Kuykian Fields

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Abstract

The Kuykian conjecture for a Hilbertian field K says that if A/K is an abelian variety, then every intermediate field of $K(A_{\mathrm{tor}})/K$ is Hilbertian. We prove the Kuykian conjecture in the following cases: (a) K is finitely generated (over its prime field). (b) $K = F_s[\boldsymbol{\sigma}]$ for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, where F is a finitely generated field. (c) $K = F_{\mathrm{ins}}$, where F is the quotient field of a complete local domain of dimension at least 2.

1 Introduction

The inverse problem of Galois theory asks whether every finite group G occurs as the Galois group $\mathrm{Gal}(f(X), \mathbb{Q})$ of a polynomial $f \in \mathbb{Q}[X]$. The most effective tool to solve the problem, at least for a large family of finite groups, is Hilbert's Irreducibility Theorem: For every polynomial $h \in \mathbb{Q}[T, X]$ which is separable in X there exist infinitely many $a \in \mathbb{Q}$ such that $\mathrm{Gal}(h(a, X), \mathbb{Q}) \cong \mathrm{Gal}(h(T, X), \mathbb{Q}(T))$. Fields that satisfy Hilbert's Irreducibility Theorem are called *Hilbertian fields.*

A theorem of Kuyk from 1970 asserts that every abelian extension of a Hilbertian field is Hilbertian [2, Thm. 16.11.3]. In analogy to that theorem, the article [6] makes the following conjecture:

Kuykian Conjecture *Let K be a Hilbertian field and A/K an abelian variety. Then every intermediate field M of $K(A_{\mathrm{tor}})/K$ is Hilbertian.*

The Kuykian Conjecture is proved in the special case where K is a number field [6, Theorem 10]. The proof makes use of Haran's diamond theorem, of the fact that each closed subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$ is finitely generated, and of the following theorem of Serre: *If A is an abelian variety over a number field K, then there is a finite extension K'/K such that $(K'(A[l^\infty]))_{l \text{ prime}}$ is a linearly disjoint sequence of extensions of K'.* The analogous statement is false for a finitely generated ground field K of positive characteristic, because already adjoining roots of unity of prime orders violates the linear disjointness. We don't know if Serre's theorem is true when K is an arbitrary finitely generated extension of $\mathbb{Q}$, except when A

is an elliptic curve. (In the case of an elliptic curve A/K with transcendental j-invariant it is known classically that $\mathrm{Gal}(K(A[n])/K) \cong \mathrm{GL}(2, \mathbb{Z}/n\mathbb{Z})$ for each positive integer n which is coprime to a fixed positive integer n_0.) However, applying different methods, we establish the following result in the function field case.

Main Theorem *Let K be a finitely generated infinite field and A/K an abelian variety. Then every intermediate field M of $K(A_{\mathrm{tor}})/K$ is Hilbertian.*

Following the Main Theorem, we call a field K *Kuykian* if for each abelian variety A/K, every intermediate field M of $K(A_{\mathrm{tor}})/K$ is Hilbertian. Thus, by the Main Theorem, every finitely generated infinite field is Kuykian. Considering the zero abelian variety, we find that every Kuykian field is Hilbertian. The Kuykian Conjecture is equivalent to the converse of this statement, namely "every Hilbertian field is Kuykian".

In the last section we give a variety of examples of non finitely generated Kuykian fields. In particular, we prove that if K is a countable Kuykian field, then for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ (in the sense of the Haar measure), the field $K_s[\boldsymbol{\sigma}]$ (see Section 4 for a definition) is Kuykian. We also prove that if K is the quotient field of a complete local domain of dimension at least 2, then the maximal purely inseparable extension K_{ins} of K is Kuykian.

2 Notation

We use the following variant of Hilbert's Irreducibility Theorem to define Hilbertian fields: A field K is **Hilbertian** if for every irreducible polynomial $h \in K[T, X]$ which is separable in X there exists $a \in K$ such that $h(a, X)$ is irreducible in $K[X]$ [2, Proposition 13.2.2].

Every number field and every finitely generated transcendental extension of an arbitrary field is Hilbertian [2, Theorem 13.4.2]. Moreover, every finite extension of a Hilbertian field and every purely inseparable extension of a Hilbertian field is Hilbertian [2, Proposition 12.3.3].

