\mathbf{X}]$. Conclude that $\mathfrak{P} = \mathfrak{p}L[\mathbf{X}]$.

Conversely, suppose that $\mathfrak{P} = \mathfrak{p}L[\mathbf{X}]$. Let $M_1(\mathbf{X}), \ldots, M_r(\mathbf{X})$ a finite set of monomials in $\mathbf{X}$ such that $M_1(\mathbf{x}), \ldots, M_r(\mathbf{x})$ are linearly independent over K. It suffices to prove that the $M_j(\mathbf{x})$ remain linearly independent over L, since the set of all monomials generates the vector space $K[\mathbf{x}]$ over K.

Assume we had a linear relation $\sum a_j M_j(\mathbf{x}) = 0$ with $a_j \in L$ and $a_j \neq 0$. Then $\sum a_j M_j(\mathbf{X}) \in \mathfrak{P}$ and we can write

$$\sum_{j=1}^{r} a_j M_j(\mathbf{X}) = \sum_{i=1}^{s} b_i f_i(\mathbf{X}) \tag{1}$$

where $f_i(\mathbf{X}) \in \mathfrak{P} \cap K[\mathbf{X}]$ are linearly independent over K and $b_i \in L$. Since $M_1, \ldots, M_r, f_1, \ldots, f_s$ are formal polynomials, they must be linearly dependent over K (Lemma 25.2). Thus there exists $z_j, y_i \in K$, not all 0 such that

$$\sum z_j M_j(\mathbf{X}) + \sum y_i f_i(\mathbf{X}) = 0.$$

Hence $\sum z_j M_j(\mathbf{x}) = 0$ and therefore $z_j = 0$ for all j. It follows that $\sum y_i f_i(\mathbf{X}) = 0$ and therefore $y_i = 0$ for all i. This contradiction proves the lemma. ∎

If V is a K-irreducible algebraic set and L is an extension of K, then V is also an L-algebraic set. But it may decompose over L. The following Proposition gives conditions for this not to happen.

Proposition 29.2: *Let $\mathfrak{p}$ be a prime ideal in $K[\mathbf{X}]$. Put $V = V(\mathfrak{p})$ for the corresponding K-irreducible algebraic set, and let $\mathbf{x}$ be a generic point of V over K. Then the following statements are equivalent:*

(a) *$K(\mathbf{x})/K$ is a regular extension.*

(b) *$\mathfrak{p}L[\mathbf{X}]$ is a prime ideal in $L[\mathbf{X}]$ for every extension L of K.*

(c) *$L \otimes_K K[\mathbf{x}]$ is an integral domain for every extension L of K.*

(d) *V is irreducible over L and $I_L(V) = \mathfrak{p}L[\mathbf{X}]$ for every extension L of K.*

(e) *$\mathfrak{p}\widetilde{K}[\mathbf{X}]$ is a prime ideal in $\widetilde{K}[\mathbf{X}]$.*

(f) *$\widetilde{K} \otimes_K K[\mathbf{x}]$ is an integral domain.*

(g) *V is irreducible over $\widetilde{K}$ and $I_{\widetilde{K}}(V) = \mathfrak{p}\widetilde{K}[\mathbf{X}]$.*

(h) *V is irreducible over $\widetilde{K}$ and $\widetilde{K} \otimes_K K[\mathbf{x}]$ is a reduced ring (i.e. it contains no nilpotent elements).*

If these statements hold we say that V is **absolutely irreducible** *and* **defined over** *K.*

Proof of (a) implies (b): Let L be an extension of K. Change the generic point $\mathbf{x}$ of V over K such that L will be free from $K(\mathbf{x})$ over K (Lemma 27.9). Then, according to Lemma 27.9, $L, K(\mathbf{x})$ are linearly disjoint over K. By Lemma 29.3, $L[\mathbf{X}]\mathfrak{p}$ is the prime ideal determined by $\mathbf{x}$ in $L[\mathbf{X}]$.

Proof of (b) is equivalent to (c): Tensor the short exact sequence

$$0 \to \mathfrak{p} \to K[\mathbf{X}] \to K[\mathbf{x}] \to 0$$

with L to obtain the short exact sequence

$$0 \to L \otimes_K \mathfrak{p} \to L \otimes_K K[\mathbf{X}] \to L \otimes_K K[\mathbf{x}] \to 0$$

(Proposition 13.5). Note that since $X_1, \ldots, X_n$ are algebraically independent over L, we have by Lemma 25.2, that $L, K(\mathbf{X})$ are linearly disjoint over L. Hence, the natural map $L \otimes_K K(\mathbf{X}) \to L[K(\mathbf{X})]$ is an isomorphism. This isomorphism maps $L \otimes_K \mathfrak{p}$ onto $L[\mathbf{X}]\mathfrak{p}$ and $L \otimes_K K[\mathbf{X}]$ onto $L[\mathbf{X}]$. Thus

$$0 \to L[\mathbf{X}]\mathfrak{p} \to L[\mathbf{X}] \to L \otimes_K K[\mathbf{x}] \to 0$$

is a short exact sequence. Conclude that $L[\mathbf{X}]\mathfrak{p}$ is a prime ideal of $L[\mathbf{X}]$ if and only if $L \otimes_K K[\mathbf{x}]$ is an integral domain, as contended.

