P-ADICALLY PROJECTIVE GROUPS AS ABSOLUTE GALOIS GROUPS*

by

Dan Haran
School of Mathematics, Tel Aviv University
Ramat Aviv, Tel Aviv 69978, Israel
e-mail: haran@post.tau.ac.il

and

Moshe Jarden
School of Mathematics, Tel Aviv University
Ramat Aviv, Tel Aviv 69978, Israel
e-mail: jarden@post.tau.ac.il

and

Florian Pop
Department of Mathematics, University of Pennsylvania
Philadelphia, PA 19104-6395, USA
e-mail: pop@math.upenn.edu

MR Classification: 12E30

Directory: \Jarden\Works\Paper80

2 June, 2006

* Research supported by the Minkowski Center for Geometry at Tel Aviv University, established by the Minerva Foundation and by the European Community's Human Potential Programme under Contract HPRN-CT-2000-00114, GTEM

Table of Contents

Introduction

We address one of the major problems of Galois theory: the characterization of absolute Galois groups among all profinite groups. Specifically, we consider a profinite group G equipped with a subset $\mathcal{G}$ of subgroups each of which is isomorphic to an absolute Galois group. The problem is to characterize those pairs for which G is isomorphic to an absolute Galois group $\mathrm{Gal}(K)$ of a field K that satisfies a local-global principle for points on smooth varieties with respect to the fixed fields of the groups in $\mathcal{G}$.

In [HJPa] we extend the pairs $(G, \mathcal{G})$ to, so called, **group structures** of a general nature. We prove that a proper profinite group structure $\mathbf{G}$ is projective if and only if $\mathbf{G}$ is the absolute Galois group structure of a proper field-valuation structure with block approximation.

The introduction to [HJPa] contains an extensive historical background to the subject.

In this work we apply the general setup of [HJPa] to a classical situation. Let $\mathcal{F}$ be a finite set of classical local fields of characteristic 0. Thus, each $\mathbb{F} \in \mathcal{F}$ is either the field $\mathbb{R}$ of real numbers or a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers for some prime number p. We assume $\mathcal{F}$ is **closed under Galois isomorphism**: if $\mathbb{F}$ and $\mathbb{F}'$ are finite extensions of $\mathbb{Q}_p$, $\mathbb{F} \in \mathcal{F}$, and $\mathrm{Gal}(\mathbb{F}') \cong \mathrm{Gal}(\mathbb{F})$, then $\mathbb{F}' \in \mathcal{F}$.

MAIN THEOREM: *Let $\mathcal{F}$ be a finite set of classical local fields of characteristic* 0 *which is closed under Galois isomorphism and let G be a profinite group. Then G is isomorphic to the absolute Galois group of a P$\mathcal{F}$C field K if and only if G is $\mathcal{F}$-projective and* $\mathrm{Subgr}(G, \mathrm{Gal}(\mathbb{F}))$ *is strictly closed in* $\mathrm{Subgr}(G)$ *for each $\mathbb{F} \in \mathcal{F}$.*

The notions appearing in the Main Theorem: For each $\mathbb{F} \in \mathcal{F}$ let $\mathrm{AlgExt}(K, \mathbb{F})$ be the set of all algebraic extensions F of K (within a fixed algebraic closure $\tilde{K}$ of K) which are elementarily equivalent to $\mathbb{F}$. We say that K is P$\mathcal{F}$C (**pseudo-$\mathcal{F}$-closed**) if it satisfies the following local-global principle: Let V be a smooth absolutely irreducible variety over K. Suppose $V(F) \neq \emptyset$ for each $F \in \bigcup_{\mathbb{F} \in \mathcal{F}} \mathrm{AlgExt}(K, \mathbb{F})$. Then $V(K) \neq \emptyset$.

The notation $\mathrm{Subgr}(G)$ stands for the space of all closed subgroups of G. $\mathrm{Subgr}(G)$ is the inverse limit of the discrete finite spaces $\mathrm{Subgr}(G/N)$, where N ranges over all open normal subgroups of G. Thus, $\mathrm{Subgr}(G)$ is a profinite space. We refer to its topology as **strict** and write $\mathrm{Subgr}(G, \mathrm{Gal}(\mathbb{F})) = \{\Gamma \in \mathrm{Subgr}(G) \mid \Gamma \cong \mathrm{Gal}(\mathbb{F})\}$.

Finally, G is **$\mathcal{F}$-projective** if it satisfies the following local-global principle: Let $\alpha\colon B \to A$ be an epimorphism of finite groups and $\varphi\colon G \to A$ a homomorphism. Suppose for each $\mathbb{F} \in \mathcal{F}$ and each $\Gamma \in \mathrm{Subgr}(G, \mathrm{Gal}(\mathbb{F}))$ there is a homomorphism $\gamma_\Gamma\colon \Gamma \to B$ satisfying $\alpha \circ \gamma_\Gamma = \varphi|_\Gamma$. Then there is a homomorphism $\gamma\colon G \to B$ with $\alpha \circ \gamma = \varphi$. ∎

The Main Theorem generalizes several known special cases:

1. First suppose $\mathcal{F}$ is the empty set. Then a P$\mathcal{F}$C field is a PAC field [FrJ, Chap. 10] and an $\mathcal{F}$-projective group is a projective group. The Main Theorem is due in this case to Lubotzky-v.d.Dries [FrJ, Cor. 20.16] and Ax [FrJ, Thm. 10.17].

2. If $\mathcal{F} = \{\mathbb{R}\}$, then a P$\mathcal{F}$C field is a PRC field and an $\mathcal{F}$-projective group is a real projective group G such that the set of all involutions of G is closed. See [HaJ1, p. 450, Thm.] for the Main Theorem in this case.

3. If $\mathcal{F} = \{\mathbb{Q}_p\}$, then a P$\mathcal{F}$C field is a P$p$C field and an $\mathcal{F}$-projective group is a p-adically projective G such that the set of all subgroups of G which are isomorphic to $\mathrm{Gal}(\mathbb{Q}_p)$ is strictly closed in $\mathrm{Subgr}(G)$. See [HaJ2, p. 148, Thm.] for the Main Theorem in this case.

4. The general case is announced in [Pop2]. Unfortunately, the proofs of [Pop2] are extremely sketchy and difficult to read. The current work is based on ideas of [Pop2] and applies results of [HJPa].

5. Koengismann [Koe2] proves the Main Theorem in case that $\mathrm{AlgExt}(K, \mathbb{F})$ is finite. Ershov ([Er1], [Er2]) proves that $\mathrm{Gal}(K)$ is $\mathcal{F}$-projective (in a stronger sense) when $(K, \mathcal{F})$ is a multi-valued P$\mathcal{F}$C field satisfying certain conditions.

None of those papers goes as far as we do in this work and equips K in the Main Theorem with a set of valuations satisfying the "block approximation theorem". We refer the reader to Section 10, in particular to Theorem 10.3, for the exact result. Here it suffices to say that each minimal $F \in \mathrm{AlgExt}(K, \mathcal{F})$ is the Henselian closure of K at a valuation v_F and the family $\{v_F \mid F \in \mathrm{AlgExt}(K, \mathcal{F})\}$ satisfies a very strong independence-density property.

In the rest of the introduction we describe the structure of the proof of the Main Theorem and its stronger versions (Theorems 9.4 and 10.3) which we actually prove.

Denote the set of all separable algebraic extensions of a field K (within a fixed separable closure K_s of K) by $\mathrm{AlgExt}(K)$. The map $F \mapsto \mathrm{Gal}(F)$ is a bijection of $\mathrm{AlgExt}(K)$ onto $\mathrm{Subgr}(\mathrm{Gal}(K))$. It transfers the strict topology of $\mathrm{Subgr}(\mathrm{Gal}(K))$ to the **strict topology** of $\mathrm{AlgExt}(K)$. We prove that $\mathrm{AlgExt}(K, \mathbb{F})$ is strictly closed in $\mathrm{AlgExt}(K)$ for each $\mathbb{F} \in \mathcal{F}$. In particular, this holds if K is P$\mathcal{F}$C. In this case the existence of points on smooth varieties over K translates into $\mathcal{F}$-projectivity.

Conversely, let G be a profinite group. Suppose G is $\mathcal{F}$-projective and $\mathrm{Subgr}(G, \mathrm{Gal}(\mathbb{F}))$ is strictly closed in $\mathrm{Subgr}(G)$ for each $\mathbb{F} \in \mathcal{F}$. Put $\mathcal{C} = \{\mathrm{Gal}(\mathbb{F}) \mid \mathbb{F} \in \mathcal{F}\}$. Let $\mathcal{G} = \mathrm{Subgr}(G, \mathcal{C})$ be the set of all $H \in \mathrm{Subgr}(G)$ which are isomorphic to some Γ in $\mathcal{C}$. Denote the set of all maximal elements in $\mathcal{G}$ by $\mathcal{G}_{\max}$. For each $\Gamma \in \mathcal{C}$ we construct a finite quotient $\bar{\Gamma}$ such that the set $\{\bar{\Gamma} \mid \Gamma \in \mathcal{C}\}$ is a "system of big quotients" of $\mathcal{C}$ in a sense made precise in Section 6. We use it to prove that G is "strongly $\mathcal{G}$-projective" (Proposition 6.5). Thus, every $\mathcal{G}$-embedding problem for G which is locally solvable is globally solvable (Section 6). In particular, there is a homomorphism κ of G into the free product $B^* = \mathop{\ast\!\!\!\prod}_{\Gamma \in \mathcal{C}} \Gamma$ which maps each $H \in \mathcal{G}$ isomorphically into a conjugate of some $\Gamma \in \mathcal{C}$. By a theorem of Geyer we may identify B^* with the absolute Galois group of an algebraic extension K_0 of $\mathbb{Q}$. Denote the fixed field of $\kappa(G)$ in $\tilde{\mathbb{Q}}$ by K_1. By Proposition 6.5, $\mathbf{G} = (G, \mathcal{G}_{\max})$ is a "proper projective group structure" (Section 5). Let $\mathrm{Gal}(\mathbf{K}_1) = (\mathrm{Gal}(K_1), \mathrm{Subgr}(\mathrm{Gal}(K_1, \mathcal{C}))$. Then κ extends to a Galois cover κ: $\mathbf{G} \to \mathrm{Gal}(\mathbf{K}_1)$ of group structures (Proof of Theorem 10.3). By the main result of [HJPa], there is a field K and an isomorphism φ: $G \to \mathrm{Gal}(K)$ such that $\mathrm{res} \circ \varphi = \kappa$. Moreover, every $F \in \mathrm{AlgExt}(K, \mathcal{F})$ is either real closed or elementarily equivalent to some $\mathbb{F} \in \mathcal{F}$. In particular, F has a "P-adic valuation" v_F. The system of fields $F \in \mathrm{AlgExt}(K, \mathcal{F})$ and valuations v_F satisfies a strong version of the weak approximation theorem which we call the "block approximation theorem". In particular, K is P$\mathcal{F}$C (Theorem 10.3).

1. The Étale and the Strict Topologies of $\mathrm{Subgr}(G)$

Let G be a profinite group. Denote the collection of all closed (resp. open, open normal) subgroups of G by $\mathrm{Subgr}(G)$ (resp. $\mathrm{Open}(G)$, $\mathrm{OpenNormal}(G)$). We introduce two topologies on $\mathrm{Subgr}(G)$ and relate them to each other.

For each $H, N \in \mathrm{Open}(G)$ with $N \triangleleft G$ let

$$\mathcal{V}(H,N) = \{A \in \mathrm{Subgr}(G) \mid AN = HN\}.$$

The collection of all $\mathcal{V}(H,N)$ is a basis for a topology on $\mathrm{Subgr}(G)$ which we call the **strict topology**. When G is finite, the strict topology is the discrete topology. In general, $\mathrm{Subgr}(G) \cong \varprojlim \mathrm{Subgr}(G/N)$ with N ranging over all open normal subgroups of G. Thus, $\mathrm{Subgr}(G)$ is a profinite space under the strict topology. We use the adverb "strictly" as a replacement for "in the strict topology". For example, for a subset $\mathcal{G}$ of $\mathrm{Subgr}(G)$ we say $\mathcal{G}$ is **strictly open** (resp. **closed**, **compact**, **Hausdorff**) if it is open (resp. closed, compact, Hausdorff) in the strict topology. Likewise, for a function f from a topological space X into $\mathrm{Subgr}(G)$, we say f is **strictly continuous** if f is continuous when $\mathrm{Subgr}(G)$ is equipped with the strict topology. We denote the strict closure of a subset $\mathcal{G}$ of $\mathrm{Subgr}(G)$ by $\mathrm{StrictClosure}(\mathcal{G})$.

If $U_1, \ldots, U_m \in \mathrm{Open}(G)$, then $U = \bigcap_{i=1}^m U_i$ is open and $\mathrm{Subgr}(U)$ is $\bigcap_{i=1}^m \mathrm{Subgr}(U_i)$. Therefore $\{\mathrm{Subgr}(U) \mid U \in \mathrm{Open}(G)\}$ is a basis for a topology on $\mathrm{Subgr}(G)$ which we call the **étale topology**. As above, for a subset $\mathcal{G}$ of $\mathrm{Subgr}(G)$ we say $\mathcal{G}$ is **étale open** (**closed**, **compact**, **Hausdorff**, etc) if $\mathcal{G}$ is open (closed, compact, Hausdorff, etc) in the étale topology. Likewise, for a function f from a topological space X into $\mathrm{Subgr}(G)$ we say f is **étale continuous** if f is continuous when $\mathrm{Subgr}(G)$ is equipped with the étale topology. Note that the étale topology of $\mathrm{Subgr}(G)$ is weaker than the strict topology. Thus, every étale open subset of $\mathrm{Subgr}(G)$ is also strictly open [HJPa, Remark 1.2].

The **envelope** of a subset $\mathcal{G}$ of $\mathrm{Subgr}(G)$ is the set of all $H_0 \in \mathrm{Subgr}(G)$ which are contained in some $H \in \mathcal{G}$. We denote it by $\mathrm{Env}(\mathcal{G})$ and use it to relate the strict topology and the étale topology of $\mathrm{Subgr}(G)$ to each other:

Lemma 1.1: *A subset $\mathcal{G}$ of $\mathrm{Subgr}(G)$ is étale compact if and only if $\mathrm{Env}(\mathcal{G})$ is strictly closed.*

Proof: Suppose first $\mathrm{Env}(\mathcal{G})$ is strictly closed, hence strictly compact. Let U_i, $i \in I$, be open subgroups of G with $\mathcal{G} \subseteq \bigcup_{i\in I} \mathrm{Subgr}(U_i)$. Then $\mathrm{Env}(\mathcal{G}) \subseteq \bigcup_{i\in I} \mathrm{Subgr}(U_i)$. Since each of the sets $\mathrm{Subgr}(U_i)$ is strictly open, I has a finite subset I_0 with $\mathrm{Env}(\mathcal{G}) \subseteq \bigcup_{i\in I_0} \mathrm{Subgr}(U_i)$. Thus, $\mathcal{G} \subseteq \bigcup_{i\in I_0} \mathrm{Subgr}(U_i)$. Therefore, $\mathcal{G}$ is étale compact.

Conversely, suppose $\mathcal{G}$ is étale compact. Consider $A \in \mathrm{Subgr}(G) \smallsetminus \mathrm{Env}(\mathcal{G})$ and $H \in \mathcal{G}$. Then $A \not\leq H$. Hence, there is $N_H \in \mathrm{OpenNormal}(G)$ with $A \not\leq HN_H$. Thus, A is not in the étale open neighborhood $\mathrm{Subgr}(HN_H)$ of H.

The collection of all $\mathrm{Subgr}(HN_H)$ covers $\mathcal{G}$. Since $\mathcal{G}$ is étale compact, there are $H_1, \ldots, H_n \in \mathcal{G}$ and $N_1, \ldots, N_n \in \mathrm{OpenNormal}(G)$ with $\mathcal{G} \subseteq \bigcup_{i=1}^n \mathrm{Subgr}(H_iN_i)$ and $A \notin \bigcup_{i=1}^n \mathrm{Subgr}(H_iN_i)$. In addition, $\bigcup_{i=1}^n \mathrm{Subgr}(H_iN_i)$ is strictly closed. Hence, $\mathrm{StrictClosure}(\mathcal{G}) \subseteq \bigcup_{i=1}^n \mathrm{Subgr}(H_iN_i)$. Thus, A belongs to the strictly open set

$\mathrm{Subgr}(G) \smallsetminus \bigcup_{i=1}^n \mathrm{Subgr}(H_iN_i)$ which is disjoint from $\mathrm{Env}(\mathcal{G})$. Therefore, A is not in $\mathrm{StrictClosure}(\mathrm{Env}(\mathcal{G}))$. It follows that $\mathrm{Env}(\mathcal{G})$ is strictly closed. ∎

COROLLARY 1.2: *Let* $\mathcal{G}$ *be an étale compact subset of* $\mathrm{Subgr}(G)$. *Then* $\mathrm{StrictClosure}(\mathcal{G})$ *is contained in* $\mathrm{Env}(G)$.

For a profinite group G, a closed subgroup H, and a subset $\mathcal{G}$ of $\mathrm{Subgr}(G)$ let $\mathcal{G}^H = \{\Gamma^h \mid \Gamma \in \mathcal{G},\ h \in H\}$. Put $\mathrm{Con}(\mathcal{G}) = \mathrm{Env}(\mathcal{G}^G) = \mathrm{Env}(\mathcal{G})^G$.

LEMMA 1.3: *Let* $\mathcal{G}$ *be an étale compact subset of* $\mathrm{Subgr}(G)$. *Then each of the sets* $\mathcal{G}^G$, $\mathrm{Env}(\mathcal{G})$, *and* $\mathrm{Con}(\mathcal{G})$ *is étale compact.*

Proof: The set $\mathcal{G}^G$ is the image of the compact space $\mathcal{G} \times G$ under the étale continuous map $(\Gamma, g) \mapsto \Gamma^g$. Hence, $\mathcal{G}^G$ is étale compact.

By Lemma 1.1, $\mathrm{Env}(\mathcal{G})$ is strictly closed, hence étale compact [HJPa, Remark 1.2]. Therefore, by the first paragraph, $\mathrm{Con}(\mathcal{G}) = \mathrm{Env}(\mathcal{G})^G$ is étale compact. ∎

LEMMA 1.4: *Let* H *be a closed subgroup of* G. *Then* $\text{ÉtaleClosure}(\{H\}) = \{B \in \mathrm{Subgr}(G) \mid H \le B\}$.

Proof: First suppose $B \in \text{ÉtaleClosure}(\{H\})$. Then H belongs to each étale open neighborhood of B. In other words, if $U \in \mathrm{Open}(G)$ and $B \le U$, then $H \le U$. Hence, $H \le B$.

Conversely, suppose $H \le B$. Then, H belongs to each basic étale open neighborhood $\mathrm{Subgr}(U)$ of B. Therefore, $B \in \text{ÉtaleClosure}(\{H\})$. ∎

LEMMA 1.5: *Let* $\varphi\colon G \to H$ *be an epimorphism of profinite groups and* G_0 *a closed subgroup of* G. *The set* $\{B \in \mathrm{Subgr}(G) \mid \varphi(G_0) \le \varphi(B)\}$ *is étale closed.*

Proof: By [HJPa, Remark 1.1(b)], the map $\varphi\colon \mathrm{Subgr}(G) \to \mathrm{Subgr}(H)$ induced by φ is étale continuous. By Lemma 1.4, the set $\{C \in \mathrm{Subgr}(H) \mid \varphi(G_0) \le C\}$ is étale closed. Its inverse image in $\mathrm{Subgr}(G)$ is $\{B \in \mathrm{Subgr}(G) \mid \varphi(G_0) \le \varphi(B)\}$, so it is étale closed. ∎

LEMMA 1.6: *Let* $\mathcal{G}$ *be an étale compact subset of* $\mathrm{Subgr}(G)$. *Then each* $A \in \mathcal{G}$ *is contained in a maximal element of* $\mathcal{G}$.

Proof: By Zorn's lemma, it suffices to prove that each ascending chain $\mathcal{G}_0$ in $\mathcal{G}$ is bounded by an element of $\mathcal{G}$. Consider $B_1, \cdots, B_n \in \mathcal{G}_0$. Then the B_i are comparable. Assume $B_1 \le B_2 \le \ldots \le B_n$. By Lemma 1.4, $B_n \in \bigcap_{i=1}^m \text{ÉtaleClosure}(\{B_i\})$. Thus, $\mathcal{G} \cap \bigcap_{i=1}^n \text{ÉtaleClosure}(\{B_i\}) \neq \emptyset$. Since $\mathcal{G}$ is étale compact, $\mathcal{G} \cap \bigcap_{B \in \mathcal{G}_0} \text{ÉtaleClosure}(\{B\})$ is nonempty. Each element of the latter set is a bound of $\mathcal{G}_0$. ∎

If a subset $\mathcal{G}$ of $\mathrm{Subgr}(G)$ contains groups A and B with $A < B$, then $\mathcal{G}$ is not étale Hausdorff. Thus, removing all nonmaximal elements from $\mathcal{G}$ is the only way to make $\mathcal{G}$ étale Hausdorff while preserving the essential information stored in $\mathcal{G}$. We denote the set of all maximal elements of $\mathcal{G}$ by $\mathcal{G}_{\max}$.