An algebraic field extension N/K is said to be **small** if for each d there are only finitely many intermediate fields L of N/K with $[L : K] = d$.

We denote the algebraic closure of a field K by $\tilde{K}$, the separable closure of K in $\tilde{K}$ by K_s, and let $\mathrm{Gal}(K) = \mathrm{Gal}(K_s/K)$ be the absolute Galois group of K. That group is equipped with a unique normalized Haar measure.

For an abelian variety A over a field K we denote the set of n-torsion points of $A(\tilde{K})$ by $A[n]$, and for a prime number p let $A[p^\infty] = \bigcup_{k\in\mathbb{N}} A[p^k]$, $A_{\mathrm{tor}} = \bigcup_{n\in\mathbb{N}} A[n]$, and $T_p(A) = \varprojlim_{k\in\mathbb{N}} A[p^k]$. If $U \subseteq A(\tilde{K})$ is a set of geometric points of A, then $K(U)$ denotes the compositum of the residue fields $K(x)$, $x \in U$.

If L is an extension of a field K and V is a K-variety, we write V_L for $V \times_{\mathrm{Spec}(K)}$

Spec(L).

3 Function Fields of One Variable

The field K appearing in the Main Theorem is either a number field or an algebraic function field of one variable over a finitely generated field. The former case is handled in [6]. The latter case is generalized below to include function fields of one variable over an arbitrary field.

Proposition 3.1 *Let F be an algebraically closed field and K/F a function field of one variable. Let A/K be an abelian variety and $K(A_{\mathrm{tor}})_0 = K_s \cap K(A_{\mathrm{tor}})$. Then* $\mathrm{Gal}(K(A_{\mathrm{tor}})_0/K)$ *is a finitely generated profinite group.*

Proof. Let Γ be the unique smooth projective F-curve with function field K. By the semistable reduction theorem [4, Théorème IX.3.6, p. 351], there is a finite Galois extension K'/K such that $A_{K'}$ has semistable reduction at all points of the normalization Γ' of Γ in K'. Moreover, $A_{K'}$ has good reduction outside a finite subset $S \subset \Gamma'$ [8, Remark 20.9, p. 148]. Then for every prime number $l \neq p := \mathrm{char}(K)$, the Galois extension $K'(A[l^\infty])/K'$ is unramified outside S by the criterion of Néron-Ogg-Shafarevich (see [10, Theorem 1]). Also, $K'(A[l^\infty])/K'$ is tamely ramified at each point of Γ'.

It suffices to prove the latter statement for $p \neq 0$. Let P be a point of Γ', v the corresponding normalized discrete valuation of K'/F and (L, w) the henselization of (K', v). Since w/v is unramified, it suffices to prove that w is tamely ramified in $L(A[l^\infty])/L$. Since F is algebraically closed, $\mathrm{Gal}(L)$ is the inertia group of the unique extension of w to L_s. By [4, Corollaire IX.3.5.2, p. 350], the maximal pro-p-subgroup of $\mathrm{Gal}(L)$ acts trivially on $T_l(A_L)$. Hence, the restriction of that group to $L(A[l^\infty])$ is trivial, so $L(A[l^\infty])/L$ is tamely ramified.

It follows for $N = \prod_{l \neq \mathrm{char}(K)} K(A[l^\infty])$ and $N' = NK'$ that N'/K' is tamely ramified at each point of Γ' and unramified outside S. Thus, N' is contained in the maximal Galois extension $K'_{S,\mathrm{tr}}$ of K' which is tamely ramified at each point of Γ' and unramified outside S. But $\mathrm{Gal}(K'_{S,\mathrm{tr}}/K')$ is finitely generated[1], [3, Corollaire XIII.2.12]. Hence the quotient $\mathrm{Gal}(N'/K')$ of $\mathrm{Gal}(K'_{S,\mathrm{tr}}/K')$ is also finitely generated. It follows that $\mathrm{Gal}(N/N \cap K') \cong \mathrm{Gal}(N'/K')$ is also finitely generated. Since K'/K is a finite extension, $\mathrm{Gal}(N/N \cap K')$ is an open subgroup of $\mathrm{Gal}(N/K)$. Therefore, $\mathrm{Gal}(N/K)$ is finitely generated.