Proof of (b) is equivalent to (d): Clear.

Proof of (b) implies (e): Clear.

Proof of (e) is equivalent to (f): It is a special case of the equivalence between (b) and (c).

Proof of (e) is equivalent to (g): It is a special case of the equivalence between (b) and (d).

Proof of (e) implies (a): As $V = V(\widetilde{K}[\mathbf{X}]\mathfrak{p})$ and $\widetilde{K}[\mathbf{X}]\mathfrak{p}$ is a prime ideal of $\widetilde{K}[\mathbf{X}]$, V is $\widetilde{K}$-irreducible. Let $\mathbf{x}'$ be a generic point of V over $\widetilde{K}$. Then $\mathbf{x}'$ is a specialization of $\mathbf{x}$ over K and therefore $\dim_K \mathbf{x}' \le \dim_K \mathbf{x}$ (Lemma 28.5). Also $\mathbf{x}$ is a specialization of $\mathbf{x}'$ over $\widetilde{K}$ and therefore $\dim_{\widetilde{K}} \mathbf{x} \le \dim_{\widetilde{K}} \mathbf{x}'$. It follows that $\dim_{\widetilde{K}} \mathbf{x} = \dim_{\widetilde{K}} \mathbf{x}'$. Hence, by Lemma 28.5, $\mathbf{x}'$ is also a specialization of $\mathbf{x}$ over $\widetilde{K}$. This gives an isomorphism $\widetilde{K}[\mathbf{x}] \cong_{\widetilde{K}} \widetilde{K}[\mathbf{x}']$. Hence $\widetilde{K}[\mathbf{X}]\mathfrak{p}$ is the prime ideal defined by $\mathbf{x}$ over $\widetilde{K}$. By Lemma 29.1, $K(\mathbf{x})$ is linearly disjoint from $\widetilde{K}$ over K. This means that $K(\mathbf{x})$ is a regular extension of K, as contended.

Proof of (h) is equivalent to (g): If (g) holds, then also (f) holds. Thus, $\widetilde{K} \otimes_K K[\mathbf{x}]$ is an integral domain, hence it is also reduced.

Conversely, suppose (h) holds but $I_{\widetilde{K}}(V) \neq \mathfrak{p}\widetilde{K}[\mathbf{X}]$. By [Lang, Introduction to Algebraic Geometry, p. 74, C6], K has a purely inseparable extension such that $I_{\widetilde{K}}(V)$ has a set of generators $g_1, \ldots, g_s \in L[\mathbf{X}]$. One of them, say g_1 does not belong to $\widetilde{K} \cdot I_K[\mathbf{X}]$. It follows from the short exact sequence $0 \to \widetilde{K} \cdot I_K(V) \to \widetilde{K}[\mathbf{X}] \to \widetilde{K}[\mathbf{x}] \to 0$ that $g_1(\mathbf{x}) \notin \widetilde{K}[\mathbf{x}]$. On the other hand, there exists a power q of $\mathrm{char}(K)$ such that $g_1(X)^1 \in I_K(V)$, hence $g_1(\mathbf{x})^q = 0$. Consequently, $\widetilde{K}[\mathbf{x}]$ is not reduced, in contrast to our assumption. ∎

EXAMPLE 29.3: *A K-irreducible variety under a purely inseparable extension.* Suppose that $\mathfrak{p}$ is a prime ideal of $K[\mathbf{X}]$ and $V = V(\mathfrak{p})$. Let L be an extension of K over which

V remains irreducible. As $V = V(\mathfrak{p}L[\mathbf{X}])$, Hilbert's Nullstellensatz implies that the radical $\sqrt{\mathfrak{p}L[\mathbf{X}]}$ is prime. However, $\mathfrak{p}L[\mathbf{X}]$ need not be prime.

For example, suppose that $\mathrm{char}(K) = p$ and $K = K_0(t)$ with t transcendental over K_0. Put $\tau = \sqrt{t}$. Then $X^p - t$ is irreducible over K and $V = V(X^p - t) = \{\tau\}$ is absolutely irreducible. However over $L = K(\tau)$ we have a decomposition $X^p - t = (X - \tau)^p$. So, V is not defined over K but rather over L. ∎

29 August, 2012