Lemma 1.7: *Let $\mathcal{G}$ be an étale compact subset of* $\mathrm{Subgr}(G)$*. Then $\mathcal{G}_{\max}$ is étale compact.*

Proof: By Lemma 1.6, $\mathrm{Env}(\mathcal{G}) = \mathrm{Env}(\mathcal{G}_{\max})$. Hence, by Lemma 1.1, $\mathcal{G}_{\max}$ is étale compact. ∎

2. Relatively Projective Groups

Pairs $(G, \mathcal{G})$ consisting of a profinite group and a subset $\mathcal{G}$ of $\mathrm{Subgr}(G)$ which satisfy a local global principle for finite embedding problems naturally arise from pairs $(K, \mathcal{X})$ consisting of a field K and a set $\mathcal{X}$ of separable algebraic extensions of K which satisfy a local global principle for points on absolutely irreducible varieties (Section 3). We prove in Proposition 3.1 that G is "$\mathcal{G}$-projective" in a sense we now explain:

Let G be a profinite group and $\mathcal{G}$ a subset of $\mathrm{Subgr}(G)$. An **embedding problem** for G is a pair

$$(1) \qquad (\varphi\colon G \to A,\ \alpha\colon B \to A),$$

where φ is a homomorphism and α is an epimorphism of profinite groups. The embedding problem is **finite** if A and B are finite. We call (1) a **$\mathcal{G}$-embedding problem** if it is **locally solvable**; that is

(2) for each $\Gamma \in \mathcal{G}$ there exists a homomorphism $\gamma_\Gamma\colon \Gamma \to B$ with $\alpha \circ \gamma_\Gamma = \varphi|_\Gamma$.

We say (1) is a **rigid $\mathcal{G}$-embedding problem** if

(3) for each $\Gamma \in \mathcal{G}$ there is $B_0 \in \mathrm{Subgr}(B)$ such that $\alpha\colon B_0 \to \varphi(\Gamma)$ is an isomorphism.

A **solution** of (1) is a homomorphism $\gamma\colon G \to B$ with $\gamma \circ \alpha = \varphi$. We say G is **$\mathcal{G}$-projective** if every finite $\mathcal{G}$-embedding problem for $\mathcal{G}$ is solvable. Our definition generalizes the definition of a "projective group". Indeed, "G is $\emptyset$-projective" means "G is projective". We refer to G as **relatively projective** if G is $\mathcal{G}$-projective for a subset $\mathcal{G}$ of $\mathrm{Subgr}(G)$.

If $\mathcal{G} \subseteq \mathcal{G}' \subseteq \mathrm{Subgr}(G)$ and G is $\mathcal{G}$-projective, then G is $\mathcal{G}'$-projective. Moreover, if $\mathcal{G}$ is étale compact, then by Lemma 1.6, each group in $\mathcal{G}$ is contained in a group of $\mathcal{G}_{\max}$. Hence, under the assumption that $\mathcal{G}$ is étale compact, G is $\mathcal{G}$-projective if and only if G is $\mathcal{G}_{\max}$-projective. Hence, G is $\mathcal{G}$-projective if and only if G is $\mathcal{G}^G$-projective.

Suppose (1) is a rigid $\mathcal{G}$-embedding problem and Γ and B_0 are as in (3). Put $A_0 = \varphi(\Gamma)$ and $\gamma_\Gamma = (\alpha|_{A_0})^{-1} \circ \varphi|_\Gamma$. Then $\gamma_\Gamma\colon \Gamma \to B$ is a homomorphism satisfying $\alpha \circ \gamma_\Gamma = \varphi|_\Gamma$. Thus, every rigid $\mathcal{G}$-embedding problem is a $\mathcal{G}$-embedding problem. The next lemma establishes a sort of converse to this statement:

Lemma 2.1: *Let G be a profinite group and $\mathcal{G}$ an étale compact subset of* $\mathrm{Subgr}(G)$*. Let (1) be a finite $\mathcal{G}$-embedding problem for G. Then:*

(a) *There exists a commutative diagram*

$$\begin{tikzcd} & G \arrow[d, "\hat{\varphi}"'] \arrow[dd, bend left, "\varphi"] \\ \hat{B} \arrow[r, "\hat{\alpha}"] \arrow[d, "\beta"'] & \hat{A} \arrow[d, "\bar{\varphi}"'] \\ B \arrow[r, "\alpha"] & A \end{tikzcd}$$

in which $\hat{\varphi}$ is an epimorphism and ($\hat{\varphi}$: $G \to \hat{A}$, $\hat{\alpha}$: $\hat{B} \to \hat{A}$) is a finite rigid $\mathcal{G}$-embedding problem.

(b) *If every finite rigid $\mathcal{G}$-embedding problem (1) for G in which φ is an epimorphism is solvable, then G is $\mathcal{G}$-projective.*

Proof of (a): Consider $\Gamma \in \mathcal{G}$. Choose a homomorphism γ_Γ with $\alpha \circ \gamma_\Gamma = \varphi|_\Gamma$. Then $\mathrm{Ker}(\gamma_\Gamma)$ is an open subgroup of Γ. Choose $N_\Gamma \in \mathrm{OpenNormal}(G)$ with $N_\Gamma \le \mathrm{Ker}(\varphi)$ and $\Gamma \cap N_\Gamma \le \mathrm{Ker}(\gamma_\Gamma)$. Then $\mathrm{Subgr}(\Gamma N_\Gamma)$ is an étale open neighborhood of Γ in $\mathrm{Subgr}(G)$ and γ_Γ extends to a homomorphism γ'_Γ: $\Gamma N_\Gamma \to B$ with kernel $\mathrm{Ker}(\gamma_\Gamma)N_\Gamma$.

Since $\mathcal{G}$ is étale compact, there are $\Gamma_1, \ldots, \Gamma_m \in \mathcal{G}$ with $\mathcal{G} \subseteq \bigcup_{i=1}^m \mathrm{Subgr}(\Gamma_i N_{\Gamma_i})$. Then $N = \bigcap_{i=1}^m N_{\Gamma_i} \in \mathrm{OpenNormal}(G)$ and $N \le \mathrm{Ker}(\varphi)$. Let $\hat{A} = G/N$, $\hat{\varphi}$: $G \to \hat{A}$ the quotient map, and $\bar{\varphi}$: $\hat{A} \to A$ the map induced by φ. Then $\varphi = \bar{\varphi} \circ \hat{\varphi}$. Now consider the fiber product $\hat{B} = B \times_A \hat{A}$ with the projection maps $\hat{\alpha}$: $\hat{B} \to \hat{A}$ and β: $\hat{B} \to B$ on the coordinates. Since α is surjective, so is $\hat{\alpha}$.

For each i put $N_i = N_{\Gamma_i}$ and $\gamma_i = \gamma'_{\Gamma_i}$: $\Gamma_i N_i \to B$. Then there is a homomorphism $\hat{\gamma}_i$: $\Gamma_i N_i \to \hat{B}$ satisfying $\hat{\alpha} \circ \hat{\gamma}_i = \hat{\varphi}|_{\Gamma_i N_i}$ and $\beta \circ \hat{\gamma}_i = \gamma_i$ [FrJ, Prop. 20.6]. Put $\hat{B}_i = \hat{\gamma}_i(\Gamma_i N_i)$.

Claim: $\hat{\alpha}$: $\hat{B}_i \to \hat{\varphi}(\Gamma_i N_i)$ *is an isomorphism.* It suffices to prove that $\hat{\alpha}$ is injective on $\hat{B}_i$. Indeed, consider $b \in \hat{B}_i$ with $\hat{\alpha}(b) = 1$. Choose $g \in \Gamma_i N_i$ with $\hat{\gamma}_i(g) = b$. Then $\hat{\varphi}(g) = \hat{\alpha}(\hat{\gamma}_i(g))) = 1$, so $g \in N \le N_i$. Thus, $\beta(b) = \gamma_i(g) = 1$. Therefore, $b = 1$, as desired.

Now consider $\Gamma \in \mathcal{G}$. Choose i with $\Gamma \le \Gamma_i N_i$. Then $\hat{\varphi}(\Gamma) \le \hat{\varphi}(\Gamma_i N_i)$. By the Claim, $\hat{\alpha}$ maps $\hat{\alpha}^{-1}(\hat{\varphi}(\Gamma)) \cap \hat{B}_i$ isomorphically onto $\hat{\varphi}(\Gamma)$. Hence, ($\hat{\varphi}$: $G \to \hat{A}$, $\hat{\alpha}$: $\hat{B} \to \hat{A}$) is a finite $\mathcal{G}$-embedding problem for G satisfying the rigidity condition.

Proof of (b): Consider a finite $\mathcal{G}$-embedding problem (1) for G. Let ($\hat{\varphi}$: $G \to \hat{A}$, $\hat{\alpha}$: $\hat{B} \to \hat{A}$) be the embedding problem given by (a). By assumption, it has a solution $\hat{\gamma}$. Then $\gamma = \beta \circ \hat{\gamma}$ solves (1). ∎

3. Pseudo Closed Fields

Let K be a field and $\mathcal{X}$ a subset of the set $\mathrm{SepAlgExt}(K)$ of all separable algebraic extensions of K. By an **absolutely irreducible variety over** K we mean a nonempty geometrical integral scheme of finite type over K. We say K is **pseudo-$\mathcal{X}$-closed** (abbreviated **P$\mathcal{X}$C**) if it satisfies the following condition:

(1) Every smooth absolutely irreducible variety over K, with an F-rational point for each $F \in \mathcal{X}$, has a K-rational point.

If V is an arbitrary absolutely irreducible variety over K, then the Zariski open subset $V_{\rm simp}$ of all simple points of V is also an absolutely irreducible variety over K. Hence, (1) is equivalent to the following condition:

(2) Every absolutely irreducible variety over K, with a simple F-rational point for each $F \in \mathcal{X}$, has a K-rational point.

Note that K is P$\emptyset$C if and only if K is PAC [FrJ, Chap. 10].

Under a mild topological assumption on $\mathcal{X}$, the P$\mathcal{X}$C property of K results in a relative projectivity of $\mathrm{Gal}(K)$. The topology in question is the étale topology of the space $\mathrm{SepAlgExt}(K)$. This space stands in a bijective correspondence with $\mathrm{Subgr}(\mathrm{Gal}(K))$. Thus, $\mathrm{SepAlgExt}(K)$ inherits the étale topology from that of $\mathrm{Subgr}(\mathrm{Gal}(K))$. Basic étale open subsets of $\mathrm{SepAlgExt}(K)$ are $\mathrm{SepAlgExt}(L)$ with L/K finite and separable.

If $\mathcal{X} \subseteq \mathcal{X}' \subseteq \mathrm{SepAlgExt}(K)$ and K is P$\mathcal{X}$C, then K is P$\mathcal{X}'$C. Denote the set of all minimal fields in $\mathcal{X}$ by $\mathcal{X}_{\rm min}$. If $\mathcal{X}$ is étale compact, then by Lemma 1.6, each field in $\mathcal{X}$ contains a minimal field in $\mathcal{X}$. Hence, K is P$\mathcal{X}$C if and only if K is P$\mathcal{X}_{\rm min}$C.

Proposition 3.1: *Let K be a field and $\mathcal{X}$ a subset of* $\mathrm{SepAlgExt}(K)$. *Put $\mathcal{G} = \{\mathrm{Gal}(K') \mid K' \in \mathcal{X}\}$. Suppose $\mathcal{X}$ is étale compact and K is P$\mathcal{X}$C. Then* $\mathrm{Gal}(K)$ *is $\mathcal{G}$-projective.*

Proof: By Lemma 1.3, $\mathcal{G}^G$ is étale compact. If we prove that $\mathrm{Gal}(K)$ is $\mathcal{G}^G$-projective, it will follow that $\mathrm{Gal}(K)$ is $\mathcal{G}$-projective. We may therefore assume, $\mathcal{G}$ is $\mathrm{Gal}(K)$-invariant.

By Lemma 2.1, it suffices to solve every finite rigid $\mathcal{G}$-embedding problem

$$(\varphi\colon \mathrm{Gal}(K) \to A,\ \alpha\colon B \to A) \tag{3}$$

where φ is an epimorphism.

Let L be the fixed field of $\mathrm{Ker}(\varphi)$ in K_s. Then identify A with $\mathrm{Gal}(L/K)$ and φ with $\mathrm{res}_{K_s/L}$. Next use [HJPa, Lemma 6.2] to construct a finitely generated regular extension E of K and a finite Galois extension F of E containing L with these properties:

(4a) $B = \mathrm{Gal}(F/E)$ and α is the restriction map $\mathrm{res}_{F/L}\colon \mathrm{Gal}(F/E) \to \mathrm{Gal}(L/K)$.

(4b) Let L_0 be a field between K and L and F_0 a field between E and F which contains L_0. Suppose $\mathrm{res}_{F/L}\colon \mathrm{Gal}(F/F_0) \to \mathrm{Gal}(L/L_0)$ is an isomorphism. Then F_0 is a purely transcendental extension of L_0.

Since E/K is finitely generated and regular, one may view E as the function field of an absolutely irreducible smooth affine variety V over K [FrJ, Cor. 9.23].

Now let $\{L_i \mid i \in I\} = \{K' \cap L \mid K' \in \mathcal{X}\}$. By rigidity, choose for each $i \in I$ a field F_i between E and F containing L_i such that $\mathrm{res}_{F/L}\colon \mathrm{Gal}(F/F_i) \to \mathrm{Gal}(L/L_i)$ is an isomorphism. By (4b), F_i is a purely transcendental extension of L_i. Hence, $V(L_i) \neq \emptyset$. Therefore, $V(K') \neq \emptyset$ for each $K' \in \mathcal{X}$. Since K is P$\mathcal{X}$C, V has a K-rational point, which by assumption is simple. By [JaR, Cor. A2], E has a valuation which is trivial on K and with K as its residue field. [HJPa, Lemma 7.4] gives an algebraic extension E' of E such that $\mathrm{res}_{E'_s/K_s}\colon \mathrm{Gal}(E') \to \mathrm{Gal}(K)$ is an isomorphism. Denote its inverse by γ'. Then $\gamma = \mathrm{res}_{E'_s/F} \circ \gamma'$ solves (3). ∎

Again, let K be a field and $\mathcal{X}$ a subset of $\mathrm{SepAlgExt}(K)$. For each algebraic extension L of K let $\mathcal{X}_L = \{K'L \mid K' \in \mathcal{X}\}$. If $[L : K] < \infty$, then L is P$\mathcal{X}_L$C [Jar1, Lemma 7.2]. The same result holds for arbitrary L if $\mathcal{X}$ is strictly closed [Jar1, Lemma 7.4]. Here we prove that L is P$\mathcal{X}_L$C under the weaker condition that $\mathcal{X}$ is étale compact.

PROPOSITION 3.2 (Extension theorem): *Let K be a field, $\mathcal{X}$ an étale compact subset of $\mathrm{SepAlgExt}(K)$, and L a separable algebraic extension of K. Suppose K is P$\mathcal{X}$C. Then $\mathcal{X}_L$ is étale compact and L is P$\mathcal{X}_L$C.*

Proof: The map $K' \mapsto K'L$ from $\mathcal{X}$ to $\mathcal{X}_L$ is étale continuous. Since $\mathcal{X}$ is étale compact, so is $\mathcal{X}_L$.

Next let V be a smooth absolutely irreducible variety over L with $V(K'L) \neq \emptyset$ for each $K' \in \mathcal{X}$. Choose a finite subextension K_1 of L/K over which V is already defined. Denote the set of all finite subextensions of L/K_1 by $\mathcal{E}$. For each $E \in \mathcal{E}$ let $\mathcal{T}_E = \{K' \in \mathcal{X} \mid V(K'E) \neq \emptyset\}$.

CLAIM: *$\mathcal{T}_E$ is étale open in $\mathcal{X}$.* Indeed, let $K' \in \mathcal{T}_E$. Then $V(K'E) \neq \emptyset$. Hence, K'/K has a finite subextension K'_0/K with $V(K'_0E) \neq \emptyset$. The open neighborhood $\mathcal{X} \cap \mathrm{SepAlgExt}(K'_0)$ of K' in $\mathcal{X}$ is contained in $\mathcal{T}_E$. Therefore, $\mathcal{T}_E$ is open in $\mathcal{X}$.

Since $L = \bigcup_{E\in\mathcal{E}} E$, we have $\bigcup_{E\in\mathcal{E}} \mathcal{T}_E = \mathcal{X}$. Since $\mathcal{X}$ is étale compact, $\mathcal{E}$ has a finite subset $\mathcal{E}_0$ with $\bigcup_{E\in\mathcal{E}_0} \mathcal{T}_E = \mathcal{X}$. Let F be the union of all $E \in \mathcal{E}_0$. Then $\mathcal{T}_E \subseteq \mathcal{T}_F$ for each $E \in \mathcal{E}_0$, so $\mathcal{X} = \mathcal{T}_F$. Thus, $V(K'F) \neq \emptyset$ for each $K' \in \mathcal{X}$. By [Jar1, Lemma 7.4] F is P$\mathcal{X}_F$C. Hence, $V(F) \neq \emptyset$, so $V(L) \neq \emptyset$. Consequently, L is P$\mathcal{X}_L$C. ∎

4. Strongly Projective Groups

Consider a profinite group G and a subset $\mathcal{G}$ of $\mathrm{Subgr}(G)$. Suppose G is $\mathcal{G}$-projective. If $\mathcal{G}$ is empty, then G is projective, so every embedding problem for G is solvable [FrJ, Lemma 20.4]. Unfortunately, we are able to solve an arbitrary $\mathcal{G}$-embedding problem in the general case only if we impose a strong condition on the global solution of each finite embedding problem: The solution has to map every local group $\Gamma \in \mathcal{G}$ into a subset of $\mathrm{Subgr}(B)$, given in advance, which is closed under conjugations and taking subgroups. In addition, we have to assume that every $\Gamma \in \mathcal{G}$ is maximal in $\mathcal{G}$ and $1 \notin \mathrm{ÉtaleClosure}(\mathcal{G})$. Section 5 shows that when these conditions are fulfilled, $G, \mathcal{G}$ naturally give rise to a proper projective group structure $\mathbf{G} = (G, X, G_x)_{x\in X}$ in the sense of [HJPa, Section 4]. By [HJPa, Prop. 4.2], every embedding problem for $\mathbf{G}$ is solvable.

Consider again a profinite group G and a subset $\mathcal{G}$ of $\mathrm{Subgr}(G)$. A $\mathcal{G}$-embedding problem for G with **local data** is a triple

$$(\varphi\colon G \to A,\ \alpha\colon B \to A,\ \mathcal{B}) \tag{1}$$

where A and B are profinite groups, $\mathcal{B}$ is a strictly closed subset of $\mathrm{Subgr}(B)$ which is closed under conjugations and taking closed subgroups, φ is a homomorphism, and α is an epimorphism. In addition, we assume α is **$\mathcal{B}$-rigid**. That is:
(2) $\varphi(\mathcal{G}) \subseteq \alpha(\mathcal{B})$ and α is injective on each $B_0 \in \mathcal{B}$.

Call (1) **finite** if B is finite.

Occasionally we construct $\mathcal{B}$ as above in the following way. Let $\mathcal{B}_0$ be a subset of $\mathrm{Subgr}(B)$. Then $\mathcal{B} = \mathrm{Con}(\mathcal{B}_0)$ is the set of all subgroup of B which are contained in B_0^b for some $B_0 \in \mathcal{B}_0$ and $b \in B$. By Lemma 1.1, $\mathcal{B}$ is strictly compact (hence closed) if $\mathcal{B}_0$ is étale compact. In particular, this is the case if $\mathcal{B}_0$ is finite.

A **solution** of (1) is a homomorphism $\gamma\colon G \to B$ with $\gamma(\mathcal{G}) \subseteq \mathcal{B}$. Call $\mathcal{G}$ **strongly $\mathcal{G}$-projective** if every finite $\mathcal{G}$-embedding problem (1) for G with local data has a solution. If in addition $\mathcal{G}$ is étale compact, then G is $\mathcal{G}$-projective.

Indeed, let $(\varphi\colon G \to A,\ \alpha\colon B \to A)$ be a rigid $\mathcal{G}$-embedding problem in the sense of Section 2, (3). For each $\Gamma \in \mathcal{G}$ choose $B_\Gamma \in \mathrm{Subgr}(B)$ such that $\alpha\colon B_\Gamma \to \varphi(\Gamma)$ is an isomorphism. Let $\mathcal{B} = \mathrm{Con}(B_\Gamma \mid \Gamma \in \mathcal{G})$. Then (1) is a $\mathcal{G}$-embedding problem for G with local data and α is $\mathcal{B}$-rigid. By assumption, there exists a homomorphism $\gamma\colon G \to B$ with $\alpha \circ \gamma = \varphi$. By Lemma 2.1(b), G is $\mathcal{G}$-projective. Moreover, by Lemmas 1.6 and 1.7, $\mathcal{G}$ is étale compact and G is strongly $\mathcal{G}_{\max}$-projective.