If $\mathrm{char}(K) = 0$, then $N = K(A_{\mathrm{tor}}) = K(A_{\mathrm{tor}})_0$, hence the claim is proved. Therefore, assume that $\mathrm{char}(K) = p > 0$ and let $N_p = K_s \cap K(A[p^\infty])$. Then the action of $\mathrm{Gal}(N_p/K)$ on $A[p^\infty]$ identifies $\mathrm{Gal}(N_p/K)$ with a closed subgroup of $\mathrm{Aut}(T_p(A))$ in a natural way. Since $T_p(A) = \mathbb{Z}_p^n$ for some integer $0 \leq n \leq \dim(A)$, see [9, p. 179], $\mathrm{Gal}(N_p/K)$ is isomorphic to a closed subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$. Therefore, by [6, Lemma 5], every closed subgroup of $\mathrm{Gal}(N_p/K)$ is finitely generated.

[1] It can be generated by $|S| + 2\mathrm{genus}(\Gamma')$ elements.

Finally observe that $K(A_{\mathrm{tor}}) = K(A[p^\infty])N$, so $K(A_{\mathrm{tor}})/N_pN$ is purely inseparable, hence $N_pN = K_s \cap K(A_{\mathrm{tor}}) = K(A_{\mathrm{tor}})_0$. Since $\mathrm{Gal}(K(A_{\mathrm{tor}})_0/N) \cong \mathrm{Gal}(N_p/N_p \cap N)$ is a closed subgroup of $\mathrm{Gal}(N_p/K)$, it is finitely generated. Since also $\mathrm{Gal}(N/K)$ is finitely generated by the first paragraph of the proof, and the class of finitely generated profinite groups is closed under group extensions, $\mathrm{Gal}(K(A_{\mathrm{tor}})_0/K)$ is finitely generated. $\square$

Remark 3.2 *It follows from [6, Lemma 4] that in the setup of Proposition 3.1 every intermediate field M of $K(A_{\mathrm{tor}})_0/K$ is Hilbertian.*

Lemma 3.3 *Let K be a Hilbertian field and N/K a small algebraic extension. Then N is Hilbertian.*

Proof. Since a purely inseparable extension of a Hilbertian field is Hilbertian, it suffices to prove that $N \cap K_s$ is Hilbertian. We may thus assume that N/K is separable.

Let $f \in N[T, X]$ be an irreducible polynomial which is separable in X. Replacing K by a finite extension in N, if necessary, we may assume that $f \in K[T, X]$. Let $m = \deg_X(f)$ and let M be the compositum of all intermediate fields of N/K of degree at most $m!$ over K. Then M/K is a finite separable extension. By [2, Corollary 12.2.3], there exists $a \in K$ such that $f(a, X)$ is irreducible over M. We claim that $f(a, X)$ is even irreducible in $N[X]$.

Otherwise $f(a, X) = g(X)h(X)$ with $1 \le \deg(g), \deg(h) \le m$ and $g, h \in N[X]$ monic. Let L be the splitting field of $f(a, X)$ over K. Then $[L : K] \le m!$ and the coefficients of g and h lie in L. Hence $g, h \in (L \cap N)[X] \subseteq M[X]$ - a contradiction. $\square$

For a field extension M/K we denote the set of all intermediate fields L with $[L : K] = d$ by $S_d(M/K)$.

Lemma 3.4 *Let K be a field and let N/K and M/K be linearly disjoint algebraic extensions. If MN/N is small, then M/K is small.*

Proof. If M' is an intermediate field of M/K with $[M' : K] = d$, then $[M'N : N] = d$ because of the linear disjointness. Moreover, the map

$$S_d(M/K) \to S_d(MN/N),\ M' \mapsto M'N$$

is injective. In fact, if $M_1, M_2 \in S_d(M/K)$ and $M_1N = M_2N$, then $[M_1M_2 : K] = [(M_1M_2)N : N] = [M_1N : N] = d$, hence $M_1 = M_2$. $\square$

Remark 3.5 *Let L/K be a finite purely inseparable extension such that L is Hilbertian. Then K is Hilbertian. Indeed, assuming* $\mathrm{char(K)} > 0$*, there exists a power q of* $\mathrm{char(K)}$ *such that $L^q \subseteq K$. Then $L^q \cong L$ is Hilbertian and K is a purely inseparable extension of L^q. Hence, K is Hilbertian.*

Now we can treat the case of an arbitrary algebraic function field.