Example 4.1: Free product of finitely many profinite groups. Consider a free product $G = \mathop{\ast\!\!\!\prod}_{i=1}^n G_i$ of finitely many profinite groups. Put $\mathcal{G} = \{G_1, \ldots, G_n\}$ and $\mathcal{B} = \mathrm{Con}(G_1, \ldots, G_n)$. Then G is strongly $\mathcal{G}$-projective.

Indeed, let (1) be a finite embedding problem for G with local data. Then φ maps each G_i onto a subgroup A_i of A and there is $B_i \in \mathcal{B}$ which α maps isomorphically onto A_i. Then $\gamma_i = (\alpha|_{B_i})^{-1} \circ (\varphi|_{G_i})$ is an epimorphism of G_i onto B_i. Extend $\gamma_1, \ldots, \gamma_n$ to a homomorphism $\gamma\colon G \to B$. Then γ solves embedding problem (1). ∎

Remark 4.2: Suppose $\mathcal{G}$ is étale compact and G is a strongly $\mathcal{G}$-projective group. An obvious modification of Lemma 2.1 proves (1) is solvable even if α is not necessarily rigid but satisfies the weaker condition instead:

(3) For each $\Gamma \in \mathcal{G}$ there is $B_0 \in \mathcal{B}$ and a homomorphism $\gamma_0\colon \Gamma \to B_0$ with $\alpha \circ \gamma_0 = \varphi|_\Gamma$.

However, we do not use (3) in the definition of strong projectivity because all embedding problems which we use in this work satisfy the condition "α is $\mathcal{B}$-rigid". ∎

LEMMA 4.3: *Let G be a strongly $\mathcal{G}$-projective group with $\mathcal{G} \subseteq \mathrm{Subgr}(G)$. Then every embedding problem with local data (1) such that A is finite and $\mathrm{rank}(B) \le \aleph_0$ is solvable.*

Proof: Put $N_0 = \mathrm{Ker}(\alpha)$ and identify A with B/N_0 and α with the quotient map $B \to B/N_0$. Choose a descending sequence $N_i \in \mathrm{OpenNormal}(B)$ with $N_i \le \mathrm{Ker}(\alpha)$, $i = 1, 2, 3, \ldots$, and $\bigcap_{i=1}^n N_i = 1$. For $j \ge i$ let $\alpha_{ji}\colon B/N_j \to B/N_i$ and $\beta_i\colon B \to B/N_i$ be the quotient maps. For each i, $\mathcal{B}/N_i = \beta_i(\mathcal{B})$ is closed under conjugation and taking subgroups. The map α is injective on each $B_0 \in \mathcal{B}$, so $\alpha_{i+1,i}$ is injective on B_0/N_{i+1}. Therefore, we may inductively construct a sequence of homomorphisms $\gamma_i\colon G \to B/N_i$ satisfying: $\gamma_0 = \varphi$, $\gamma_i(\mathcal{G}) \subseteq \mathcal{B}/N_i$, and $\alpha_{i+1,i} \circ \gamma_{i+1} = \gamma_i$, $i = 1, 2, 3, \ldots$.

The γ_i's define a homomorphism $\gamma\colon G \to B$ with $\beta_i \circ \gamma = \gamma_i$, $i = 0, 1, 2, \ldots$. Since $\mathcal{B}$ is strictly closed, $\mathcal{B} = \varprojlim \mathcal{B}/N_i$. Hence, $\gamma(\mathcal{G}) \subseteq \mathcal{B}$. Therefore, γ is a solution of (1). ∎

Free products of finitely many profinite groups have some nice properties:

LEMMA 4.4 ([HeR, Prop. 2 and Thm. B']): *Let $G = \mathop{\ast\!\!\!\prod}_{i\in I} G_i$ be the free profinite product of finitely many profinite groups G_i. Then $G_i^g \cap G_j \neq 1$ implies $i = j$ and $g \in G_i$.*

Lemma 4.4 carries over to strongly $\mathcal{G}$-projective groups:

PROPOSITION 4.5: *Let G be a profinite group and $\mathcal{G}$ an étale compact subset of* $\mathrm{Subgr}(G)$ *which is closed under conjugation. Suppose G is strongly $\mathcal{G}$-projective. Then:*
(a) $\Gamma_1 \cap \Gamma_2 = 1$ *for all distinct* $\Gamma_1, \Gamma_2 \in \mathcal{G}_{\max}$.
(b) $N_G(\Gamma) = \Gamma$ *for each nontrivial* $\Gamma \in \mathcal{G}_{\max}$.

Proof: Consider an epimorphism $\varphi\colon G \to A$ with A finite. Write $\varphi(\mathcal{G}) = \{A_i \mid i \in I\}$ with I finite. For each $i \in I$ choose an isomorphic copy B_i of A_i. Choose a large positive integer e and put $B = \hat{F}_e * \mathop{\ast\!\!\!\prod}_{i\in I} B_i$. Then there is an epimorphism $\alpha\colon B \to A$ which maps B_i isomorphically onto A_i. Let $\mathcal{B} = \mathrm{Con}(B_1, \ldots, B_n)$. Then, (1) is a $\mathcal{G}$-embedding problem for G with local data. By Lemma 4.3, there is a homomorphism $\gamma\colon G \to B$ with $\alpha \circ \gamma = \varphi$ and $\gamma(\mathcal{G}) \subseteq \mathcal{B}$.

Proof of (a): Assume $\Gamma_1 \cap \Gamma_2 \neq 1$. Choose $N_0 \in \mathrm{OpenNormal}(G)$ with $\Gamma_1 N_0 \neq \Gamma_2 N_0$. Consider $N \in \mathrm{OpenNormal}(G)$ with $N \le N_0$. Put $A = G/N$ and let $\varphi\colon G \to G/N$ be the quotient map. Then let B, B_i, $\mathcal{B}$, α, and γ be as above. In particular, $\gamma(\Gamma_i) \in \mathcal{B}$, $i = 1, 2$. Hence, there are $j, k \in I$ and $b_j, b_k \in B$ with $\gamma(\Gamma_1) \le B_j^{b_j}$ and $\gamma(\Gamma_2) \le B_k^{b_k}$. Also, $\alpha\big(\gamma(\Gamma_1) \cap \gamma(\Gamma_2)\big) \subseteq \varphi(\Gamma_1) \cap \varphi(\Gamma_2) \neq 1$, hence $\gamma(\Gamma_1) \cap \gamma(\Gamma_2) \neq 1$, so $B_j^{b_j} \cap B_k^{b_k} \neq 1$. By Lemma 4.4, $B_j^{b_j} = B_k^{b_k}$. Consider $\Gamma_N \in \mathcal{G}$ with $\varphi(\Gamma_N) = \alpha(B_j^{b_j})$. Then, $\varphi(\Gamma_i) \le \varphi(\Gamma_N)$, $i = 1, 2$. It follows that the set $\mathcal{G}_N = \{\Gamma \in \mathcal{G} \mid \varphi(\Gamma_1), \varphi(\Gamma_2) \le \varphi(\Gamma)\}$ is nonempty. By Lemma 1.5, $\mathcal{G}_N$ is étale closed. If $N_1, \ldots, N_m$ are open normal subgroups of G and $N = \bigcap_{j=1}^m N_j$, then $\mathcal{G}_N \subseteq \bigcap_{j=1}^m \mathcal{G}_{N_j}$. Hence, since $\mathcal{G}$ is weakly compact, $\bigcap_{N \in \mathrm{OpenNormal}(G)} \mathcal{G}_N \neq \emptyset$. Each Γ in this intersection belongs to $\mathcal{G}$ and satisfies $\Gamma_1 \cap \Gamma_2 \le \Gamma$. Since Γ_1 and Γ_2 are maximal in $\mathcal{G}$, $\Gamma_1 = \Gamma = \Gamma_2$, in contradiction to assumption.

Proof of (b): Let $g \in G$ with $\Gamma^g = \Gamma$. Choose $N_0 \in \mathrm{OpenNormal}(G)$ with $\Gamma \not\le N_0$. Consider $N \in \mathrm{OpenNormal}(G)$ with $N \le N_0$. Put $A = G/N$ and let $\varphi\colon G \to G/N$ be the quotient epimorphism. Then let B, B_i, $\mathcal{B}$, and α be as in the first paragraph of the proof. In particular, there is $i \in I$ and $b \in B$ with $\gamma(\Gamma) \le B_i^b$. Also, $\gamma(\Gamma)^{\gamma(g)} = \gamma(\Gamma)$ and $\gamma(\Gamma) \neq 1$ so $B_i^b \cap B_i^{b\gamma(g)} \neq 1$. By Lemma 4.4, $\gamma(g) \in B_i^b$. Choose $\Gamma_N \in \mathcal{G}$ with $\varphi(\Gamma_N) = \alpha(B_i^b)$. Then $\varphi(\Gamma) \le \varphi(\Gamma_N)$ and $\varphi(g) \in \varphi(\Gamma_N)$.

Again, by Lemma 1.5, the nonempty set

$$\mathcal{G}'_N = \{\Gamma' \in \mathcal{G} \mid \varphi(\Gamma) \le \varphi(\Gamma'),\ \varphi(g) \in \varphi(\Gamma')\}$$

is étale closed. Since $\mathcal{G}$ is étale compact, there is Γ' which belongs to all $\mathcal{G}'_N$. It satisfies $\Gamma \le \Gamma'$ and $g \in \Gamma'$. Since Γ is maximal in $\mathcal{G}$, we have $\Gamma = \Gamma'$. Therefore, $g \in \Gamma$. ∎

LEMMA 4.6: *Let G be a profinite group and $\mathcal{G}$ an étale compact subset of* $\mathrm{Subgr}(G)$ *which is closed under conjugation. Suppose $1 \notin$* $\mathrm{StrictClosure}(\mathcal{G})$ *and G is strongly $\mathcal{G}$-projective. Then:*

(a) *G is strongly $\mathcal{G}_{\max}$-projective.*
(b) *$\mathcal{G}_{\max}$ is étale compact Hausdorff.*
(c) *$N_G(\Gamma) = \Gamma$ for each $\Gamma \in \mathcal{G}_{\max}$.*

Proof of (a): By Lemma 1.7, $\mathcal{G}_{\max}$ is étale compact. Suppose (1) is a finite $\mathcal{G}_{\max}$-embedding problem with local data for G. In particular, $\varphi(\mathcal{G}_{\max}) \subseteq \alpha(\mathcal{B})$ and α is injective on each $B_0 \in \mathcal{B}$. We prove (1) is a $\mathcal{G}$-embedding problem with local data for G. To this end let $\Gamma_0 \in \mathcal{G}$. By Lemma 1.6, Γ_0 is contained in some $\Gamma \in \mathcal{G}_{\max}$. Choose $B_1 \in \mathcal{B}$ with $\alpha(B_1) = \varphi(\Gamma)$. Let $B_0 = B_1 \cap \alpha^{-1}(\varphi(\Gamma_0))$. Then $B_0 \in \mathcal{B}$, and α maps B_0 isomorphically onto $\varphi(\Gamma_0)$, as needed.

Since G is strongly $\mathcal{G}$-projective, there is a homomorphism $\gamma\colon G \to B$ with $\alpha\circ\gamma = \varphi$ and $\gamma(\mathcal{G}) \subseteq \mathcal{B}$, so $\gamma(\mathcal{G}_{\max}) \subseteq \mathcal{B}$. Thus, G is strongly $\mathcal{G}_{\max}$-projective.

Proof of (b): By (a) and Proposition 4.5(a), $\Gamma_1 \cap \Gamma_2 = 1$ for all distinct $\Gamma_1, \Gamma_2 \in \mathcal{G}_{\max}$. Hence, by [HJPa, Cor. 1.4], $\mathcal{G}_{\max}$ is étale Hausdorff.

Proof of (c): By assumption, each $\Gamma \in \mathcal{G}_{\max}$ is nontrivial. Hence, by Proposition 4.5(b), $N_G(\Gamma) = \Gamma$. ∎

Our next goal is to prove under the assumptions of Lemma 4.6 that $\mathcal{G}_{\max}$ is a profinite space in the étale topology. By definition, a **profinite space** X is an inverse limit of discrete finite spaces. In particular, X has a basis consisting of open-closed sets. Conversely, every compact Hausdorff space which has a basis consisting of open-closed sets is profinite (See also [RiZ, Thm. 1.1.12] for the connection with condition "X is totally disconnected".)

LEMMA 4.7: *Let X be a compact Hausdorff space and G a profinite group which acts continuously on X. Suppose X/G has a basis consisting of open-closed sets. Then X is profinite.*

Proof: Let $x \in X$ and W an open neighborhood of x. We have to find an open-closed neighborhood of x in W.

PART A: *G is finite.* Let $S = \{\sigma \in G \mid x^\sigma = x\}$. Write $G = \dot\bigcup_{i=1}^m S\sigma_i$ with $\sigma_1 = 1$. Then $x^{\sigma_1}, \ldots, x^{\sigma_m}$ are the distinct conjugates of x. Since X is Hausdorff, there are open neighborhoods $V_1, \ldots, V_m$ of x in W such that $V_1^{\sigma_1}, \ldots, V_m^{\sigma_m}$ are disjoint. Put $V = \bigcap_{i=1}^m \bigcap_{\sigma\in S} V_i^\sigma$. This is an S-invariant open neighborhood of x in W and $V^{\sigma_1}, \ldots, V^{\sigma_m}$ are disjoint.

The quotient map $\pi\colon X \to X/G$ is continuous and open. In particular, $\pi(V)$ is an open neighborhood of $\pi(x)$ in X/G. By assumption, there is an open-closed neighborhood $\bar U$ of $\pi(x)$ in X/G with $\bar U \subseteq \pi(V)$. Then, $U = \pi^{-1}(\bar U)$ is an open-closed G-invariant neighborhood of x in X and $U \subseteq \pi^{-1}(\pi(V)) = \dot\bigcup_{i=1}^m V^{\sigma_i}$. Therefore $U = \dot\bigcup_{i=1}^m U \cap V^{\sigma_i}$. Since the sets $U \cap V^{\sigma_i}$ are open, they are also closed in U, and hence in X. Thus $U \cap V = U \cap V^{\sigma_1}$ is an open-closed neighborhood of x contained in W.

PART B: *G is arbitrary.* The action $X \times G \to X$ is continuous and $x^1 \in W$. Hence, x has an open neighborhood V and G has an open normal subgroup N with $V^N \subseteq W$. Let $\nu\colon X \to X/N$ be the quotient map. Then $\nu(V)$ is an open neighborhood of $\nu(x)$ in

X/N. Since X is compact Hausdorff, so is X/N [Bre, Thm. 3.1(1)]. The finite group G/N acts on X/N continuously and $X/G = (X/N)/(G/N)$. Thus, by Part A, X/N is profinite. Therefore, $\nu(x)$ has an open-closed neighborhood $\bar{U}$ in X/N with $\bar{U} \subseteq \nu(V)$. Therefore $U = \nu^{-1}(\bar{U})$ is an open-closed neighborhood of x in X and $U \subseteq V^N \subseteq W$, as desired. ∎

PROPOSITION 4.8: *Let G be a profinite group and $\mathcal{G}$ an étale compact G-invariant subset of* $\mathrm{Subgr}(G)$. *Suppose* $1 \notin \mathrm{StrictClosure}(\mathcal{G})$ *and G is strongly $\mathcal{G}$-projective. Then $\mathcal{G}_{\max}$ is étale profinite.*

Proof: By Lemma 4.6, G is strongly $\mathcal{G}_{\max}$-projective and $\mathcal{G}_{\max}$ is étale Hausdorff compact. We may therefore replace $\mathcal{G}$ by $\mathcal{G}_{\max}$, if necessary, to assume $\mathcal{G} = \mathcal{G}_{\max}$ and prove that the étale topology of $\mathcal{G}$ has a basis consisting of étale open-closed sets.

Let $\pi\colon \mathrm{Subgr}(G) \to \mathrm{Subgr}(G)/G$ be the quotient map modulo conjugation. Put a bar over each group in $\mathrm{Subgr}(G)$ and each subset of $\mathrm{Subgr}(G)$ to denote their images under π. By Lemma 4.7, it suffices to prove that the étale topology of $\bar{\mathcal{G}}$ has a basis consisting of étale open-closed sets. Thus, given $\Gamma_0 \in \mathcal{G}$ and $H \in \mathrm{Open}(G)$ with $\Gamma_0 \le H$, it suffices to find a G-invariant étale open-closed subset $\mathcal{U}_0$ of $\mathrm{Subgr}(G)$ with $\bar{\Gamma}_0 \in \bar{\mathcal{U}}_0 \subseteq \overline{\mathrm{Subgr}(H)}$. The construction of $\mathcal{U}_0$ breaks up into three parts.

PART A: *An open normal subgroup of G.* Since $1 \notin \acute{\mathrm{E}}\mathrm{taleClosure}(\mathcal{G})$, there is $N_0 \in \mathrm{OpenNormal}(G)$ which contains no $\Gamma \in \mathcal{G}$. Consider the étale open subset $\mathcal{H} = \bigcup_{g\in G} \mathrm{Subgr}(H^g)$ of $\mathrm{Subgr}(G)$ and the étale closed subset $\mathcal{H}' = \mathcal{G} \smallsetminus \mathcal{H}$ of $\mathcal{G}$. Both $\mathcal{H}$ and $\mathcal{H}'$ are G-invariant. Since $\mathcal{G}$ is étale compact and Hausdorff, so is $\mathcal{H}'$. Each $\Gamma \in \mathcal{H}'$ is not contained in H, so $\Gamma \ne \Gamma_0$. Since $\mathcal{G} = \mathcal{G}_{\max}$, $\Gamma_0 \not\le \Gamma$. Therefore, there is $N_\Gamma \in \mathrm{OpenNormal}(G)$ with $N_\Gamma\Gamma_0 \not\le N_\Gamma\Gamma$ and $N_\Gamma\Gamma \not\le N_\Gamma H$.

The set $\mathrm{Subgr}(N_\Gamma\Gamma)$ is an étale open neighborhood of Γ in $\mathrm{Subgr}(G)$. Since $\mathcal{H}'$ is étale compact, there are $\Delta_1, \ldots, \Delta_m \in \mathcal{H}'$ with $\mathcal{H}' \subseteq \bigcup_{i=1}^m \mathrm{Subgr}(N_{\Delta_i}\Delta_i)$. Then $N = N_0 \cap \bigcap_{i=1}^m N_{\Delta_i}$ is an open normal subgroup of G, $N\Gamma \ne N$ for each $\Gamma \in \mathcal{G}$, and $N\Gamma_0 \not\le N\Gamma$, $N\Gamma \not\le NH$ for each $\Gamma \in \mathcal{H}'$.

PART B: *$\mathcal{G}$-embedding problem for G with local data.* Put $A = G/N$ and let $\varphi\colon G \to A$ be the quotient map. By Part A,

(4a) $\varphi(\Gamma) \ne 1$ for each $\Gamma \in \mathcal{G}$ and

(4b) $\varphi(\Gamma_0) \not\le \varphi(\Gamma)$ and $\varphi(\Gamma) \not\le \varphi(H)$ for each $\Gamma \in \mathcal{H}'$.

Now choose $\Gamma_1, \ldots, \Gamma_n \in \mathcal{H}'$ such that $\varphi(\Gamma_1), \ldots, \varphi(\Gamma_n)$ represent the conjugacy classes in A of the maximal elements of $\varphi(\mathcal{H}')$. Let B_0 be an isomorphic copy of $\varphi(H)$ and B_i an isomorphic copy of $\varphi(\Gamma_i)$, $i = 1, \ldots, n$. Choose a positive integer $e \ge \mathrm{rank}(A)$. Put $B = \hat{F}_e * \mathop{\ast}_{i=0}^n B_i$. Then B is finitely generated and there is an epimorphism $\alpha\colon B \to A$ which maps B_0 isomorphically onto $\varphi(H)$ and B_i isomorphically onto $\varphi(\Gamma_i)$, $i = 1, \ldots, n$. Let $\mathcal{B} = \mathrm{Con}(B_0, \ldots, B_n)$. Then

$$(\varphi\colon G \to A,\ \alpha\colon B \to A,\ \mathcal{B}) \tag{5}$$

is a $\mathcal{G}$-embedding problem for G with local data. By Lemma 4.3, there is a homomorphism $\gamma\colon G \to B$ with $\alpha \circ \gamma = \varphi$ and $\gamma(\mathcal{G}) \subseteq \mathcal{B}$.

Part C: *Partition of* $\mathcal{G}$. For each i let B'_i be an identical copy of B_i. Let $B' = \prod_{i=0}^n B'_i$ be the direct product of $B_0, \ldots, B_n$. Let $\beta\colon B \to B'$ be the epimorphism which maps $\hat{F}_e$ to 1 and each B_i identically onto B'_i. Put $\gamma' = \beta \circ \gamma$. For each i put $\mathcal{U}_i = \{\Gamma \in \mathcal{G} \mid \gamma'(\Gamma) \le B'_i\}$. Then $\mathcal{U}_i$ is a G-invariant étale open subset of $\mathcal{G}$. Therefore, $\bar{\mathcal{U}}_i$ is an étale open subset of $\bar{\mathcal{G}}$.

Claim C1: $\bar{\mathcal{G}} = \dot{\bigcup}_{i=0}^m \bar{\mathcal{U}}_i$. Indeed, since $\gamma(\mathcal{G}) \subseteq \mathcal{B}$, there are for each $\Gamma \in \mathcal{G}$ an i between 0 and m and a $b \in B$ with $\gamma(\Gamma) \le B_i^b$. Hence, $\gamma'(\Gamma) \le B'_i$ and $\bar{\Gamma} \in \bar{\mathcal{U}}_i$.

Moreover, by (4a), $\alpha(\gamma(\Gamma)) = \varphi(\Gamma) \ne 1$. Hence, $\gamma(\Gamma) \ne 1$. Since β is injective on B_i^b, we have $\gamma'(\Gamma) \ne 1$. Hence, $\gamma'(\Gamma) \not\le B'_j$, so $\bar{\Gamma} \notin \bar{\mathcal{U}}_j$ for all $j \ne i$.

It follows that each $\bar{\mathcal{U}}_i$ is an étale open-closed subset of $\bar{\mathcal{G}}$. In particular, so is $\bar{\mathcal{U}}_0$.

Claim C2: $\bar{\Gamma}_0 \in \bar{\mathcal{U}}_0$. Indeed, $\gamma(\Gamma_0) \le B_i^b$ with $0 \le i \le n$ and $b \in B$ (Claim C1). Assume $i \ge 1$. Then $\varphi(\Gamma_0) = \alpha(\gamma(\Gamma_0)) \le \alpha(B_i)^{\alpha(b)} = \varphi(\Gamma_i^g)$ for some $g \in G$ and $\Gamma_i^g \in \mathcal{H}'$ (by the choice of Γ_i). Hence, by (4b), $\varphi(\Gamma_0) \not\le \varphi(\Gamma_i^g)$. This contradiction proves that $i = 0$, $\gamma'(\Gamma) \le B'_0$, and $\bar{\Gamma}_0 \in \bar{\mathcal{U}}_0$.

Claim C3: $\bar{\mathcal{U}}_0 \subseteq \bar{\mathcal{H}}$. Indeed, consider $\Gamma \in \mathcal{U}_0$. By Part B, $\gamma(\Gamma) \le B_i^b$ with $0 \le i \le n$ and $b \in B$. If $i \ge 1$, then $\gamma'(\Gamma) \le B'_i$ and $\bar{\Gamma} \in \bar{\mathcal{U}}_i$, in contradiction to Claim C1. Hence, $\gamma(\Gamma) \le B_0^b$. Therefore, $\varphi(\Gamma^g) \le \varphi(H)$ for some $g \in G$ with $\varphi(g) = \alpha(b)^{-1}$. Since $\mathcal{H}'$ is G-invariant, (4b) implies $\Gamma \notin \mathcal{H}'$. Consequently, $\Gamma \in \mathcal{H}$, as desired.

Finally, observe that $\bar{\mathcal{H}} = \overline{\mathrm{Subgr}(H)}$ to conclude the proof of the proposition. ∎

5. Projective Group Structures

The crucial step of going from solvability of finite $\mathcal{G}$-embedding problems for a profinite group G to solvability of arbitrary $\mathcal{G}$-embedding problems occurs in the category of "profinite group structures". We recall the definition of this concept from [HJPa, Section 2].

A **profinite group structure** is a data $\mathbf{G} = (G, X, G_x)_{x \in X}$ where G is a profinite group, X is a profinite space on which G acts continuously from the right, and G_x is a closed subgroup of G, $x \in X$. These objects must satisfy the following conditions:
(1a) The map $x \mapsto G_x$ from X into $\mathrm{Subgr}(G)$ is étale continuous.
(1b) $G_{x^g} = G_x^g$ for all $x \in X$ and $g \in G$.
(1c) $\{g \in G \mid x^g = x\} \le G_x$ for each $x \in X$.

The structure $\mathbf{G}$ is **finite** if both G and X are finite.

A **morphism** of group structures $\varphi\colon (G, X, G_x)_{x \in X} \to (H, Y, H_y)_{y \in Y}$ is a couple consisting of a homomorphism $\varphi\colon G \to H$ and a continuous map $\varphi\colon X \to Y$ such that $\varphi(G_x) \le H_{\varphi(x)}$ and $\varphi(x^g) = \varphi(x)^{\varphi(g)}$ for all $x \in X$ and $g \in G$. The morphism φ is an **epimorphism** if $\varphi(G) = H$, $\varphi(X) = Y$, and for each $y \in Y$, there is $x \in X$ with $\varphi(G_x) = H_y$. We call φ a **cover** if $\varphi(G) = H$, $\varphi(X) = Y$, $\varphi\colon G_x \to H_{\varphi(x)}$ is an isomorphism for each $x \in X$, and $\varphi(x) = \varphi(x')$ implies $x^k = x'$ for some $k \in \mathrm{Ker}(\varphi)$.

An **embedding problem** for $\mathbf{G}$ is a pair $(\varphi\colon \mathbf{G} \to \mathbf{A}, \alpha\colon \mathbf{B} \to \mathbf{A})$ where $\mathbf{A}$ and $\mathbf{B}$ are profinite group structures, φ is a morphism, and α is a cover. A **solution** of the problem is a morphism $\gamma\colon \mathbf{G} \to \mathbf{B}$ satisfying $\alpha \circ \gamma = \varphi$. The problem is **finite** if both

$\mathbf{B}$ and $\mathbf{A}$ are finite. We say $\mathbf{G}$ is **projective** if every finite embedding problem for $\mathbf{G}$ is solvable. Then every embedding problem for $\mathbf{G}$ is solvable [HJPa, Prop. 4.2].

LEMMA 5.1: *Let $\mathbf{G} = (G, X, G_x)_{x\in X}$ be a projective group structure. Put $\mathcal{G} = \{G_x \mid x \in X\}$. Then G is strongly $\mathcal{G}$-projective.*

Proof: By definition, $\mathcal{G}$ is the image of the compact space X under the étale continuous map $x \mapsto G_x$. Hence, $\mathcal{G}$ is étale compact.

Now consider a finite $\mathcal{G}$-embedding problem

$$(\varphi\colon G \to A,\ \alpha\colon B \to A,\ \mathcal{B}) \tag{2}$$

for G with local data. Replace A by $A_0 = \varphi(G)$, B by $B_0 = \alpha^{-1}(\varphi(G))$, and $\mathcal{B}$ by $\mathcal{B}_0 = \mathcal{B} \cap \mathrm{Subgr}(B_0)$, if necessary, to assume φ is surjective. By [HJPa, Lemma 3.8], $\varphi\colon G \to A$ extends to an epimorphism φ of $\mathbf{G}$ onto a finite group structure $\mathbf{A} = (A, I, A_i)_{i\in I}$. Choose a set of representatives I_0 for the A-orbits of I. For each $i \in I_0$ there exists $x \in X$ with $\varphi(x) = i$ and $\varphi(G_x) = A_i$. The rigidity condition (2) of Section 4 gives $B' \in \mathcal{B}$ which α maps isomorphically onto A_i. Hence, by [HJPa, Lemma 4.5], there is a group structure $\mathbf{B} = (B, J, B_j)_{j\in J}$ and $\alpha\colon B \to A$ extends to a cover $\alpha\colon \mathbf{B} \to \mathbf{A}$ with $B_j \in \mathcal{B}$ for all $j \in J$. In particular, $\varphi\colon J \to I$ is an epimorphism with finite fibers, so J is finite, hence $\mathbf{B}$ is finite. Since $\mathbf{G}$ is projective, there is a morphism $\gamma\colon \mathbf{G} \to \mathbf{B}$ with $\alpha \circ \gamma = \varphi$. Its group component $\gamma\colon G \to B$ solves embedding problem (2). Consequently, G is strongly $\mathcal{G}$-projective. ∎

LEMMA 5.2: *Let G be a profinite group and $\mathcal{G}$ an étale compact subset of $\mathrm{Subgr}(G)$. Suppose $\mathcal{G} = \dot\bigcup_{i\in I} \mathcal{G}_i$ is a partition of $\mathcal{G}$ into finitely many disjoint open-closed subsets. Then there exists an open normal subgroup N of G such that if $\varphi\colon G \to A$ is an epimorphism with $\mathrm{Ker}(\varphi) \le N$ and $\Gamma, \Gamma' \in \mathcal{G}$ satisfy $\varphi(\Gamma) \le \varphi(\Gamma')$, then there is $i \in I$ with $\Gamma, \Gamma' \in \mathcal{G}_i$.*

Proof: Let $\Gamma \in \mathcal{G}$. There Γ belongs to a unique $\mathcal{G}_i$. Since $\mathcal{G}_i$ is open in $\mathcal{G}$, there is an open normal subgroup N_Γ of G with $\mathcal{G} \cap \mathrm{Subgr}(\Gamma N_\Gamma) \subseteq \mathcal{G}_i$. Thus, $\mathcal{G} \subseteq \bigcup_{\Gamma\in\mathcal{G}} \mathrm{Subgr}(\Gamma N_\Gamma)$. Since $\mathcal{G}$ is étale compact, there are $\Gamma_1, \ldots, \Gamma_m \in \mathcal{G}$ with $\mathcal{G} \subseteq \bigcup_{j=1}^m \mathrm{Subgr}(\Gamma_j N_{\Gamma_j})$.

For each $1 \le j \le m$ there is a unique $i(j) \in I$ with $\mathcal{G} \cap \mathrm{Subgr}(\Gamma_j N_{\Gamma_j}) \subseteq \mathcal{G}_{i(j)}$. Put $N = \bigcap_{j=1}^m N_{\Gamma_j}$. Let $\varphi\colon G \to A$ be an epimorphism with $\mathrm{Ker}(\varphi) \le N$ and let $\Gamma, \Gamma' \in \mathcal{G}$ with $\varphi(\Gamma) \le \varphi(\Gamma')$. Choose j between 1 and m with $\Gamma' \in \mathrm{Subgr}(\Gamma_j N_{\Gamma_j})$. Then $\Gamma' \in \mathcal{G}_{i(j)}$. Hence, $\Gamma \le \Gamma'\mathrm{Ker}(\varphi) \le \Gamma' N \le \Gamma' N_{\Gamma_j} \le \Gamma_j N_{\Gamma_j}$. Therefore, $\Gamma \in \mathcal{G}_{i(j)}$, as desired. ∎

Let $\mathbf{G} = (G, X, G_x)_{x\in X}$ be a group structure. Put $\mathcal{G} = \{G_x \mid x \in X\}$. We say $\mathbf{G}$ is **proper** if the map $x \mapsto G_x$ of X onto $\mathcal{G}$ is an étale homeomorphism.

PROPOSITION 5.3: *Let $\mathbf{G} = (G, X, G_x)_{x\in X}$ be a proper group structure. Let $\mathcal{G} = \{G_x \mid x \in X\}$. Suppose G is strongly $\mathcal{G}$-projective. Then $\mathbf{G}$ is projective.*

Proof: Consider a finite embedding problem

$$(\varphi\colon \mathbf{G} \to \mathbf{A},\ \alpha\colon \mathbf{B} \to \mathbf{A}) \tag{3}$$

for $\mathbf{G}$ with $\mathbf{A} = (A, I, A_i)_{i\in I}$. The solution of this problem breaks up into three parts.

Part A: *A partition of* $\mathcal{G}$. Consider the partition $X = \bigcup_{i\in I}^{\cdot} \varphi^{-1}(i)$ into open-closed sets. For each $i \in I$ let $\mathcal{G}_i = \{G_x \mid \varphi(x) = i\}$. Since the map $x \mapsto G_x$ is an étale homeomorphism, $\mathcal{G} = \bigcup_{i\in I}^{\cdot} \mathcal{G}_i$ is a partition of $\mathcal{G}$ into étale open-closed sets. Lemma 5.2 gives an open normal subgroup N of G such that if $\hat\varphi$: $G \to \hat A$ is an epimorphism with $\ker(\hat\varphi) \le N$, then

(4) $x, y \in X$ and $\hat\varphi(G_x) \le \hat\varphi(G_y)$ imply $\varphi(x) = \varphi(y)$.

By [HJPa, Lemma 3.8] there are a morphism $\bar\varphi$: $\hat{\mathbf{A}} \to \mathbf{A}$ of finite group structures and an epimorphism $\hat\varphi$: $\mathbf{G} \to \hat{\mathbf{A}}$ such that $\varphi = \bar\varphi \circ \hat\varphi$ and $\mathrm{Ker}(\hat\varphi) \le N$. In particular, (4) holds.

The fiber product $\hat{\mathbf{B}} = (\hat B, \hat J, \hat B_j)_{j\in\hat J} = \mathbf{B} \times_{\mathbf{A}} \hat{\mathbf{A}}$ fits into a commutative diagram

$$\begin{tikzcd} & \mathbf{G} \arrow[d, "\hat\varphi"] \\ \hat{\mathbf{B}} \arrow[r, "\hat\alpha"] \arrow[d, "\beta"'] & \hat{\mathbf{A}} \arrow[d, "\bar\varphi"] \\ \mathbf{B} \arrow[r, "\alpha"] & \mathbf{A} \end{tikzcd}$$

in which $\hat\alpha$ is a cover [HJPa, Lemma 2.12(c)]. Let $\hat{\mathcal{B}} = \mathrm{Con}\big(\{\hat B_j \mid j \in \hat J\}\big)$. Then $(\hat\varphi$: $G \to \hat A$, $\hat\alpha$: $\hat B \to \hat A$, $\hat{\mathcal{B}})$ is an embedding problem for G with a local data. Since G is strongly $\mathcal{G}$-projective, there exists a homomorphism $\hat\gamma$: $G \to \hat B$ such that $\hat\alpha \circ \hat\gamma = \hat\varphi$ and

(5) for each $x \in X$ there is $j \in \hat J$ with $\hat\gamma(G_x) \le \hat B_j$.

Part B: *The map* $\hat\gamma$: $X \to \hat J$. Consider the open normal subgroup $K = \mathrm{Ker}(\hat\gamma)$ of G. For each $y \in X$, the open subgroup G_yK of G contains $S_y = \{\sigma \in G \mid y^\sigma = y\}$. Also, $V_y = \{x \in X \mid G_x \le G_yK\}$ of y is an open neighborhood of y in X which is G_yK-invariant. Indeed, if $\sigma \in G_y$, $\kappa \in K$, and $x \in V_y$, then

$$G_{x^{\sigma\kappa}} = G_x^{\sigma\kappa} \le (G_yK)^{\sigma\kappa} = (G_y^\sigma K^\sigma)^\kappa = (G_yK)^\kappa = G_yK,$$

whence $x^{\sigma\kappa} \in V_y$.

By [HJPa, Lemma 3.6], there are $y_1, \ldots, y_m \in X$ and open-closed subsets $X_1, \ldots, X_m$ of X such that the following holds for each k between 1 and m:

(6a) X_k is $G_{y_k}K$-invariant and $y_k \in X_k \subseteq V_{y_k}$.

(6b) $X = \bigcup_{k=1}^{\cdot\, m} \bigcup_{\tau\in T_k}^{\cdot} X_k^\tau$, where $G = \bigcup_{\tau\in T_k}^{\cdot} G_{y_k}K\tau$ and $1 \in T_k \subseteq G$.

Define $\hat\gamma$: $X \to \hat J$ as follows. For each k between 1 and m use (5) to choose $\hat\gamma(y_k) \in \hat J$ with $\hat\gamma(G_{y_k}) \le \hat B_{\hat\gamma(y_k)}$. Then let

$$\hat\gamma(y^\tau) = \hat\gamma(y_k)^{\hat\gamma(\tau)} \text{ for all } y \in X_k \text{ and } \tau \in T_k. \tag{7}$$

By (6b), $\hat\gamma$: $X \to \hat J$ is well defined. In addition,

(8) $\hat\gamma$ is constant on each X_k^τ with $\tau \in T_k$.

Hence, by (6b), $\hat{\gamma}$ is continuous.

Taking $\tau = 1$ in (7), gives $\hat{\gamma}(y) = \hat{\gamma}(y_k)$ for all $y \in X_k$. Hence, by (7),

$$\hat{\gamma}(y^\tau) = \hat{\gamma}(y)^{\hat{\gamma}(\tau)} \text{ for all } y \in X_k \text{ and } \tau \in T_k. \tag{9}$$

We claim that

(10) $\hat{\gamma}(G_x) \le \hat{B}_{\hat{\gamma}(x)}$ for every $x \in X$.

Indeed, $x = y^\tau$, where $y \in X_k$, $\tau \in T_k$. By (6a), $y \in V_{y_k}$, that is, $G_y \le G_{y_k}K$, so $\hat{\gamma}(G_y) \le \hat{\gamma}(G_{y_k}) \le \hat{B}_{\hat{\gamma}(y_k)} = \hat{B}_{\hat{\gamma}(y)}$. By (9), $\hat{\gamma}(y)^{\hat{\gamma}(\tau)} = \hat{\gamma}(x)$. Hence, $\hat{\gamma}(G_x) = \hat{\gamma}(G_y^\tau) = \hat{\gamma}(G_y)^{\hat{\gamma}(\tau)} \le \hat{B}_{\hat{\gamma}(y)}^{\hat{\gamma}(\tau)} = \hat{B}_{\hat{\gamma}(y)^{\hat{\gamma}(\tau)}} = \hat{B}_{\hat{\gamma}(x)}$, as claimed.

We know that $\hat{\alpha} \circ \hat{\gamma} = \hat{\varphi}$ on G. But we do not know that $\hat{\alpha} \circ \hat{\gamma} = \hat{\varphi}$ on X. Therefore, we define $\gamma = \beta \circ \hat{\gamma}\colon G \to B$ and $\gamma = \beta \circ \hat{\gamma}\colon X \to J$ and prove directly that $\gamma\colon \mathbf{G} \to \mathbf{A}$ is a morphism which solves embedding problem (3).

Part C: *The morphism* $\gamma\colon \mathbf{G} \to \mathbf{B}$. An application of β on (8), (9), and (10) implies:

(11a) γ is constant on each X_k^τ with $\tau \in T_k$, so, by (6b), $\gamma\colon X \to J$ is continuous.

(11b) $\gamma(y^\tau) = \gamma(y)^{\gamma(\tau)}$ for all $y \in X_k$ and $\tau \in T_k$

(11c) $\gamma(G_x) \le B_{\gamma(x)}$ for every $x \in X$.

Claim C1: $\alpha \circ \gamma = \varphi$. That $\alpha \circ \gamma = \varphi$ on G follows from the equality $\hat{\alpha} \circ \hat{\gamma} = \hat{\varphi}$ on G. Consider therefore $x \in X$. Since $\hat{\varphi}\colon \mathbf{G} \to \hat{\mathbf{A}}$ is an epimorphism, there is $y \in X$ such that $\hat{\varphi}(y) = \hat{\alpha}(\hat{\gamma}(x))$ and $\hat{\varphi}(G_y) = \hat{A}_{\hat{\alpha}(\hat{\gamma}(x))}$. By (10),

$$\hat{\varphi}(G_x) = \hat{\alpha}(\hat{\gamma}(G_x)) \le \hat{\alpha}(\hat{B}_{\hat{\gamma}(x)}) = \hat{A}_{\hat{\alpha}(\hat{\gamma}(x))} = \hat{\varphi}(G_y).$$

By (4), $\varphi(x) = \varphi(y)$. In addition,

$$\varphi(y) = \bar{\varphi}(\hat{\varphi}(y)) = \bar{\varphi}(\hat{\alpha}(\hat{\gamma}(x))) = \alpha(\beta(\hat{\gamma}(x))) = \alpha(\gamma(x)).$$

Hence, $\varphi(x) = \alpha(\gamma(x))$, as claimed.

Claim C2: $\gamma(G_{y_k})$ *is contained in the stabilizer* $S_{\gamma(y_k)}$ *of* $\gamma(y_k)$ *in* B. Let $j = \gamma(y_k)$. By Claim C1, $\alpha(j) = \varphi(y_k)$. By [HJPa, Remark 2.1], $G_{y_k} = S_{y_k}$. Hence,

$$\alpha(\gamma(G_{y_k})) = \varphi(G_{y_k}) = \varphi(S_{y_k}) \le S_{\varphi(y_k)} = S_{\alpha(j)}. \tag{12}$$

By (11c), $\gamma(G_{y_k}) \le B_j$. Since $\alpha\colon \mathbf{B} \to \mathbf{A}$ is a cover, $\alpha\colon B_j \to A_{\alpha(j)}$ is an isomorphism that maps S_j onto $S_{\alpha(j)}$ [HJPa, Lemma 2.2]. Therefore, by (12), $\gamma(G_{y_k}) \le S_{\gamma(y_k)}$.