Proposition 3.6 *Let K/F be a finitely generated extension of fields with* $\mathrm{tr.deg}(K/F) = 1$. *Then K is Kuykian.*

Proof. We consider an abelian variety A/K and an intermediate field M of $K(A_{\mathrm{tor}})/K$ and prove that M is Hilbertian.

First we prove the claim under the assumption that K/F is separable. Let $K(A_{\mathrm{tor}})_0 = K(A_{\mathrm{tor}}) \cap K_s$ and $M_0 = M \cap K_s = M \cap K(A_{\mathrm{tor}})_0$. Then M/M_0 is a purely inseparable extension. If we prove that M_0 is Hilbertian, it will follow that M is Hilbertian. Replacing M by M_0, if necessary, we may assume that M/K is separable.

Let F' be the algebraic closure of F in M. Then KF' is a finitely generated transcendental extension of F', hence KF' is Hilbertian [2, Theorem 13.4.2].

$$\begin{tikzcd}
 & K(A_{\mathrm{tor}})_0 \arrow[r, dash] \arrow[d, dash] & K(A_{\mathrm{tor}})_0\tilde{F} \arrow[d, dash] \\
 & M \arrow[r, dash] \arrow[d, dash] & M\tilde{F} \arrow[d, dash] \\
K \arrow[r, dash] \arrow[d, dash] & KF' \arrow[r, dash] \arrow[d, dash] & K\tilde{F} \arrow[d, dash] \\
F \arrow[r, dash] & F' \arrow[r, dash] & \tilde{F}
\end{tikzcd}$$

Moreover, M/KF' is separable, hence M/F' is separable. By definition, F' is algebraically closed in M, so M/F' is regular. Since $F' \subseteq KF' \subseteq M$ and $KF' \cdot \tilde{F} = K\tilde{F}$, [2, Lemma 2.5.3] implies that M and $K\tilde{F}$ are linearly disjoint over KF'. Hence, by Lemma 3.3 and Lemma 3.4 it suffices to show that $M\tilde{F}/K\tilde{F}$ is small. Indeed, $M\tilde{F}$ is an intermediate field of $K(A_{\mathrm{tor}})_0\tilde{F}/K\tilde{F}$. By Proposition 3.1, $K(A_{\mathrm{tor}})_0\tilde{F}/K\tilde{F}$ is a Galois extension with a finitely generated Galois group, hence $M\tilde{F}/K\tilde{F}$ is small as desired.

In the general case F has a finite purely inseparable extension F_1 such that $K_1 = KF_1$ is a separable extension of F_1. By the separable case $M_1 = MK_1$ is Hilbertian. In addition, M_1 is a finite purely inseparable extension of M. Hence, by Remark 3.5, M is Hilbertian. $\square$

Remark 3.7 *A field K is said to be* fully Hilbertian *if for every absolutely irreducible polynomial $f \in K[T, X]$ which is separable in X there exist* $\mathrm{card}(K)$ *many $a \in K$ and b_a a root of $f(a, X)$ such that $f(a, X)$ is irreducible in $K[X]$ and the $K(b_a)$ are linearly disjoint over K. Using results of [1], we could adjust the proof of Proposition 3.6 to prove, in the notation used there, that M is not only Hilbertian but even fully Hilbertian. However, since our Main Theorem* *deals with finitely generated, hence countable fields, and for countable fields the notions Hilbertian and fully Hilbertian coincide, we refrain from doing so.*

Everything is now set to prove our Main Theorem.

Theorem 3.8 *Every infinite finitely generated field K is Kuykian.*

Proof. The case where K is a number field is covered by [6, Theorem 10]. In all other cases K is finitely generated and transcendental over its prime field, hence over some subfield F with $\mathrm{tr.deg}(K/F) = 1$. Thus, K is Kuykian by Proposition 3.6. □

4 Non finitely generated Hilbertian fields

By Theorem 3.8, every infinite finitely generated field is Kuykian. By Proposition 3.6, every finitely generated transcendental extension of an arbitrary field is Kuykian. In this section we supply more evidence to the Kuykian Conjecture by giving a variety of examples of non finitely generated Hilbertian fields that are Kuykian.