Claim C3: γ *preserves the action.* We prove first that $\gamma(y^\sigma) = \gamma(y)^{\gamma(\sigma)}$ for all $y \in X_k$ and $\sigma \in G$. To this end we use (6b) to write $\sigma = \lambda\tau$ with $\lambda \in G_{y_k}K$ and $\tau \in T_k$. Then $\gamma(\lambda) \in \gamma(G_{y_k}K) = \gamma(G_{y_k})$. Hence, by Claim C2, $\gamma(y_k)^{\gamma(\lambda)} = \gamma(y_k)$. Whence, by (11a), $\gamma(y)^{\gamma(\lambda)} = \gamma(y)$. By (6a), $y^\lambda \in X_k^\lambda = X_k$. Hence, by (11a), $\gamma(y^\lambda) = \gamma(y)$, and by (11b), $\gamma((y^\lambda)^\tau) = \gamma(y^\lambda)^{\gamma(\tau)}$. Therefore, $\gamma(y^\sigma) = \gamma((y^\lambda)^\tau) = \gamma(y^\lambda)^{\gamma(\tau)} = \gamma(y)^{\gamma(\tau)} = \gamma(y)^{\gamma(\lambda)\gamma(\tau)} = \gamma(y)^{\gamma(\sigma)}$.

Now consider $x \in X_k^{\tau'}$ with $\tau' \in T_k$. Write $x = y^{\tau'}$ with $y \in X_k$. Let $g \in G$. By the preceding paragraph, $\gamma(x^g) = \gamma(y^{\tau' g}) = \gamma(y)^{\gamma(\tau' g)} = \gamma(y)^{\gamma(\tau')\gamma(g)} = \gamma(y^{\tau'})^{\gamma(g)} = \gamma(x)^{\gamma(g)}$, as claimed.

Thus, γ is a solution of embedding problem (3). ∎

Let G be a profinite group and $\mathcal{G}$ an étale profinite G-invariant subset of $\mathrm{Subgr}(G)$. Suppose $N_G(\Gamma) = \Gamma$ for each $\Gamma \in \mathcal{G}$. Choose a homeomorphic copy X of $\mathcal{G}$ and a homeomorphism $x \mapsto G_x$ of X onto $\mathcal{G}$. The action of G on $\mathcal{G}$ induces an action on X making $\mathbf{G} = (G, X, G_x)_{x \in X}$ a proper group structure. In this case we also refer to $(G, \mathcal{G})$ as a **proper group structure**. We call $(G, \mathcal{G})$ **projective** if $\mathbf{G}$ is projective.

PROPOSITION 5.4: *Let G be a profinite group and $\mathcal{G}$ an étale compact G-invariant subset of* $\mathrm{Subgr}(G)$. *Suppose* $1 \notin \mathrm{StrictClosure}(\mathcal{G})$ *and G is strongly $\mathcal{G}$-projective. Then $(G, \mathcal{G}_{\max})$ is a proper projective group structure.*

Proof: By Lemma 4.6, $N_G(\Gamma) = \Gamma$ for each $\Gamma \in \mathcal{G}_{\max}$ and G is strongly $\mathcal{G}_{\max}$-projective. By Proposition 4.8, $\mathcal{G}_{\max}$ is étale profinite. It follows, $(G, \mathcal{G}_{\max})$ is a proper group structure. By Proposition 5.3, $(G, \mathcal{G}_{\max})$ is projective. ∎

Remark 5.5: Relatively projective groups. Let G and $\mathcal{G}$ be as in Proposition 5.4. Then $\mathcal{G}_{\max}$ is étale profinite. Let Γ_1, Γ_2 be distinct groups in $\mathcal{G}_{\max}$. By Lemma 4.5, $\Gamma_1 \cap \Gamma_2 = 1$. Choose étale open-closed neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of Γ_1 and Γ_2 in $\mathcal{G}_{\max}$, respectively, with $\mathcal{G} = \mathcal{U}_1 \mathbin{\dot\bigcup} \mathcal{U}_2$. By [HJPa, Lemma 2.3], the union of all Γ in $\mathcal{U}_i$ is a closed subset of G. Thus, $\mathcal{G}$ is separated in the sense of [Har, Def. 3.1]. In addition, G is strongly $\mathcal{G}_{\max}$-projective. Consequently, G is projective relative to $\mathcal{G}$ in the sense of [Har, Def. 4.2]. ∎

We interpret the notions of a "morphism" and a "cover" of proper group structures in terms of the pairs $(G, \mathcal{G})$: Let $(H, \mathcal{H})$ and $(G, \mathcal{G})$ be proper group structures. Then a morphism $\varphi\colon (H, \mathcal{H}) \to (G, \mathcal{G})$ is just a homomorphism $\varphi\colon H \to G$ which maps $\mathcal{H}$ into $\mathrm{Con}(\mathcal{G})$. In other words, for each $\Delta \in \mathcal{H}$ there is $\Gamma \in \mathcal{G}$ with $\varphi(\Delta) \le \Gamma$.

The morphism φ is a **cover** if

(13a) $\varphi(H) = G$, $\varphi(\mathcal{H}) = \mathcal{G}$,

(13b) φ is injective on each $\Delta \in \mathcal{H}$, and

(13c) if $\Delta, \Delta' \in \mathcal{H}$ and $\varphi(\Delta) = \Delta'$, then there exists $\kappa \in \mathrm{Ker}(\varphi)$ with $\Delta^\kappa = \Delta'$.

A **sub-group-structure** of $(H, \mathcal{H})$ is a proper group structure $(H_0, \mathcal{H}_0)$ with $H_0 \le H$ and $\mathcal{H}_0 \subseteq \mathcal{H}$. Specializing [HJPa, Cor. 4.3] to proper group structures gives the following result:

PROPOSITION 5.6: *Let $\varphi\colon (H, \mathcal{H}) \to (G, \mathcal{G})$ be a cover of proper group structures. Suppose $(G, \mathcal{G})$ is projective. Then $(H, \mathcal{H})$ has a sub-group-structure $(H_0, \mathcal{H}_0)$ which φ maps isomorphically onto $(G, \mathcal{G})$.*

6. Big Quotients

Let G be a profinite group and $\mathcal{G}$ a subset of $\mathrm{Subgr}(G)$. We have already mentioned in Section 4 that if G is strongly $\mathcal{G}$-projective, then G is $\mathcal{G}$-projective. We show in this section that the converse is also true if there are only finitely many isomorphism types of groups in $\mathcal{G}$ and they have a "system of big quotients".

Let $\mathcal{C}$ be a finite set of finitely generated profinite groups. Each profinite group Δ which is isomorphic to a group in $\mathcal{C}$ is of **type** $\mathcal{C}$. A set $\mathcal{G}$ of profinite groups is said to be of **type** $\mathcal{C}$ if each $H \in \mathcal{G}$ is of type $\mathcal{C}$.

Let G be a profinite group and $\mathcal{G}$ a subset of $\mathrm{Subgr}(G)$. For each $\Gamma \in \mathcal{C}$ let $\mathcal{G}_\Gamma = \{H \in \mathcal{G} \mid H \cong \Gamma\}$. We prove an analog of Lemma 4.3 for $\mathcal{G}$-projective groups:

LEMMA 6.1: *Let G be a profinite group and $\mathcal{G}$ a subset of* $\mathrm{Subgr}(G)$ *of type $\mathcal{C}$. Suppose G is $\mathcal{G}$-projective and*

$$(\varphi\colon G \to A,\ \alpha\colon B \to A) \tag{1}$$

is a $\mathcal{G}$-embedding problem with A finite and $\mathrm{rank}(B) \le \aleph_0$*. Then (1) is solvable.*

Proof: There exists a descending sequence $\mathrm{Ker}(\alpha) = N_0 \ge N_1 \ge N_2 \ge \cdots$ of open normal subgroups of B with trivial intersection. Identify A with B/N_0 and α with the quotient map $B \to B/N_0$. Let $\varphi_0 = \varphi$ and $\alpha_0 = \alpha$. For each i and j with $j \ge i \ge 0$ let $\alpha_i\colon B \to B/N_i$ and $\alpha_{ji}\colon B/N_j \to B/N_i$ be the quotient maps.

CLAIM: *Let $i \ge 0$ and let $\varphi_i\colon G \to B/N_i$ be a homomorphism such that $(\varphi_i\colon G \to B/N_i,\ \alpha_i\colon B \to B/N_i)$ is a $\mathcal{G}$-embedding problem for G. Then there is a homomorphism $\varphi_{i+1}\colon G \to B/N_{i+1}$ such that $\alpha_{i+1,i} \circ \varphi_{i+1} = \varphi_i$ and $(\varphi_{i+1}\colon G \to B/N_{i+1},\ \alpha_{i+1}\colon B \to B/N_{i+1})$ is a $\mathcal{G}$-embedding problem for G.*

Once the claim has been proved, we may inductively construct for each $i \ge 0$ a homomorphism $\varphi_{i+1}\colon G \to B/N_{i+1}$ with $\alpha_{i+1,i} \circ \varphi_{i+1} = \varphi_i$. The maps φ_i define a $\gamma \in \mathrm{Hom}(G, B)$ with $\alpha \circ \gamma = \varphi$.

Without loss we prove the claim for $i = 0$. To this end note that for each j, $(\varphi\colon G \to A,\ \alpha_{j,0}\colon B/N_j \to A)$ is a finite $\mathcal{G}$-embedding problem of G. Indeed, given $\Gamma \in \mathcal{G}$, there is a homomorphism $\gamma'\colon \Gamma \to B$ with $\alpha \circ \gamma' = \varphi|_\Gamma$. Thus, $\alpha_{j,0} \circ (\alpha_j \circ \gamma') = \varphi|_\Gamma$, as desired.

For each $\beta \in \mathrm{Hom}(G, B/N_j)$ let

$$\beta \circ \prod_{\Gamma \in \mathcal{C}} \mathrm{Hom}(\Gamma, G) = \{(\beta \circ \psi_\Gamma)_{\Gamma \in \mathcal{C}} \mid \psi_\Gamma \in \mathrm{Hom}(\Gamma, G) \text{ for each } \Gamma \in \mathcal{C}\}.$$

This is a subset of $\prod_{\Gamma \in \mathcal{C}} \mathrm{Hom}(\Gamma, B/N_j)$. Since $\mathcal{C}$ is finite, each $\Gamma \in \mathcal{C}$ is finitely generated, and B/N_j is finite, $\prod_{\Gamma \in \mathcal{C}} \mathrm{Hom}(\Gamma, B/N_j)$ is finite. Hence, the collection of subsets

$$\mathcal{H}_j = \{\beta \circ \prod_{\Gamma \in \mathcal{C}} \mathrm{Hom}(\Gamma, G) \mid \beta \in \mathrm{Hom}(G, B/N_j),\ \alpha_{j,0} \circ \beta = \varphi\}$$

of $\prod_{\Gamma \in \mathcal{C}} \mathrm{Hom}(\Gamma, B/N_j)$ is finite. Since G is $\mathcal{G}$-projective, $\mathcal{H}_j$ is nonempty.

The map $\beta \circ \prod_{\Gamma \in \mathcal{C}} \mathrm{Hom}(\Gamma, G) \mapsto \alpha_{j+1,j} \circ \beta \circ \prod_{\Gamma \in \mathcal{C}} \mathrm{Hom}(\Gamma, G)$ maps $\mathcal{H}_{j+1}$ into $\mathcal{H}_j$. Hence, $\varprojlim \mathcal{H}_j \ne \emptyset$. Thus, there are homomorphisms $\beta_j\colon G \to B/N_j$ with $\alpha_{j,0} \circ \beta_j = \varphi$ and

$$\alpha_{j+1,j} \circ \beta_{j+1} \circ \prod_{\Gamma \in \mathcal{C}} \mathrm{Hom}(\Gamma, G) = \beta_j \circ \prod_{\Gamma \in \mathcal{C}} \mathrm{Hom}(\Gamma, G), \quad j = 0, 1, 2, \ldots. \tag{2}$$

In particular, $\alpha_{1,0} \circ \beta_1 = \varphi$.

We prove that $(\beta_1\colon G \to B/N_1,\ \alpha_1\colon B \to B/N_1)$ is a $\mathcal{G}$-embedding problem for G. To this end consider $\Gamma \in \mathcal{C}$ and $H \in \mathcal{G}_\Gamma$. Then $H \cong \Gamma$. Hence, by (2),

$$\alpha_{j+1,j} \circ \beta_{j+1} \circ \mathrm{Hom}(H,G) = \beta_j \circ \mathrm{Hom}(H,G), \quad j = 0,1,2,\ldots, . \tag{3}$$

Use (3) to inductively construct homomorphisms $\eta_j\colon H \to B/N_j$, $j = 1,2,\ldots$ with $\eta_1 = \beta_1|_H$ and $\alpha_{j+1,j} \circ \eta_{j+1} = \eta_j$. The η_j's define a homomorphism $\eta\colon H \to B$ with $\alpha_1 \circ \eta = \beta_1$, as needed. This concludes the proof of the claim. ∎

Lemma 6.2: *Let $\mathcal{C}$ be a finite set of finitely generated profinite groups, G a profinite group, and $\mathcal{G}$ a subset of* $\mathrm{Subgr}(G)$ *of type $\mathcal{C}$. Consider a finite $\mathcal{G}$-embedding problem with local data for G*

$$(\varphi\colon G \to A,\ \alpha\colon B \to A,\ \mathcal{B}). \tag{4}$$

Then there are

(5a) *a positive integer e,*
(5b) *a finite set $\{\Delta_\lambda \mid \lambda \in \Lambda\}$ of groups of type $\mathcal{C}$, and*
(5c) *an epimorphism $\beta\colon B^* = \hat{F}_e * \mathop{\ast}_{\lambda\in\Lambda} \Delta_\lambda \to B$ such that*

$$(\varphi\colon G \to A,\ \alpha \circ \beta\colon B^* \to A) \tag{6}$$

is a $\mathcal{G}$-embedding problem for G with A finite, $\mathrm{rank}(B^*) \le \aleph_0$*, and $\beta(\Delta_\lambda) \in \mathcal{B}$ for each $\lambda \in \Lambda$.*

Proof: The proof has two parts.

Part A: *Free product.* For each $\Gamma \in \mathcal{C}$ let Λ_Γ be the set of all homomorphisms $\lambda\colon \Gamma \to B$ satisfying
(7) $\lambda(\Gamma) \in \mathcal{B}$ and there is an embedding $\varepsilon\colon \Gamma \to G$ such that $\alpha \circ \lambda = \varphi \circ \varepsilon$.

Since Γ is finitely generated and B is finite, Λ_Γ is a finite set. For each $\lambda \in \Lambda_\Gamma$ choose an isomorphic copy Δ_λ of Γ and an isomorphism $\delta_\lambda\colon \Delta_\lambda \to \Gamma$. Let $\Lambda = \dot{\bigcup}_{\Gamma\in\mathcal{C}} \Lambda_\Gamma$. Then $\{\Delta_\lambda \mid \lambda \in \Lambda\}$ is a finite set of groups of type $\mathcal{C}$. Put $e = \mathrm{rank}(B)$. Choose an epimorphism $\beta_e\colon \hat{F}_e \to B$. Then consider the free product $B^* = \hat{F}_e * \mathop{\ast}_{\lambda\in\Lambda} \Delta_\lambda$. Let $\beta\colon B^* \to B$ be the unique epimorphism whose restriction to $\hat{F}_e$ is β_e and to Δ_λ is $\lambda \circ \delta_\lambda$. By (7), $\beta(\Delta_\lambda) \in \mathcal{B}$ for all $\lambda \in \Lambda$.

Part B: *$\mathcal{G}$-embedding problem.* Let $\Gamma \in \mathcal{C}$ and $H \in \mathcal{G}_\Gamma$. Then, there is an isomorphism $\theta\colon \Gamma \to H$. The definition of embedding problems with local data (Section 4) gives a homomorphism η of H into B such that $\eta(H) \in \mathcal{B}$ and $\alpha \circ \eta = \varphi \circ \iota$, where ι is the inclusion $H \to G$. Put $\lambda = \eta \circ \theta$ and $\varepsilon = \iota \circ \theta$. Then $\alpha \circ \lambda = \varphi \circ \varepsilon$, so $\lambda \in \Lambda_\Gamma$. Thus, $\delta_\lambda^{-1} \circ \theta^{-1}$ maps H onto the subgroup Δ_λ of B^*. Furthermore, $\alpha \circ \beta \circ (\delta_\lambda^{-1} \circ \theta^{-1}) = \alpha \circ (\lambda \circ \delta_\lambda) \circ (\delta_\lambda^{-1} \circ \theta^{-1}) = \alpha \circ \lambda \circ \theta^{-1} = \alpha \circ \eta \circ \theta \circ \theta^{-1} = \varphi \circ \iota$. Therefore, (6) is a $\mathcal{G}$-embedding problem for G. ∎

LEMMA 6.3: *Let G be a profinite group and $\mathcal{G}$ a subset of* $\mathrm{Subgr}(G)$ *of type $\mathcal{C}$. For each $\Gamma \in \mathcal{C}$ let $\bar{\Gamma}$ be a finite quotient of Γ. Suppose $\mathcal{G}_\Gamma$ is strictly closed in* $\mathrm{Subgr}(G)$. *Then G has an open normal subgroup N satisfying: For each $\Gamma \in \mathcal{C}$ and each $H \in \mathcal{G}_\Gamma$ the group $\bar{\Gamma}$ is a quotient of $H/H \cap N = HN/N$.*

Proof: Let $\Gamma \in \mathcal{C}$ and $H \in \mathcal{G}_\Gamma$. Then $H \cong \Gamma$, so H has an open normal subgroup M_H with $H/M_H \cong \bar{\Gamma}$. Choose $N_H \in \mathrm{OpenNormal}(G)$ with $H \cap N_H \le M_H$. Let $\mathcal{U}_H = \{H' \in \mathcal{G}_\Gamma \mid H'N_H = HN_H\}$. Then $\mathcal{U}_H$ is a strictly open neighborhood of H in $\mathcal{G}_\Gamma$. By assumption, $\mathcal{G}_\Gamma$ is strictly compact. Hence, there are $H_{\Gamma,1}, \ldots, H_{\Gamma,m(\Gamma)} \in \mathcal{G}_\Gamma$ with $\mathcal{G}_\Gamma = \bigcup_{i=1}^{m(\Gamma)} \mathcal{U}_{H_{\Gamma,i}}$.

Let $N = \bigcap_{\Gamma\in\mathcal{C}} \bigcap_{i=1}^{m(\Gamma)} N_{H_{\Gamma,i}}$. Consider $\Gamma \in \mathcal{C}$ and $H \in \mathcal{G}_\Gamma$. Then there is i with $HN_{H_{\Gamma,i}} = H_{\Gamma,i}N_{H_{\Gamma,i}}$. By construction, $N \le N_{H_{\Gamma,i}}$. This gives a sequence $H/H \cap N \longrightarrow H/H \cap N_{H_{\Gamma,i}} \cong HN_{H_{\Gamma,i}}/N_{H_{\Gamma,i}} = H_{\Gamma,i}N_{H_{\Gamma,i}}/N_{H_{\Gamma,i}} \cong H_{\Gamma,i}/H_{\Gamma,i} \cap N_{H_{\Gamma,i}} \longrightarrow H_{\Gamma,i}/M_{H_{\Gamma,i}} \cong \bar{\Gamma}$ where the arrows are epimorphisms. Therefore, $\bar{\Gamma}$ is a quotient of $H/H \cap N$. ∎

Definition 6.4: Big quotients. For each $\Gamma \in \mathcal{C}$ let $\bar{\Gamma}$ be a finite quotient of Γ. We say $\{\bar{\Gamma} \mid \Gamma \in \mathcal{C}\}$ is a **system of big quotients** for $\mathcal{C}$ if it has the following property: Let e be a nonnegative integer, J a finite set, and for each $j \in J$ let Δ_j be a profinite group of type $\mathcal{C}$. Consider the free product $B^* = \hat{F}_e * \coprod^*_{j\in J} \Delta_j$. Let $\Gamma \in \mathcal{C}$ and let Δ be a closed subgroup of B^* with epimorphisms $\Gamma \xrightarrow{\gamma} \Delta \to \bar{\Gamma}$. Then Δ is conjugate to a closed subgroup of some Δ_j and γ is an isomorphism. ∎

PROPOSITION 6.5: *Let $\mathcal{C}$ be a finite set of finitely generated groups, G a profinite group, and $\mathcal{G}$ a G-invariant subset of* $\mathrm{Subgr}(G)$ *of type $\mathcal{C}$. Suppose $\mathcal{C}$ has a system of finite big quotients and $\mathcal{G}_\Gamma$ is strictly closed in* $\mathrm{Subgr}(G)$ *for each $\Gamma \in \mathcal{C}$, and G is $\mathcal{G}$-projective. Then:*

(a) *G is strongly $\mathcal{G}$-projective.*

(b) *There is a homomorphism $\delta\colon G \to \coprod^*_{\Gamma\in\mathcal{C}} \Gamma$ which maps each $H \in \mathcal{G}$ injectively into a conjugate of some $\Gamma \in \mathcal{C}$.*

(c) *Suppose in addition, $1 \notin \mathcal{C}$. Then $(G, \mathcal{G}_{\max})$ is a proper projective group structure.*

Proof of (a): By assumption, $\mathcal{G} = \bigcup_{\Gamma\in\mathcal{C}} \mathcal{G}_\Gamma$ is strictly closed. By [HJPa, Remark 1.2], $\mathcal{G}$ is étale compact. It remains to solve a finite $\mathcal{G}$-embedding problem for G with local data (4). Let $\{\bar{\Gamma} \mid \Gamma \in \mathcal{C}\}$ be a system of finite big quotients for $\mathcal{C}$.

PART A: *$\bar{\Gamma}$ is a quotient of $\varphi(H)$ for each $\Gamma \in \mathcal{C}$ and each $H \in \mathcal{G}$.* Lemma 6.3 gives $N \in \mathrm{OpenNormal}(G)$ such that $\bar{\Gamma}$ is a quotient of $H/H \cap N$ for all $\Gamma \in \mathcal{C}$ and $H \in \mathcal{G}_\Gamma$. We may assume $N \le \mathrm{Ker}(\varphi)$, otherwise replace N with $N \cap \mathrm{Ker}(\varphi)$. Put $A' = G/N$. Let $\varphi'\colon G \to A'$ be the quotient map and $\bar{\varphi}\colon A' \to A$ the map induced by φ. Then $\bar{\Gamma}$ is a quotient of $\varphi'(H)$ for all $\Gamma \in \mathcal{C}$ and $H \in \mathcal{G}_\Gamma$. Also, $\varphi = \bar{\varphi} \circ \varphi'$. Put $B' = B \times_A A'$ and let $\alpha'\colon B' \to A'$ and $\beta\colon B' \to B$ be the canonical projections.

Put $\mathcal{B}'_0 = \{B_0 \times_A \varphi'(H) \mid B_0 \in \mathcal{B},\ H \in \mathcal{G},\ \alpha(B_0) = \varphi(H)\}$ and $\mathcal{B}' = \mathrm{Con}(\mathcal{B}'_0)$. Then $\varphi'(\mathcal{G}) \subseteq \alpha'(\mathcal{B}')$. By definition, α is injective on each $B_0 \in \mathcal{B}$. Therefore, α' is injective on each $B'_0 \in \mathcal{B}'_0$, hence on each $B'_0 \in \mathcal{B}'$. Thus,

$$(\varphi'\colon G \to A',\ \alpha'\colon B' \to A',\ \mathcal{B}') \tag{8}$$

is a finite $\mathcal{G}$-embedding problem with local data for $\mathcal{G}$.

Since $\beta(\mathcal{B}') \subseteq \mathcal{B}$, any solution γ' of (8) gives rise to a solution $\beta \circ \gamma'$ of embedding problem (4). Thus, replacing (4) with (8), if necessary, we may assume $\bar{\Gamma}$ is a quotient of $\varphi(H)$ for each $\Gamma \in \mathcal{C}$ and each $H \in \mathcal{G}$.

PART B: *Solving embedding problem (6).* Lemma 6.2 gives a $\mathcal{G}$-embedding problem (6) for G with $\mathrm{rank}(B^*) \le \aleph_0$, and $\beta(\Delta_\lambda) \in \mathcal{B}$ for each $\lambda \in \Lambda$. Since G is $\mathcal{G}$-projective, Lemma 6.1 gives a homomorphism $\gamma^*\colon G \to B^*$ with $\alpha \circ \beta \circ \gamma^* = \varphi$. We claim that $\beta \circ \gamma^*$ solves (4).

Let $\Gamma \in \mathcal{C}$ and $H \in \mathcal{G}_\Gamma$. Put $\Delta = \gamma^*(H)$. Then Δ is a subgroup of B^* as well as a quotient of Γ. Moreover, $\alpha(\beta(\Delta)) = \varphi(H)$. Therefore, by Part A, $\bar{\Gamma}$ is a quotient of Δ. By the definition of big quotients, Δ is conjugate to a closed subgroup of Δ_λ for some $\lambda \in \Lambda$ and γ^* is injective on H. Since $\beta(\Delta_\lambda) \in \mathcal{B}$, we have $\beta \circ \gamma^*(H) \in \mathcal{B}$.

Proof of (b): Consider embedding problem (4) with $B = A = 1$. The proof of (a) gives a homomorphism $\gamma^*\colon G \to \hat{F}_e * \coprod_{\lambda \in \Lambda} \Delta_\lambda$ which maps each $H \in \mathcal{G}$ isomorphically into a conjugate of some Δ_λ. Define a homomorphism $\delta^*\colon \hat{F}_e * \coprod_{\lambda \in \Lambda} \Delta_\lambda \to \coprod_{\Gamma \in \mathcal{C}} \Gamma$ which is the trivial map on $\hat{F}_e$ and maps each $\Delta_\lambda \in \mathcal{G}_\Gamma$ isomorphically onto Γ. Then $\delta = \delta^* \circ \gamma^*$ is a homomorphism of G into $\coprod_{\Gamma \in \mathcal{C}} \Gamma$ which maps each $H \in \mathcal{G}$ injectively into a conjugate of some $\Gamma \in \mathcal{C}$.

Proof of (c): Since $\Gamma \neq 1$ for each $\Gamma \in \mathcal{C}$ and $\mathcal{G}$ is strictly closed, $1 \notin \mathrm{StrictClosure}(\mathcal{G})$. By (a), G is strongly $\mathcal{G}$-projective. Therefore, by Proposition 5.4, $(G, \mathcal{G}_{\max})$ is a proper projective group structure. ∎

7. P-adically Closed Fields

We prove in the next section that any finite family of absolute Galois groups of P-adically closed fields has a system of big quotients. This section gives the necessary prerequisites for the proof.

Let p be a prime number. Denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{Q}_p$ by $\mathbb{Q}_{p,\mathrm{abs}}$. It is well defined up to an isomorphism.

Let (L, v) be a valued field. Call (L, v) **P-adic** if there is a prime number p satisfying these conditions:

(1a) The residue field $\bar{L}_v$ is finite, say with $q = p^f$ elements.

(1b) There is $\pi \in L$ with a smallest positive value $v(\pi)$ in $v(L^\times)$. Call π a **prime element** of (L, v).

(1c) There is a positive integer e with $v(p) = ev(\pi)$.

Refer to (p, e, f) as the **type** of (L, v) and to p as the **residue characteristic** of (L, v). We say (L, v) is **P-adically closed** if (L, v) admits no finite proper P-adic extension of the same type. Refer to a field L as **P-adically closed** if L admits a valuation v with (L, v) P-adically closed.

Remark 7.1: Comparison with former definitions. Prestel and Roquette [PrR] use "p-adically closed" instead of "P-adically closed of residue characteristic p". The same expression, "p-adically closed", is used in [HaJ2] for "P-adically closed field of type $(p,1,1)$". ∎

The proposition below summarizes well known facts about P-adically closed fields (see also [Pop1, Sec. 1]). We use $F \equiv F'$ to denote elementary equivalence between fields and $(F,v) \equiv (F',v')$ to denote elementary equivalence between valued fields.

PROPOSITION 7.2:
(a) *$\mathbb{Q}_p$ is P-adically closed of type $(p,1,1)$.*
(b) *Every P-adically closed valued field (L,v) is Henselian of characteristic 0.*
(c) *A field L is P-adically closed for at most one P-adic valuation.*
(d) *Suppose (K,v) is a P-adic field. Then (K,v) has a P-adically closed algebraic extension (L,w) of the same type. Call (L,w) a* **P-adic closure** *of (K,v). If (K,v) is discrete, then (L,w) is uniquely determined up to a K-isomorphism.*
(e) *In the notation of (d), L is minimal among all P-adically closed extensions of K.*
(f) *Let K be a subfield of a P-adically closed field L. Suppose K is algebraically closed in L. Then, K is a P-adically closed field of the same type as L and $K \equiv L$. Moreover, the restriction of the P-adic valuation of L to K is the P-adic valuation of K.*
(g) *A P-adic field (L,v) is P-adically closed if and only if (L,v) is Henselian and $v(L^\times)/nv(L^\times) \cong \mathbb{Z}/n\mathbb{Z}$ for every positive integer n.*
(h) *Suppose a field L is elementarily equivalent (in the language of fields) to a P-adically closed field F. Then L is a P-adically closed field of the same type as F. Moreover, let v (resp. w) be the P-adic valuation of L (resp. F). Then $(L,v) \equiv (F,w)$.*
(i) *Every finite extension of a P-adically closed field is a P-adically closed field of the same residue characteristic.*
(j) *Every P-adically closed field of residue characteristic p is elementarily equivalent to a finite extension of $\mathbb{Q}_{p,\mathrm{abs}}$ and also to a finite extension of $\mathbb{Q}_p$.*
(k) *Let L be a P-adically closed field. Then $\mathrm{Gal}(L)$ is a finitely generated prosolvable group.*
(l) *Let L be a P-adically closed field and $L_0 = L \cap \tilde{\mathbb{Q}}$. Then $\mathrm{res}\colon \mathrm{Gal}(L) \to \mathrm{Gal}(L_0)$ is an isomorphism.*
(m) *Let F be a P-adically closed field and F' an arbitrary field. Suppose $F' \equiv F$. Then $\mathrm{Gal}(F') \cong \mathrm{Gal}(F)$.*
(n) *Let F be a P-adically closed field and F' an arbitrary field. Suppose $\mathrm{Gal}(F') \cong \mathrm{Gal}(F)$. Then F' is a P-adically closed field of the same type as F.*

Proof of (a): Let K be a finite proper extension of $\mathbb{Q}_p$. Then $[K:\mathbb{Q}_p] = ef$ where e is the ramification index and f is the residue degree [CaF, p. 19, Prop. 3]. In particular, $e = v(p)$, where v is the unique normalized p-adic valuation of K. Also, the residue fields of $\mathbb{Q}_p$ and K are $\mathbb{F}_p$ and $\mathbb{F}_{p^f}$, respectively. Hence, $e > 1$ or $f > 1$. This proves (a).

Proof of (b): That (L,v) is Henselian is stated in [PrR, p. 34, Thm. 3.1]. By (1a), $p = \mathrm{char}(\bar{L}_v)$ is a prime number. Hence, either $\mathrm{char}(L) = 0$ or $\mathrm{char}(L) = p$. By (1b) and (1c), $v(p) \neq 0$. Therefore, $\mathrm{char}(L) = 0$.

Proof of (c): Suppose v and v' are P-adic valuations of L. By (b), both are Henselian. Since their residue fields are not separably closed (by (1a)), F.K. Schmidt - Engler [Jar2, Prop. 13.4] implies v is equivalent to v'.

Proof of (d): See [PrR, p. 37, Thm. 3.2].

Proof of (e): Let (L_0, w_0) be a P-adically closed field with $K \subseteq L_0 \subseteq L$. Denote the unique P-adically closed valuation of L by w. By (b), both (L, w) and (L_0, w_0) are Henselian. Extend w_0 to a valuation w_1 of L. Then (L, w_1) is Henselian.

Assume w_1 is inequivalent to w. By (1a), both $\bar{L}_w$ and $\bar{L}_{w_1}$ are algebraic extensions of finite fields. Hence, none of them is the residue field of a nontrivial valuation of the other. This means, w and w_1 are incomparable. Hence, by F.K. Schmidt - Engler [Jar, Prop. 13.4], $\bar{L}_w$ is separably closed , in contradiction to (1a). Therefore, w and w_1 are equivalent.

Thus, (L, w) extends (L_0, w_0) and (L_0, w_0) extends (K, v). Since (L, w) and (K, v) have the same type, also (L_0, w_0) has the same type. In particular, the residue characteristic of (L_0, w_0) is p. Since (L_0, w_0) is P-adically closed, $(L_0, w_0) = (L, w)$, as contended.

Proof of (f): [PrR, p. 38, Thm. 3.4] says K is P-adically closed of the same type as L. Moreover, the P-adic valuation of K is the restriction of the P-adic valuation of L. By [PrR, p. 86, Thm. 5.1], $K \equiv L$.

Proof of (g): See [PrR, p. 34, Thm. 3.1].

Proof of (h): Denote the P-adic valuation of F by v. Let (p, e, f) be the type of (F, v) and π a prime element F. Consider the Kochen operator

$$\gamma(X) = \frac{1}{\pi} \frac{X^q - X}{(X^q - X)^2 - 1},$$

with $q = p^f$. Put $\gamma(F) = \{\gamma(x) \mid x \in F \text{ and } x^q - x \neq \pm 1\}$. By [JaR, Lemma 4.1(iii)], $\gamma(F)$ is the valuation ring of v.

Since $L \equiv F$, $\gamma(L)$ is a valuation ring of L. Denote the corresponding valuation by w. Then $(L, w) \equiv (F, v)$. Since (F, v) satisfies (1), so does (L, w). Thus, (L, w) is P-adic.

Finally note: The conditions of (g) on a P-adic field to be P-adically closed are elementary in the language of valued fields. Consequently, (L, w) is P-adically closed.

Proof of (i): Let (L, v) be a P-adically closed field and L' a finite extension of L. Since L is Henselian, v uniquely extends to a valuation v of L' and (L', v) is Henselian. Since both $[\bar{L}'_v : \bar{L}_v]$ and $\big(v((L')^\times) : v(L^\times)\big)$ are finite, (L', v) is a P-adic valued field and $v((L')^\times)/nv((L')^\times)) \cong \mathbb{Z}/n\mathbb{Z}$ for every positive integer n. By (g), L' is P-adically closed.

Proof of (j): Let (L, v) be a P-adically closed field of residue characteristic p. By (b), $\mathrm{char}(L) = 0$. Put $L_0 = L \cap \tilde{\mathbb{Q}}$. By (f), L_0 is a P-adically closed field of the same type as L and $L_0 \equiv L$.

Let v_0 be the P-adic valuation of L_0. By (b), (L_0, v_0) is Henselian. Moreover, $v_0|_{\mathbb{Q}}$ is the p-adic valuation v_p of $\mathbb{Q}$. Hence, $\mathbb{Q}_{p,\mathrm{abs}} \subseteq L_0$. The relation $[K : \mathbb{Q}_{p,\mathrm{abs}}] = ef$ for finite extensions $K/\mathbb{Q}_{p,\mathrm{abs}}$ and the finiteness of the type of L_0 imply $[L_0 : \mathbb{Q}_{p,\mathrm{abs}}] < \infty$.

It follows that $F = L_0\mathbb{Q}_p$ is a finite extension of $\mathbb{Q}_p$ with $F \cap \tilde{\mathbb{Q}} = L_0$. By (i), F is P-adically closed. By (f), $F \equiv L_0$. Consequently, $F \equiv L$.

Proof of (k) and (l): Let L_0 and F be as in the proof of (j). By [Jan, Satz 3.6], $\mathrm{Gal}(F)$ is finitely generated (see also [JRi2, p. 2]). By [CaF, p. 31, Cor. 1], $\mathrm{Gal}(F)$ is prosolvable. By Krasner's Lemma, $\tilde{\mathbb{Q}}\mathbb{Q}_p = \tilde{\mathbb{Q}}_p$, so res: $\mathrm{Gal}(F) \to \mathrm{Gal}(L_0)$ is an isomorphism and $\mathrm{Gal}(L_0)$ is finitely generated. Since $L \equiv L_0$, every finite quotient G of $\mathrm{Gal}(L)$ is a finite quotient of $\mathrm{Gal}(L_0)$ (as the proof of [FrJ, Prop. 18.12] shows). It follows from [FrJ, Props. 15.3 and 15.4] that the epimorphism res: $\mathrm{Gal}(L) \to \mathrm{Gal}(L_0)$ is an isomorphism. Consequently, $\mathrm{Gal}(L)$ is prosolvable and finitely generated.

Proof of (m): By (h), F' is a P-adically closed field. Let $F_0 = F \cap \tilde{\mathbb{Q}}$ and $F_0' = F' \cap \tilde{\mathbb{Q}}$. Then $F_0 \equiv F_0'$. Hence, $F_0 \cong F_0'$ [FrJ, Lemma 18.19]. By (l), $\mathrm{Gal}(F) \cong \mathrm{Gal}(F_0)$ and $\mathrm{Gal}(F') \cong \mathrm{Gal}(F_0')$. Therefore, $\mathrm{Gal}(F) \cong \mathrm{Gal}(F')$.

Proof of (n): Efrat [Efr, Thm. A] (in the case $p \neq 2$) and Koenigsmann [Koe1, Thm. 4.1] (in general) construct a Henselian valuation v' of F' with $\mathrm{char}(\bar{F}'_{v'}) \neq 0$. It follows from [Pop1, E9] that F' is P-adically closed. Moreover, if F is a finite extension of $\mathbb{Q}_p$, then so is F'. ∎

LEMMA 7.3: *For each prime number p the group $\mathrm{Gal}(\mathbb{Q}_p)$ is torsion free.*

Proof: For $p \neq 2$ there is $x \in \mathbb{Q}_p$ with $x^2 + p - 1 = 0$ (Hensel's lemma). For $p = 2$ there is $x \in \mathbb{Q}_2$ with $x^2 + 7 = 0$ (use Hensel-Rychlik). Since both $p - 1$ and 7 are sums of squares (namely $1^2 + \cdots + 1^2$), $\mathbb{Q}_p$ is not formally real. Therefore, by Artin-Schreier, $\mathrm{Gal}(\mathbb{Q}_p)$ is torsion free. ∎

We summarize some well known facts about real closed fields and algebraically closed fields of characteristic 0.

Remark 7.4: Algebraically closed and real closed fields. Let $\mathbb{F}$ be a finite extension of $\mathbb{R}$. Then either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Suppose $F \equiv \mathbb{F}$. If $\mathbb{F} = \mathbb{R}$, then F is real closed and $\mathrm{Gal}(F) = \mathbb{Z}/2\mathbb{Z}$. If $\mathbb{F} = \mathbb{C}$, then F is algebraically closed and $\mathrm{Gal}(F)$ is trivial. Now let K be a subfield of F. Then res: $\mathrm{Gal}(F) \to \mathrm{Gal}(F \cap \tilde{K})$ is an isomorphism and $F \equiv F \cap \tilde{K}$ [Pre, p. 53, Cor. 5.6 and p. 51, Cor. 5.3]. Conversely, if F' extends F and $F' \equiv F$, then $F' \cap \tilde{F} = F$. ∎

8. Construction of Big Quotients for Classical Groups

We say that a field $\mathbb{F}$ is **classical local of characteristic** 0 if $\mathbb{F}$ is either $\mathbb{R}$, $\mathbb{C}$, or a finite extension of $\mathbb{Q}_p$ for some p. A profinite group G is **classical local of characteristic** 0 if G is isomorphic to the absolute Galois group of a classical local field of characteristic 0.

Let $\mathcal{F}$ be a finite set of classical local fields of characteristic 0. Put

$$\mathcal{C} = \{\mathrm{Gal}(\mathbb{F}) \mid \mathbb{F} \in \mathcal{F}\}.$$

We have already mentioned that each $\Gamma \in \mathcal{C}$ is finitely generated and prosolvable (Proposition 7.2(k)). We use the next result together with Lemma 7.3 to equip $\mathcal{C}$ with a system of big quotients.

Notation 8.1: Let p be a prime number and G a profinite group. Denote the maximal pro-p quotient of G by $G(p)$. Let G_p be a p-Sylow subgroup of G. ∎

PROPOSITION 8.2: *Let p, l be prime numbers and $\mathbb{F}$ be a finite extension of $\mathbb{Q}_p$. Then $\mathbb{F}$ has a finite Galois extension $\mathbb{F}'$ with the following properties:*

(a) *Let Δ be a quotient of* $\mathrm{Gal}(\mathbb{F})$ *which has* $\mathrm{Gal}(\mathbb{F}'/\mathbb{F})$ *as a quotient. Then, Δ_p is not a free pro-p group and Δ_l is not a free pro-l group.*

(b) *Let p' be a prime number, L an algebraic extension of $\mathbb{Q}_{p'}$, and γ:* $\mathrm{Gal}(\mathbb{F}) \to \mathrm{Gal}(L)$ *an epimorphism. Suppose there is an epimorphism β:* $\mathrm{Gal}(L) \to \mathrm{Gal}(\mathbb{F}'/\mathbb{F})$*. Then $p = p'$, γ is an isomorphism, and* $[\mathbb{F} : \mathbb{Q}_p] = [L : \mathbb{Q}_p]$.

Proof: By Proposition 7.2(k), $\mathrm{Gal}(\mathbb{F})$ is finitely generated.

CONSTRUCTION OF $\mathbb{F}'$: Denote the compositum of all extensions of $\mathbb{F}$ of degree at most $\max(p-1, l-1)$ by E_0. In particular, E_0 contains the roots of unity ζ_p and ζ_l of order p and l, respectively.

By [HaJ2, Lemma 11.1], E_0 has a finite extension E_1 with this property:

(1) $\mathrm{rank}(\mathrm{Gal}(E_1'/E_0)(p)) = \mathrm{rank}(\mathrm{Gal}(E_0)(p))$ for every Galois extension E_1' of E_0 which contains E_1.

Since $\zeta_p \in E_0$, $\mathrm{Gal}(E_0)(p)$ is not a free pro-p group [Koc, p. 96, Satz 10.3]. By [HaJ2, Lemma 11.2], E_0 has a proper finite p-extension $E_{2,p}$ with this property:

(2a) For every Galois extension E_2' of E_0 containing $E_{2,p}$, the group $\mathrm{Gal}(E_2'/E_0)$ is not a free pro-p group.