Let N/K be a field extension and A/N an abelian variety. We say that A *descends to* K if there exists an abelian variety B/K such that $B_N \cong A$. In this case, $A(N') = B(N')$ for every extension N' of N, and by abuse of language we will treat A as an abelian variety over K.

Lemma 4.1 *Every finite extension L of a Kuykian field K is Kuykian.*

Proof. Let B/L be an abelian variety and M an extension of L in $L(B_{\mathrm{tor}})$. We have to prove that M is Hilbertian. It suffices to prove this statement under the assumption that L/K is separable or purely inseparable.

Case A: L/K is separable. Applying restriction of scalars to B yields an abelian variety $A = \mathrm{res}_{L/K} B$ over K and an epimorphism $\lambda : A_L \to B$, see [11, Section 1.3] or [7, §1]. If we let C be an abelian subvariety of A_L such that $\lambda|_C : C \to B$ is an isogeny, [8, Proposition 12.1], we see that $\lambda(A_{\mathrm{tor}}) \supseteq \lambda(C_{\mathrm{tor}}) = B_{\mathrm{tor}}$. Consequently, $L(B_{\mathrm{tor}}) \subseteq L(A_{\mathrm{tor}})$.

Let $L_0 = K(A_{\mathrm{tor}}) \cap L$. Then $K(A_{\mathrm{tor}})$ and L are linearly disjoint over L_0 and $K(A_{\mathrm{tor}})L = L(A_{\mathrm{tor}})$. Hence, $M_0 = K(A_{\mathrm{tor}}) \cap M$ satisfies $M_0L = M$. By our assumption on K, M_0 is Hilbertian. Also, M/M_0 is finite, hence M is Hilbertian.

Case B: L/K is purely inseparable. Assuming $\mathrm{char}(K) > 0$, there exists a power q of $\mathrm{char}(K)$ such that $L^q \subseteq K$. Then the q-Frobenius on L transforms B/L onto an abelian variety A/L^q that satisfies $L^q \subseteq M^q \subseteq L^q(A_{\mathrm{tor}})$. Hence, $K \subseteq M^qK \subseteq K(A_{\mathrm{tor}})$. By our assumption on K, M^qK is Hilbertian. Since M/M^qK is purely inseparable, M is also Hilbertian. □

Lemma 4.2 *If K is a Kuykian field and A/K is an abelian variety, then $K(A_{\mathrm{tor}})$ is also Kuykian.*

Proof. Let $N = K(A_{\mathrm{tor}})$ and let B/N be an abelian variety. Then N/K has a finite subextension K'/K such that B descends to K'. It follows that $N(B_{\mathrm{tor}}) = K'((A_{K'} \times B)_{\mathrm{tor}})$. By Lemma 4.1, K' is Kuykian. Therefore, every intermediate field of $N(B_{\mathrm{tor}})/N$ is Hilbertian. □

Lemma 4.3 *Let K be a field and let L_1, L_2 be algebraic extensions of K. Then $L_1L_2 \cap K_s = (L_1 \cap K_s)(L_2 \cap K_s)$.*

Proof. Our statement follows from the observation that $(L_1 \cap K_s)(L_2 \cap K_s)/K$ is a separable extension and $L_1L_2/(L_1 \cap K_s)(L_2 \cap K_s)$ is a purely inseparable extension. □

Proposition 4.4 *Let L_1, L_2 be normal algebraic extensions of a Hilbertian field K and let M be a separable extension of K in L_1L_2. Suppose $M \not\subseteq L_1$ and $M \not\subseteq L_2$. Then M is Hilbertian.*

Proof. We set $L = L_1L_2$, $L_0 = L \cap K_s$, $L_{1,0} = L_1 \cap K_s$, and $L_{2,0} = L_2 \cap K_s$. Then $L_{1,0}$ and $L_{2,0}$ are Galois extensions of K. By Lemma 4.3, $L_{1,0}L_{2,0} = L_0$. In addition, M is contained in L_0 but neither in $L_{1,0}$ nor in $L_{2,0}$. Hence, by Haran's diamond theorem [2, Theorem 3.8.3], M is Hilbertian. □

Proposition 4.5 *Let N be a normal algebraic extension of a Kuykian field K and suppose that N is Hilbertian. Then N is Kuykian.*

Proof. Let A/N an abelian variety and M an intermediate field of $N(A_{\mathrm{tor}})/N$. Then K has a finite extension K' such that A descends to K'. By Lemma 4.1, K' is Kuykian. Hence, we may assume that $K' = K$.