Similarly, E_0 has a proper finite l-extension $E_{2,l}$ satisfying this:

(2b) For every Galois extension E_2' of E_0 containing $E_{2,l}$, the group $\mathrm{Gal}(E_2'/E_0)$ is not a free pro-l group.

Put $E_2 = E_{2,p}E_{2,l}$.

Since $\mathrm{Gal}(\mathbb{Q}_p)$ is finitely generated, $\mathbb{Q}_p$ has only finitely many extensions of degree $[\mathbb{F} : \mathbb{Q}_p]$. Let $L_1, \ldots, L_k$ be all extensions of $\mathbb{Q}_p$ satisfying this:

(3) $[L_j : \mathbb{Q}_p] = [\mathbb{F} : \mathbb{Q}_p]$, $\mathrm{Gal}(L_j)$ is a quotient of $\mathrm{Gal}(\mathbb{F})$, but $\mathrm{Gal}(L_j) \not\cong \mathrm{Gal}(\mathbb{F})$, $j = 1, \ldots, k$ (k may be 0).

For each j choose a finite Galois extension F_j of $\mathbb{F}$ such that $\mathrm{Gal}(F_j/\mathbb{F})$ is not a quotient of $\mathrm{Gal}(L_j)$ [FrJ, Prop. 15.4]. Let $\mathbb{F}'$ be the compositum of all extensions of $\mathbb{F}$ of degree at most $m = \max([E_i : \mathbb{F}], [F_j : \mathbb{F}])_{i=1,2;\, j=1,\ldots,k}$. Then $\mathbb{F}'$ is a finite Galois extension of $\mathbb{F}$ which contains $E_1, E_2, F_1, \ldots, F_k$.

PROOF OF (a): Let Δ be as in (a). Then $\Delta \cong \mathrm{Gal}(M/\mathbb{F})$ for some Galois extension M of $\mathbb{F}$. Since $\mathrm{Gal}(\mathbb{F}'/\mathbb{F})$ is a quotient of Δ, there is a Galois extension $\mathbb{F}''$ of $\mathbb{F}$ in M with $\mathrm{Gal}(\mathbb{F}''/\mathbb{F}) \cong \mathrm{Gal}(\mathbb{F}'/\mathbb{F})$. In particular, $\mathbb{F}''$ is the compositum of extensions of $\mathbb{F}$ of degree at most m, so $\mathbb{F}'' \subseteq \mathbb{F}'$. Since $[\mathbb{F}'' : \mathbb{F}] = [\mathbb{F}' : \mathbb{F}]$, we have $\mathbb{F}' = \mathbb{F}'' \subseteq M$.

Assume $\mathrm{Gal}(M/\mathbb{F})_p$ is a free pro-p group. Then $\mathrm{Gal}(M/E_0)_p$ is also a free pro-p group [FrJ, Cor. 20.38], so $\mathrm{cd}_p\mathrm{Gal}(M/E_0) \leq 1$ [Rib, p. 235, Thm. 6.5]. Therefore, by

[Rib, p, 255, Thm. 3.2], $\mathrm{Gal}(M/E_0)(p)$ is pro-p free. This contradiction to (2a) proves that $\mathrm{Gal}(M/\mathbb{F})_p$ is not a free pro-p group. Similarly, $\mathrm{Gal}(M/\mathbb{F})_l$ is not a free pro-l group.

Proof of (b): Let p', L, γ, and β be as in (b). Denote the fixed field of $\mathrm{Ker}(\gamma)$ (resp. $\mathrm{Ker}(\beta\circ\gamma)$) in $\tilde{\mathbb{Q}}_p$ by N (resp. F'). Then F' is a Galois extension of $\mathbb{F}$ in N satisfying $\mathrm{Gal}(F'/\mathbb{F}) \cong \mathrm{Gal}(\mathbb{F}'/\mathbb{F})$. In particular, F' is a compositum of extensions of $\mathbb{F}$ of degree at most m. Hence, $F' \subseteq \mathbb{F}'$. Since $[F' : \mathbb{F}] = [\mathbb{F}' : \mathbb{F}]$, we have $F' = \mathbb{F}'$.

By construction, $E_0 \subset E_{2,p} \subseteq \mathbb{F}' \subseteq N$ and $E_{2,p}/E_0$ is a proper p-extension, so p divides $[N : E_0]$. Let $E_0^{(p)}$ be the maximal pro-p extension of E_0. Then $\mathrm{Gal}(N \cap E_0^{(p)}/E_0)$ is the maximal pro-p quotient of $\mathrm{Gal}(N/E_0)$. Also, $E_{2,p} \subseteq N \cap E_0^{(p)}$. Hence, by (2a), $\mathrm{Gal}(N\cap E_0^{(p)}/E_0)$ is not a free pro-p group. It follows from [Rib, p. 255] that $\mathrm{cd}_p\mathrm{Gal}(N/E_0) > 1$.

Let L_0 be the fixed field of $\gamma(\mathrm{Gal}(E_0))$ in $\tilde{\mathbb{Q}}_{p'}$. Then $\mathrm{Gal}(N/E_0) \cong \mathrm{Gal}(L_0)$. Hence, by the preceding paragraph, $\mathrm{cd}_p\mathrm{Gal}(L_0) > 1$. This implies, $p^\infty \nmid [L_0 : \mathbb{Q}_{p'}]$ [Rib, p. 291–292]. Also, L_0 is the compositum of all extensions of L of degree at most $\max(p-1, l-1)$. In particular, $\zeta_p \in L_0$. By (1) applied to N instead of to E_1' and by [Neu, Satz 4]

(4) $\quad \mathrm{rank}(\mathrm{Gal}(L_0)(p)) = \mathrm{rank}(\mathrm{Gal}(N/E_0)(p)) = \mathrm{rank}(\mathrm{Gal}(E_0)(p)) = 2 + [E_0 : \mathbb{Q}_p].$

In particular, $\mathrm{rank}(\mathrm{Gal}(L_0)(p)) \geq 3$. Hence, $p' = p$ and $\mathrm{rank}(\mathrm{Gal}(L_0)(p)) = 2 + [L_0 : \mathbb{Q}_p]$ [Neu, Satz 4]. It follows from (4) that $[E_0 : \mathbb{Q}_p] = [L_0 : \mathbb{Q}_p]$. Since $[E_0 : \mathbb{F}] = [L_0 : L]$, this implies $[\mathbb{F} : \mathbb{Q}_p] = [L : \mathbb{Q}_p]$.

Finally assume $\mathrm{Gal}(L) \not\cong \mathrm{Gal}(\mathbb{F})$. By assumption, $\mathrm{Gal}(L)$ is a quotient of $\mathrm{Gal}(\mathbb{F})$. Hence, $L = L_j$ with $1 \leq j \leq m$. By construction, $\mathrm{Gal}(F_j/\mathbb{F})$ is not a quotient of $\mathrm{Gal}(L)$. Since $\mathbb{F} \subseteq F_j \subseteq \mathbb{F}'$, this implies $\mathrm{Gal}(\mathbb{F}'/\mathbb{F})$ is not a quotient of $\mathrm{Gal}(L)$, in contradiction to our assumption. Thus, $\mathrm{Gal}(\mathbb{F}) \cong \mathrm{Gal}(L)$. Consequently, by [FrJ, Prop. 15.3], γ is an isomorphism. ∎

The following result gives sufficient conditions for a finite set $\mathcal{C}$ of finitely generated profinite groups to have a system of big quotients (Definition 6.4):

Proposition 8.3: *Let $\mathcal{C}$ be a finite set of finitely generated profinite groups. Suppose each $\Gamma \in \mathcal{C}$ is finite or prosolvable. For each infinite $\Gamma \in \mathcal{C}$ let $\bar{\Gamma}$ be a finite quotient of Γ and for each finite $\Gamma \in \mathcal{C}$ let $\bar{\Gamma} = \Gamma$. Suppose there exists a prime number l such that for every infinite $\Gamma \in \mathcal{C}$ and every profinite group Δ with epimorphisms $\Gamma \xrightarrow{\gamma} \Delta \longrightarrow \bar{\Gamma}$ the following holds:*

(5a) *Γ_l is torsion free.*
(5b) *Δ_l is not a free pro-l group.*
(5c) *There is a prime number $p \neq l$ such that Δ_p is not a free pro-p group.*
(5d) *If Δ is isomorphic to a subgroup of some $\Gamma' \in \mathcal{C}$, then γ is an isomorphism.*

Then $\{\bar{\Gamma} \mid \Gamma \in \mathcal{C}\}$ is a system of big quotients for $\mathcal{C}$.

Proof: Let $\{\Delta_j \mid j \in J\}$ be a finite collection of profinite groups of type $\mathcal{C}$ and let e be a positive integer. Put $B^* = \hat{F}_e * \mathop{\ast}_{j\in J} \Delta_j$. Let Γ be a group in $\mathcal{C}$ and Δ a closed subgroup of B^* with epimorphisms $\Gamma \xrightarrow{\gamma} \Delta \to \bar{\Gamma}$.

First suppose Δ is finite. By [HeR, p. 160, Thm. 1], Δ is conjugate to a closed subgroup of some Δ_j. By (5d), γ is an isomorphism.

Now suppose Δ is prosolvable. Let l be a prime number as in the proposition. By the first paragraph and (5a) (applied to Δ_j instead of to Γ), no element of B^* has order l. In particular, Δ_l is torsion free. By [Pop3, Thm. 2(2)], Δ is conjugate to a subgroup of some Δ_j. Again, by (5d), γ is an isomorphism. ∎

LEMMA 8.4: *Let $\mathcal{F}$ be a finite set of classical local fields of characteristic* 0. *Put* $\mathcal{C} = \{\mathrm{Gal}(\mathbb{F}) \mid \mathbb{F} \in \mathcal{F}\}$. *Then $\mathcal{C}$ has a system of big quotients.*

Proof: Let S be the set of all residue characteristics of $\mathbb{F} \in \mathcal{F}$. Choose a prime number l not in $S \cup \{2\}$. By Proposition 7.2(k) and Remark 7.4, each $\Gamma \in \mathcal{C}$ is finitely generated and prosolvable. Moreover, Γ_l is torsion free (Lemma 7.3). Omit $\mathbb{C}$ from $\mathcal{F}$, if necessary, to assume $1 \notin \mathcal{C}$. For each $\Gamma \in \mathcal{C}$ choose $\mathbb{F} \in \mathcal{F}$ with $\Gamma \cong \mathrm{Gal}(\mathbb{F})$. If Γ is finite (i.e. $\mathbb{F} = \mathbb{R}$) choose $\bar{\Gamma} = \Gamma$. If Γ is infinite and $\mathbb{F}$ is a finite extension of $\mathbb{Q}_p$, let $\bar{\Gamma} = \mathrm{Gal}(\mathbb{F}'/\mathbb{F})$, where $\mathbb{F}'$ is the finite extension of $\mathbb{F}$ given by Proposition 8.2. We apply Lemma 7.3 to prove that $\{\bar{\Gamma} \mid \Gamma \in \mathcal{C}\}$ is a system of big quotients for $\mathcal{C}$ in the sense of Definition 6.4:

Since each $\Gamma \in \mathcal{C}$ is an absolute Galois group, Γ_l is torsion free. Let $\mathbb{F} \in \mathcal{F}$, $\Gamma = \mathrm{Gal}(\mathbb{F})$, Δ be a profinite group, and $\Gamma \xrightarrow{\gamma} \Delta \to \bar{\Gamma}$ epimorphisms. Let p be the residue characteristic of $\mathbb{F}$. By Proposition 8.2(a), Δ_p is not a free pro-p group and Δ_l is not a free pro-l group. Finally, suppose Δ is isomorphic to a subgroup of some $\Gamma' \in \mathcal{C}$. Identify Δ with $\mathrm{Gal}(L)$ where L is an algebraic extension of $\mathbb{Q}_{p'}$. By Proposition 8.2, γ is an isomorphism. Thus, all parts of Condition (4) hold. By Proposition 8.3, $\mathcal{C}$ has a system of big quotients. ∎

9. Spaces of Classically Local Fields

Let $\mathcal{F}$ be a finite set of classical local fields of characteristic 0 and let K be a field. For each $\mathbb{F} \in \mathcal{F}$ let $\mathrm{AlgExt}(K,\mathbb{F})$ be the set of all algebraic extensions of K which are elementarily equivalent to $\mathbb{F}$. Put $\mathrm{AlgExt}(K,\mathcal{F}) = \bigcup_{\mathbb{F}\in\mathcal{F}} \mathrm{AlgExt}(K,\mathbb{F})$. Call a field K **pseudo-$\mathcal{F}$-closed** (abbreviated P$\mathcal{F}$C) if it is pseudo-$\mathrm{AlgExt}(K,\mathcal{F})$-closed; that is, $V(K) \neq \emptyset$ for each smooth absolutely irreducible variety V satisfying $V(F) \neq \emptyset$ for all $F \in \mathrm{AlgExt}(K,\mathcal{F})$.

We call a profinite group G (**strongly**) **$\mathcal{F}$-projective**, if G is (strongly) $\mathcal{G}$-projective, where $\mathcal{G} = \bigcup_{\mathbb{F}\in\mathcal{F}} \{H \in \mathrm{Subgr}(G) \mid H \cong \mathrm{Gal}(\mathbb{F})\}$.

Our first main result is that "K is P$\mathcal{F}$C" implies "$\mathrm{Gal}(K)$ is strongly $\mathcal{F}$-projective". The only still missing ingredient of the proof is the strict closedness of $\mathrm{AlgExt}(K,\mathbb{F})$.

LEMMA 9.1: *Let K be a field and $\mathbb{F}$ a finite extension of $\mathbb{Q}_p$ or of $\mathbb{R}$. Then* $\mathrm{AlgExt}(K,\mathbb{F})$ *is strictly closed in* $\mathrm{AlgExt}(K)$.

Proof: We prove the theorem in the case where $\mathbb{F}$ is a finite extension of $\mathbb{Q}_p$. The same proof applies to the case where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. We only have to replace the references to Proposition 7.2 by references to Remark 7.4. By Proposition 7.2(f) and Remark 7.4, $\mathbb{F} \equiv \mathbb{F} \cap \tilde{\mathbb{Q}}$. Hence, we may replace $\mathbb{F}$ by $\mathbb{F} \cap \tilde{\mathbb{Q}}$.

So, assume without loss, $\mathbb{F}$ is a finite extension of $\mathbb{Q}_{p,\mathrm{alg}}$. By [FrJ, Lemma 18.19], $\mathrm{AlgExt}(\mathbb{Q},\mathbb{F}) = \{\mathbb{F}^\sigma \mid \sigma \in \mathrm{Gal}(\mathbb{Q})\}$. Thus, $\mathrm{AlgExt}(\mathbb{Q},\mathbb{F})$ is the image of the strictly continuous map $\mathrm{Gal}(\mathbb{Q}) \to \mathrm{AlgExt}(\mathbb{Q})$ given by $\sigma \mapsto \mathbb{F}^\sigma$. Since both spaces are profinite, $\mathrm{AlgExt}(\mathbb{Q},\mathbb{F})$ is strictly closed in $\mathrm{AlgExt}(\mathbb{Q})$.

Now put $\mathcal{X} = \mathrm{AlgExt}(K,\mathbb{F})$. Consider $F \in \mathrm{StrictClosure}(\mathcal{X})$. Then, for every finite Galois extension N of K, $\mathcal{W}_N = \{E \in \mathcal{X} \mid E \cap N = F \cap N\} \neq \emptyset$. By Proposition 7.2(m), $\mathrm{Gal}(E) \cong \mathrm{Gal}(\mathbb{F})$ for each $E \in \mathcal{X}$. Therefore,
(1) every finite quotient of $\mathrm{Gal}(F)$ is a finite quotient of $\mathrm{Gal}(\mathbb{F})$.

Conversely, let $F_0 = F \cap \tilde{\mathbb{Q}}$. Observe that the map φ: $\mathrm{AlgExt}(K) \to \mathrm{AlgExt}(\mathbb{Q})$ given by $L \mapsto L \cap \tilde{\mathbb{Q}}$ is strictly continuous. It maps $\mathrm{AlgExt}(K,\mathbb{F})$ into $\mathrm{AlgExt}(\mathbb{Q},\mathbb{F})$ (by Proposition 7.2(f)). Hence, $F_0 = \varphi(F) \in \mathrm{StrictClosure}(\mathrm{AlgExt}(\mathbb{Q},\mathbb{F}))$. By the second paragraph of the proof, $F_0 \in \mathrm{AlgExt}(\mathbb{Q},\mathbb{F})$. Thus, $F_0 \equiv \mathbb{F}$. Hence, by Proposition 7.2(m), $\mathrm{Gal}(F_0) \cong \mathrm{Gal}(\mathbb{F})$, so $\mathrm{Gal}(\mathbb{F})$ is an image of $\mathrm{Gal}(F)$. In particular, every finite quotient of $\mathrm{Gal}(\mathbb{F})$ is a finite quotient of $\mathrm{Gal}(F)$. Combining with (1), we conclude that $\mathrm{Gal}(F)$ and $\mathrm{Gal}(\mathbb{F})$ have the same finite quotients. By Proposition 7.2(k), $\mathrm{Gal}(\mathbb{F})$ is finitely generated. Hence, by [FrJ, Prop. 15.4], $\mathrm{Gal}(F) \cong \mathrm{Gal}(\mathbb{F})$. It follows that $\mathrm{Gal}(F)$ is finitely generated and isomorphic to $\mathrm{Gal}(F_0)$. Since res: $\mathrm{Gal}(F) \to \mathrm{Gal}(F_0)$ is surjective, it is bijective [FrJ, Prop. 15.3].

Next observe that the intersection of finitely many sets $\mathcal{W}_N$ contains a set of this form. Hence the intersection is nonempty. Therefore, there is an ultrafilter $\mathcal{D}$ of $\mathcal{X}$ which contains each $\mathcal{W}_N$. Put $F^* = \prod_{E\in\mathcal{X}} E/\mathcal{D}$. By the fundamental property of ultraproducts, $F^* \equiv \mathbb{F}$ [FrJ, Cor. 6.12].

Embed F in F^* by mapping each $x \in F$ onto the element $(x_E)/\mathcal{D}$ where x_E is x if $x \in E$ and $x_E = 0$ otherwise. Put $F_0^* = F^* \cap \tilde{\mathbb{Q}}$. Then, by Proposition 7.2(f), $F_0 \equiv \mathbb{F} \equiv F^* \equiv F_0^*$. Also, $F_0 \subseteq F_0^*$. Let $x \in F_0^*$. Put $f = \mathrm{irr}(x,\mathbb{Q})$. Since $F_0 \equiv F_0^*$, the number of roots of f in F_0 is equal to the number of roots of f in F_0^*. Hence, $x \in F_0$. Therefore, $F_0 = F_0^*$. By Proposition 7.2(l), res: $\mathrm{Gal}(F^*) \to \mathrm{Gal}(F_0)$ is an isomorphism. Hence, so is res: $\mathrm{Gal}(F^* \cap \tilde{K}) \to \mathrm{Gal}(F_0)$. Since $F \subseteq F^* \cap \tilde{K}$ and res: $\mathrm{Gal}(F) \to \mathrm{Gal}(F_0)$ is an isomorphism, we have $F = F^* \cap \tilde{K}$. Again, by Proposition 7.2(f), $F \equiv F^*$. Consequently, $F \in \mathcal{X}$, as desired. ∎

In order to formulate the first main result of this work we have to impose a certain restriction on $\mathcal{F}$.

Remark 9.2: Isomorphism of Galois groups of P-adic fields. We say $\mathcal{F}$ is **closed under Galois isomorphism** if for all classical local fields $\mathbb{F}, \mathbb{F}'$ the following holds:
(1) $\mathbb{F} \in \mathcal{F}$ and $\mathrm{Gal}(\mathbb{F}) \cong \mathrm{Gal}(\mathbb{F}')$ implies $\mathbb{F}' \in \mathcal{F}$.

Actually, by Remark 7.4, it suffices to impose Condition (2) only for a finite extension $\mathbb{F}$ of $\mathbb{Q}_p$ and a finite extension $\mathbb{F}'$ of $\mathbb{Q}_{p'}$. By Proposition 8.2, $\mathrm{Gal}(\mathbb{F}) \cong \mathrm{Gal}(\mathbb{F}')$ implies $p = p'$ and $[\mathbb{F}' : \mathbb{Q}_p] = [\mathbb{F} : \mathbb{Q}_p]$. So, for each $\mathbb{F} \in \mathcal{F}$ there are only finitely many fields $\mathbb{F}'$ with $\mathrm{Gal}(\mathbb{F}') \cong \mathrm{Gal}(\mathbb{F})$.

Section 2 of [JRi1] gives for each p examples of nonisomorphic extensions F and F' of $\mathbb{Q}_p$ with $\mathrm{Gal}(F) \cong \mathrm{Gal}(F')$. Indeed, [JRi1, p. 2, Thm.] and [Rit, p. 281, Thm.] prove for arbitrary finite extensions F, F' of $\mathbb{Q}_p$ (if $p = 2$, the theorem assumes $\sqrt{-1} \in F$)

that $\mathrm{Gal}(F) \cong \mathrm{Gal}(F')$ if and only if $[F : \mathbb{Q}_p] = [F' : \mathbb{Q}_p]$ and $F \cap \mathbb{Q}_{p,\mathrm{ab}} = F' \cap \mathbb{Q}_{p,\mathrm{ab}}$. Here $\mathbb{Q}_{p,\mathrm{ab}}$ is the maximal abelian extension of $\mathbb{Q}_p$.

Finally consider classical local fields F and F' of characteristic 0. Suppose F is elementarily equivalent to F'. Then F is isomorphic to F'. Indeed, we may assume F is a finite extension of $\mathbb{Q}_p$ and F' is a finite extension of $\mathbb{Q}_{p'}$. By 7.2(h), $p = p'$. Let $F_0 = F \cap \tilde{\mathbb{Q}}$ and $F_0' = F' \cap \tilde{\mathbb{Q}}$. By 7.2(f), $F_0 \equiv F$ and $F_0' \equiv F'$. Hence, by [FrJ, Lemma 18.19], $F_0 \cong F_0'$. We may therefore assume $F_0 = F_0'$ and F_0 is a finite extension of $\mathbb{Q}_{p,\mathrm{alg}}$. But then the isomorphism res: $\mathrm{Gal}(\mathbb{Q}_p) \to \mathrm{Gal}(\mathbb{Q}_{p,\mathrm{alg}})$ (see 7.2(l)) maps both $\mathrm{Gal}(F)$ and $\mathrm{Gal}(F')$ onto $\mathrm{Gal}(F_0)$. Consequently $F = F'$. ∎

Lemma 9.3: *Let $\mathcal{F}$ be a finite set of classical local fields of characteristic 0. Suppose $\mathcal{F}$ is closed under Galois isomorphism. Then for every field K we have*

$$(3) \qquad \bigcup_{\mathbb{F}\in\mathcal{F}} \{F \in \mathrm{AlgExt}(K) \mid F \equiv \mathbb{F}\} = \bigcup_{\mathbb{F}\in\mathcal{F}} \{F \in \mathrm{AlgExt}(K) \mid \mathrm{Gal}(F) \cong \mathrm{Gal}(\mathbb{F})\}.$$

Proof: By Proposition 7.2(m), the left hand side of (3) is contained in its right hand side. Conversely, let $F \in \mathrm{AlgExt}(K)$ and $\mathbb{F} \in \mathcal{F}$ be fields with $\mathrm{Gal}(F) \cong \mathrm{Gal}(\mathbb{F})$. If $\mathbb{F}$ is real closed, then so is F and $F \equiv \mathbb{F}$ (Remark 7.4). Otherwise, $\mathbb{F}$ is a finite extension of $\mathbb{Q}_p$ for some p. By Proposition 7.2(n), F is elementarily equivalent to a finite extension $\mathbb{F}'$ of $\mathbb{Q}_p$. Hence, by Proposition 7.2(m), $\mathrm{Gal}(\mathbb{F}') \cong \mathrm{Gal}(F) \cong \mathrm{Gal}(\mathbb{F})$. Since $\mathcal{F}$ is closed under Galois isomorphism, $\mathbb{F}' \in \mathcal{F}$. Consequently, F belongs to the left hand side of (3). ∎

Theorem 9.4: *Let $\mathcal{F}$ be a finite set of classical local fields of characteristic 0 not containing $\mathbb{C}$ which is closed under Galois isomorphism. Let K be a P$\mathcal{F}$C field. Put*

$$\mathcal{G} = \bigcup_{\mathbb{F}\in\mathcal{F}} \{\mathrm{Gal}(F) \mid F \in \mathrm{AlgExt}(K) \text{ and } \mathrm{Gal}(F) \cong \mathrm{Gal}(\mathbb{F})\}.$$

Then $\mathrm{Gal}(K)$ is strongly $\mathcal{F}$-projective and $(\mathrm{Gal}(K), \mathcal{G}_{\max})$ is a proper projective group structure.

Proof: Let $\mathcal{C} = \{\mathrm{Gal}(\mathbb{F}) \mid \mathbb{F} \in \mathcal{F}\}$. For each $\Gamma \in \mathcal{C}$ let

$$\mathcal{G}_\Gamma = \bigcup_{\substack{\mathbb{F}\in\mathcal{F} \\ \mathrm{Gal}(\mathbb{F})\cong\Gamma}} \{\mathrm{Gal}(F) \mid F \in \mathrm{AlgExt}(K, \mathbb{F})\}.$$

By Lemma 9.1, $\mathcal{G}_\Gamma$ is strictly closed in Subgr(Gal(K). By Lemma 9.3, $\mathcal{G} = \bigcup_{\Gamma\in\mathcal{C}} \mathcal{G}_\Gamma$, so $\mathcal{G}$ is strictly closed in $\mathrm{Subgr}(\mathrm{Gal}(K))$. Hence, by [HJPa, Remark 1.2], $\mathcal{G}$ is étale compact. By Proposition 3.1, $\mathrm{Gal}(K)$ is $\mathcal{G}$-projective. By Lemma 8.4, $\mathcal{C}$ has a system of big quotients. It follows from Proposition 6.5 that $\mathrm{Gal}(K)$ is strongly $\mathcal{G}$-projective and $(\mathrm{Gal}(K), \mathcal{G}_{\max})$ is a proper projective group structure. ∎

10. Realization of Strongly Projective Groups as Absolute Galois Groups

The second main result of this work is a converse to Theorem 9.4. We consider again a finite set $\mathcal{F}$ of classical local fields of characteristic 0 not containing $\mathbb{C}$. We prove that each $\mathcal{F}$-projective group G which satisfies the group theoretic analog of Lemma 9.1 is isomorphic to $\mathrm{Gal}(K)$ for some P$\mathcal{F}$C field K. Moreover, we construct K equipped with a "field-valuation structure" satisfying the "block approximation condition". We recall the definition of these concepts from [HJPa]:

A **field structure** is data $\mathbf{K} = (K, X, K_x)_{x\in X}$ where K is a field, X is a profinite space with a continuous action of $\mathrm{Gal}(K)$ on X, and for each $x \in X$, K_x is separable algebraic extension of K satisfying the following conditions:

(1a) For each finite separable extension L of K the set $X_L = \{x \in X \mid L \subseteq K_x\}$ is open.

(1b) $K_{x^\sigma} = K_x^\sigma$ for all $x \in X$ and $\sigma \in \mathrm{Gal}(K)$.

(1c) $\{\sigma \in \mathrm{Gal}(K) \mid x^\sigma = x\} \subseteq \mathrm{Gal}(K_x)$.

Thus, $\mathrm{Gal}(\mathbf{K}) = (\mathrm{Gal}(K), X, \mathrm{Gal}(K_x))_{x\in X}$ is a group structure called the **absolute Galois structure** associated with $\mathbf{K}$ [HJPa, Section 6].

Denote the set of all valuations, including the trivial one, of a field L by $\mathrm{Val}(L)$. A subbasis for the **patch topology** of $\mathrm{Val}(L)$ consists of all sets

$$\mathrm{Val}_a(K) = \{v \in \mathrm{Val}(K) \mid v(a) > 0\}, \qquad \mathrm{Val}'_a(K) = \{v \in \mathrm{Val}(K) \mid v(a) \geq 0\}$$

with $a \in K$.

A **field-valuation structure** is a structure $\mathbf{K} = (K, X, K_x, v_x)_{x\in X}$ satisfying the following conditions:

(2a) $(K, X, K_x)_{x\in x}$ is a field structure.

(2b) v_x is a valuation of K_x satisfying $v_{x^\sigma} = v_x^\sigma$ for all $x \in X$ and $\sigma \in \mathrm{Gal}(K)$. Here $v_x^\sigma(u^\sigma) = v_x(u)$ for each $u \in K_x$.

(2c) For each finite separable extension L of K define a map $\nu_L\colon X_L \to \mathrm{Val}(L)$ by $\nu_L(x) = v_x|_L$. Then ν_L is continuous.

We call **K Henselian** if (K_x, v_x) is Henselian for each $x \in X$.

The **absolute Galois structure** associated with $\mathbf{K}$ is the same associated with the underlying field structure, namely $\mathrm{Gal}(\mathbf{K}) = (\mathrm{Gal}(K), X, \mathrm{Gal}(K_x))_{x\in X}$. We call **K proper** if $\mathrm{Gal}(\mathbf{K})$ is proper.

Definition 10.1: Block approximation condition. A **block approximation problem** for a field-valuation structure $\mathbf{K} = (K, X, K_x, v_x)_{x\in X}$ is data $(V, X_i, L_i, \mathbf{a}_i, c_i)_{i\in I_0}$ satisfying this:

(3a) $(\mathrm{Gal}(L_i), X_i)_{i\in I_0}$ is a **special partition** of $\mathrm{Gal}(\mathbf{K})$: For each $i \in I_0$ the set X_i is open-closed in X, for all $x \in X_i$ we have $L_i \subseteq K_x$, $\mathrm{Gal}(L_i) = \{\sigma \in \mathrm{Gal}(K) \mid X_i^\sigma = X_i\}$, and $X = \dot\bigcup_{i\in I_0} \dot\bigcup_{\rho\in R_i} X_i^\rho$, where R_i is any subset of $\mathrm{Gal}(K)$ satisfying $\mathrm{Gal}(K) = \dot\bigcup_{\rho\in R_i} \mathrm{Gal}(L_i)\rho_i$.

(3b) V is a smooth affine variety over K.

(3c) $\mathbf{a}_i \in V(L_i)$.

(3d) $c_i \in K^\times$.

A **solution** of the problem is a point $\mathbf{a} \in V(K)$ with $v_x(\mathbf{a} - \mathbf{a}_i) > v_x(c_i)$ for all $i \in I_0$ and $x \in X_i$. We say $\mathbf{K}$ satisfies the **block approximation condition** if each block approximation problem for $\mathbf{K}$ is solvable. ∎

The block approximation condition has several interesting consequences:

PROPOSITION 10.2 ([HJPa, Proposition 12.3]): *Let $\mathbf{K} = (K, X, K_x, v_x)_{x \in X}$ be a Henselian field-valuation structure satisfying the block approximation condition.*
(a) *Put $\mathcal{K} = \{K_x \mid x \in X\}$. Then K is P$\mathcal{K}$C.*
(b) *Suppose $x_1, \ldots, x_n \in X$ lie in distinct* $\mathrm{Gal}(K)$*-orbits. Then $v_{x_1}|_K, \ldots v_{x_n}|_K$ satisfies the weak approximation theorem.*
(c) *Suppose $x, y \in X$ lie in distinct* $\mathrm{Gal}(K)$*-orbits. Then $v_x|_K$ and $v_y|_K$ are independent.*
(d) *Suppose X has more than one* $\mathrm{Gal}(K)$*-orbit. Then the trivial valuation is not in $\nu_K(X)$.*
(e) *For each $x \in X$, K is v_x-dense in K_x; and*
(f) *(K_x, v_x) is a Henselian closure of $(K, v_x|_K)$.*

THEOREM 10.3: *Let $\mathcal{F}$ be a finite set of classical local fields of characteristic* 0 *and G an $\mathcal{F}$-projective group. Let*

$$\mathcal{C} = \{\mathrm{Gal}(\mathbb{F}) \mid \mathbb{F} \in \mathcal{F}\} \quad \textit{and} \quad \mathcal{G} = \mathrm{Subgr}(G, \mathcal{C}) = \bigcup_{\Gamma \in \mathcal{C}} \mathrm{Subgr}(G, \Gamma).$$

Suppose:
(5a) $\mathbb{C} \notin \mathcal{F}$.
(5b) *$\mathcal{F}$ is closed under Galois isomorphism.*
(5c) $\mathrm{Subgr}(G, \Gamma)$ *is strictly closed in* $\mathrm{Subgr}(G)$ *for each $\Gamma \in \mathcal{C}$.*
Then there is a proper field-valuation structure $\mathbf{K} = (K, X, K_x, v_x)_{x \in X}$ such that:
(6a) **K** *satisfies the block approximation condition.*
(6b) *There is an isomorphism φ: $(G, \mathcal{G}_{\max}) \to \mathrm{Gal}(\mathbf{K})$; in particular $G \cong \mathrm{Gal}(K)$.*
(6c) $\{K_x \mid x \in X\} = \mathrm{AlgExt}(K, \mathcal{F})_{\min}$.
(6d) *K is P$\mathcal{F}$C.*

Proof: By Lemma 8.4, $\mathcal{C}$ has a system of finite big quotients. By Proposition 7.2(k), each $\Gamma \in \mathcal{C}$ is finitely generated and prosolvable. Finally, by assumption, G is $\mathcal{G}$-projective. Hence, by Proposition 6.5, $\mathbf{G} = (G, \mathcal{G}_{\max})$ is a proper projective group structure. Moreover, Proposition 6.5 gives a homomorphism δ: $G \to \mathop{\Large *}_{\Gamma \in \mathcal{C}} \Gamma$ which maps each $H \in \mathcal{G}$ injectively into a conjugate of some $\Gamma \in \mathcal{C}$. By assumption, each $\Gamma \in \mathcal{C}$ is the absolute Galois group of a Henselian algebraic extension of $\mathbb{Q}$ or a real closure of $\mathbb{Q}$. Therefore, by [Gey, Thm. 10.1], we may identify $\mathop{\Large *}_{\Gamma \in \mathcal{C}} \Gamma$ with $\mathrm{Gal}(D)$ for some algebraic extension field D of $\mathbb{Q}$. Let E be the fixed field of $\delta(G)$ in $\tilde{\mathbb{Q}}$. Then δ: $G \to \mathrm{Gal}(E)$ is an epimorphism of profinite groups which extends to a cover δ: $\mathbf{G} \to \mathrm{Gal}(\mathbf{E})$ of group structures, with $\mathbf{E}$ being a field structure whose underlying field is E. Indeed, E is the associated field structure to the quotient structure $(G, \mathcal{G}_{\min})/\mathrm{Ker}(\delta)$ [HJPa, Example 2.5]. Note that $\mathrm{Gal}(\mathbf{E})$ need not be proper.

Put $X = \mathcal{G}_{\max}$. By [HJPa, Thm. 15.4], there is a proper Henselian field-valuation structure $\mathbf{K} = (K, X, K_x, v_x)_{x\in X}$ which satisfies the block approximation condition and there is an isomorphism φ: $\mathbf{G} \to \mathrm{Gal}(\mathbf{K})$ such that

(7a) $E = K \bigcap \tilde{\mathbb{Q}}$, $E_{\delta(x)} = K_x \cap \tilde{\mathbb{Q}}$, and v_x is trivial on $E_{\delta(x)}$ for all $x \in X$.

(7b) $\mathrm{res}_{\tilde{K}/\tilde{E}} \circ \varphi = \delta$.

By Lemma 9.3, $F \in \mathrm{AlgExt}(K, \mathcal{F})$ if and only if $\mathrm{Gal}(F) \cong \mathrm{Gal}(\mathbb{F})$ for some $\mathbb{F} \in \mathcal{F}$. Therefore the isomorphism φ: $\mathbf{G} \to \mathrm{Gal}(\mathbf{K})$ establishes, via Galois correspondence, a bijection of $\mathrm{Subgr}(G, \mathcal{C})$ onto $\mathrm{AlgExt}(K, \mathcal{F})$ which maps $\mathrm{Subgr}(G, \mathcal{C})_{\max}$ onto $\mathrm{AlgExt}(K, \mathcal{F})_{\min}$. This proves (6c)

By Proposition 10.2(a), K is pseudo-$\{K_x \mid x \in X\}$-closed. Therefore, by (6c), K is P$\mathcal{F}$C. ∎

Problem 10.4: *Is it possible to remove the condition "$\mathcal{F}$ is closed under Galois isomorphism" from Theorem 10.3?*

References

[Ax] J. Ax, *A metamathematical approach to some problems in number theory,* AMS Proc. Symp. Pure Math. **XX** (1971), 161–190.

[Bre] G. E. Bredon, *Introduction to Compact Transformation Groups,* Pure and applied Mathematics **46**, Academic Press, New Your, 1972.

[CaF] J.W.S. Cassels and A. Fröhlich, *Algebraic Number Theory,* Academic Press, London, 1967.

[Efr] I. Efrat, *A Galois-theoretic characterization of p-adically closed fields,* Israel Journal of Mathematics **91** (1995), 273-284.

[Er1] Yu. L. Ershov, *Projectivity of absolute Galois groups of RC^*_ζ-fields,* Proceedings of the III International Conference on Algebra, Krasnoyarsk 1993 (Yu. L. Ershov et al., eds.), 63–80, Walter de Gruyter, Berlin 1996.

[Er2] Yu. L. Ershov, *Projectivity of the absolute Galois groups of multi-valued fields,* Siberian Advances in Mathematics **14** (2004), 1–6.

[FrJ] M. D. Fried and M. Jarden, *Field Arithmetic,* Ergebnisse der Mathematik (3) **11**, Springer-Verlag, Heidelberg, 1986.

[Gey] W.-D. Geyer, *Galois groups of intersections of local fields,* Israel Journal of Mathematics **30** (1978), 382–396.

[Har] D. Haran, *On closed subgroups of free products of profinite groups,* Proceedings of the London Mathematical Society **55** (1987), 266–289.

[HeR] W. Herfort and L. Ribes, *Torsion elements and centralizers in free products of profinite groups,* Journal für die reine und angewandte Mathematik **358** (1985), 155-161.

[HaJ1] D. Haran and M. Jarden, *The absolute Galois group of a pseudo real closed field,* Annali della Scuola Normale Superiore — Pisa, Serie IV, **12** (1985), 449–489.

[HaJ2] D. Haran and M. Jarden, *The absolute Galois group of a pseudo p-adically closed field,* Journal für die reine und angewandte Mathematik **383** (1988), 147–206.

[HJPa] D. Haran, M. Jarden, and F. Pop, *Projective group structures as absolute Galois structures with block approximation,* Memoirs of the AMS, (also http://www.math.uiuc.edu/Algebraic-Number-Theory/0367 Preprint, September 24, 2002)

[Jan] U. Jannsen, *Über Galoisgruppen lokaler Körper,* Inventiones mathematicae **70** (1982), 53–69.

[Jar1] M. Jarden, *Algebraic realization of p-adically projective groups,* Compositio Mathematica **79** (1991), 21–62.

[Jar2] M. Jarden, *Intersection of local algebraic extensions of a Hilbertian field (A. Barlotti et al., eds),* NATO ASI Series C **333** 343–405, Kluwer, Dordrecht, 1991

[JaR] M. Jarden and Peter Roquette, *The Nullstellensatz over p-adically closed fields,* Journal of the Mathematical Society of Japan **32** (1980), 425–460.

[JRi1] M. Jarden and J. Ritter, *On the characterization of local fields by their absolute Galois groups,* Journal of Number Theory **11** (1979), 1–13.

[JRi2] M. Jarden and J. Ritter, *On the Frattini subgroup of the absolute Galois group of a local field,* Israel Journal of Mathematics **74** (1991), 81–90.

[Koc] H. Koch, *Galoissche Theorie der p-Erweiterungen,* Mathematische Monographien **10**, Berlin 1970.

[Koe1] J. Koenigsmann, *From p-rigid elements to valuations (with a Galois-characterisation of p-adic fields),* Journal für die reine und angewandte Mathematik **465** (1995), 165–182.

[Koe2] J. Koenigsmann, *Relatively projective groups as absolute Galois groups,* Israel Journal of Mathematics **127** (2002), 93–129.

[Neu] J. Neukirch, *Kennzeichnung der p-adischen und der endlichen algebraischen Zahlkörper,* Inventiones mathematicae **6** (1969), 296-314.

[Pop1] F. Pop, *Galoissche Kennzeichnung p-adisch abgeschlossener Körper,* Journal für die reine und angewandte Mathematik **392** (1988), 145–175.

[Pop2] F. Pop, *Classically projective groups and pseudo classically closed fields,* in *Valuation theory and its applications,* Vol. II (Saskatoon, SK, 1999), 251–283, Fields Inst. Commun., 33, Amer. Math. Soc., Providence, RI, 2003. A printout of a preprint, Heidelberg, 1990.

[Pop3] F. Pop, *On prosolvable subgroups of profinite free products and some applications,* manuscripta mathematica **86** (1995), 125–135.

[Pre] A. Prestel, *Lectures on Formally Real Fields,* Lecture Notes in Mathematics **1093**, Springer, Berlin, 1984.

[PrR] A. Prestel and P. Roquette, *Formally p-adic Fields,* Lecture Notes in Mathematics 1050, Springer, Berlin, 1984.

[Rib] L. Ribes, *Introduction to Profinite Groups and Galois Cohomology,* Queen's papers in Pure and Applied Mathematics **24**, Queen's University, Kingston, 1970.

[RiZ] L. Ribes and P. Zalesskii, *Profinite Groups,* Ergebnisse der Mathemtik III **40**, Springer, Berlin, 2000.