Let $M_0 = M \cap K_s$. If $M_0 \subseteq N$, then $M_0 = N$ and M_0 is Hilbertian. If $M_0 \subseteq K(A_{\mathrm{tor}})$, then M_0 is Hilbertian, because K is Kuykian. Finally, if $M_0 \not\subseteq N$ and $M_0 \not\subseteq K(A_{\mathrm{tor}})$, then M_0 is Hilbertian, by Proposition 4.4.

Finally, M/M_0 is a purely inseparable extension, so M is Hilbertian. □

Remark 4.6 *Let K be a number field or a finitely generated transcendental extension of an arbitrary field. By Proposition 3.6 and Theorem 3.8, K is Kuykian. Hence, by Proposition 4.5, every Galois extension N of K with N Hilbertian, is Kuykian. By the result of Kuyk mentioned in the introduction, every abelian extension of K is Hilbertian. Hence, it is also Kuykian. In particular, $\mathbb{Q}_{\mathrm{ab}}$, the maximal abelian Galois extension of $\mathbb{Q}$, is a Kuykian field.*

If K is a field and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_e) \in \mathrm{Gal}(K)^e$, then $K_s[\boldsymbol{\sigma}]$ denotes the maximal Galois extension of K in the fixed field

$$K_s(\boldsymbol{\sigma}) = \{x \in K_s \colon \sigma_i(x) = x,\ i = 1, \ldots, e\}.$$

Corollary 4.7 *Let K be a countable Kuykian field and $e \geq 1$. Then $K_s[\boldsymbol{\sigma}]$ is Kuykian for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ (in the sense of the Haar measure).*

Proof. Since K is Kuykian, it is Hilbertian. Hence $K_s[\boldsymbol{\sigma}]$ is Hilbertian for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ by [5, Theorem 2.7]. Moreover, by definition, $K_s[\boldsymbol{\sigma}]$ is a Galois extension of K. Hence, by Proposition 4.5, $K_s[\boldsymbol{\sigma}]$ is also Kuykian. □

We denote the maximal purely inseparable extension of a field K by K_{ins}.

Proposition 4.8 *Let K be the quotient field of a complete local domain R of dimension at least 2. Then the field K_{ins} is Kuykian.*

Proof. By [1, Theorem 1.7], K_{ins} is fully Hilbertian. Let A/K_{ins} be an Abelian variety and let M an intermediate field of $K_{\mathrm{ins}}(A_{\mathrm{tor}})/K_{\mathrm{ins}}$. Then $K_{\mathrm{ins}}(A_{\mathrm{tor}})$ is obtained from K_{ins} by adjoining the countably many points of A_{tor}. Hence, K_{ins} has only countably many finite extensions in $K_{\mathrm{ins}}(A_{\mathrm{tor}})$. Therefore, K_{ins} has at most countably many finite extensions in M.

Let $\mathfrak{m}$ be the maximal ideal of R. Our assumption on R to be complete means in particular that $(R, \mathfrak{m})$ is Hausdorff. In particular $\mathfrak{m}^2 \neq \mathfrak{m}$. We choose $t \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$. Then the power series $\sum_{i=0}^{\infty} \epsilon_i t^i$ with $\epsilon_i \in \{0, 1\}$ converge to distinct elements of R. Thus, $\mathrm{card}(K_{\mathrm{ins}}) \geq \mathrm{card}(R) \geq 2^{\aleph_0} > \aleph_0$. Hence, by [1, Theorem 4.6(c)], M is Hilbertian. It follows that K_{ins} is Kuykian. □

Example 4.9 *Each of the rings $K_0[[X_1, \ldots, X_n]]$ with K_0 an arbitrary field and $n \geq 2$ satisfies the assumption of Proposition 4.8, hence $K_0((X_1, \ldots, X_n))_{\mathrm{ins}}$ is Kuykian.*

Remark 4.10 *The conclusion of Proposition 4.8 is somewhat annoying. We would rather claim that K itself is Kuykian rather than K_{ins}. This would follow if we knew that a field M is Hilbertian once M_{ins} is Hilbertian.*

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