Skolem Density Problems over Large Galois Extensions of Global Fields

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Abstract: Let K be a global field, $\mathcal{V}$ an infinite proper subset of the set of all primes of K, and $\mathcal{S}$ a finite subset of $\mathcal{V}$. Denote the maximal Galois extension of K in which each $\mathfrak{p} \in \mathcal{S}$ totally splits by $K_{\mathrm{tot},\mathcal{S}}$. Let M be an algebraic extension of K. A data for an $(\mathcal{S},\mathcal{V})$-Skolem density problem for M consists of a finite subset $\mathcal{T}$ of $\mathcal{V}$ containing $\mathcal{S}$, polynomials $f_1,\ldots,f_m \in \tilde{K}[X_1,\ldots,X_n]$ satisfying $|f_i|_{\mathfrak{q}} = 1$ for each non-archimedean prime $\mathfrak{q} \in \tilde{\mathcal{V}} \smallsetminus \tilde{\mathcal{T}}$, a point $\mathbf{a} \in M^n$, and a positive real number γ. A solution to the problem is a point $\mathbf{x} \in M^n$ such that $|x_i - a_i|_{\mathfrak{p}} < \gamma$ for each $\mathfrak{p} \in \tilde{\mathcal{T}}$ and $|x_i|_{\mathfrak{q}} \le 1$, $|f_j(\mathbf{x})|_{\mathfrak{q}} = 1$ for each non-archimedean prime $\mathfrak{q} \in \tilde{\mathcal{V}} \smallsetminus \tilde{\mathcal{T}}$, $i = 1,\ldots,n$, $j = 1,\ldots,m$.

For $\boldsymbol{\sigma} = (\sigma_1,\ldots,\sigma_e) \in \mathrm{Gal}(K)^e$ let $K_s(\boldsymbol{\sigma}) = \{x \in K_s \mid \sigma_i(x) = x,\ i = 1,\ldots,e\}$. Denote the maximal Galois extension of K inside $K_s(\boldsymbol{\sigma})$ by $K_s[\boldsymbol{\sigma}]$. Then, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ (with respect to the Haar measure), each $(\mathcal{S},\mathcal{V})$-Skolem density problem for $K_s[\boldsymbol{\sigma}] \cap K_{\mathrm{tot},\mathcal{S}}$ has a solution.

This result generalizes a previous work [JR2] in which $\mathcal{V}$ corresponds to the set of all nonzero prime ideals of $\mathcal{O}_K$ (in particular, $\mathcal{V}$ does not contain archimedean primes). There we prove for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ that each $(\mathcal{S},\mathcal{V})$-Skolem density problem for the maximal purely inseparable extension of $K_s(\boldsymbol{\sigma}) \cap K_{\mathrm{tot},\mathcal{S}}$ has a solution.

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Introduction

Let K be a global field. Fix a separable closure K_s and an algebraic closure $\tilde{K}$ of K. Denote the absolute Galois group of K by $\mathrm{Gal}(K)$.

Denote the set of all primes of K by $\mathbb{P}$, all finite (= nonarchimedean) primes by $\mathbb{P}_0$, and all infinite (= archimedean) primes by $\mathbb{P}_\infty$. Each $\mathfrak{p} \in \mathbb{P}$ is, by definition, an equivalence class of absolute values. For each $\mathfrak{p} \in \mathbb{P}$ choose a separable algebraic extension $K_\mathfrak{p}$ of K as follows: If $\mathfrak{p} \in \mathbb{P}_0$, then $K_\mathfrak{p}$ is a Henselization of K at $\mathfrak{p}$. If $\mathfrak{p} \in \mathbb{P}_\infty$ is real, then $K_\mathfrak{p}$ is a real closure of K at $\mathfrak{p}$. Finally, if $\mathfrak{p} \in \mathbb{P}_\infty$ is complex, then $K_\mathfrak{p} = K_s$. For each $\mathfrak{p} \in \mathbb{P}$ choose an absolute value $|\ |_\mathfrak{p}$ of K representing $\mathfrak{p}$ and, for each prime $\mathfrak{q}$ of $\tilde{K}$ lying over $\mathfrak{p}$, denote the absolute value of $\tilde{K}$ which represents $\mathfrak{q}$ and extends $|\ |_\mathfrak{p}$ by $|\ |_\mathfrak{q}$. For a subset $\mathcal{R}$ of $\mathbb{P}$, we denote the set of primes of $\tilde{K}$ lying over primes in $\mathcal{R}$ by $\tilde{\mathcal{R}}$.

Fix an infinite proper subset $\mathcal{V}$ of $\mathbb{P}$ and a finite subset $\mathcal{S}$ of $\mathcal{V}$. Let

$$K_{\mathrm{tot},\mathcal{S}} = \bigcap_{\mathfrak{p}\in\mathcal{S}} \bigcap_{\sigma\in\mathrm{Gal}(K)} K_\mathfrak{p}^\sigma$$

be the field of **totally $\mathcal{S}$-adic numbers**. This is the maximal Galois extension of K in which each $\mathfrak{p} \in \mathcal{S}$ totally splits.

The goal of this work is to prove the following result:

THEOREM A: *Let e be a nonnegative integer. Then, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, both $K_s(\boldsymbol{\sigma}) \cap K_{\mathrm{tot},\mathcal{S}}$ and $K_s[\boldsymbol{\sigma}] \cap K_{\mathrm{tot},\mathcal{S}}$ are S-Skolem fields with respect to $\mathcal{V}$.*

Here "almost all" is used in the sense of the Haar measure of $\mathrm{Gal}(K)^e$. For each $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_e)$, the field $K_s(\boldsymbol{\sigma})$ is the fixed field of $\sigma_1, \ldots, \sigma_e$ in K_s. The field $K_s[\boldsymbol{\sigma}]$ is the maximal Galois extension of K inside $K_s(\boldsymbol{\sigma})$.

An algebraic extension M of K is an **$\mathcal{S}$-Skolem field with respect to $\mathcal{V}$** if the following holds: Let $\mathcal{T}$ be a finite subset of $\mathcal{V}$ containing $\mathcal{S}$. Put $\mathcal{U} = (\mathcal{V} \smallsetminus \mathcal{T}) \cap \mathbb{P}_0$. Let $f_1, \ldots, f_m \in \tilde{K}[X_1, \ldots, X_n]$ be $\mathfrak{q}$-primitive polynomials for each $\mathfrak{q} \in \tilde{\mathcal{U}}$. Here a polynomial is **$\mathfrak{q}$-primitive** if its coefficients are $\mathfrak{q}$-integrals and at least one of them is a $\mathfrak{q}$-unit. Let $\mathbf{a} = (a_1, \ldots, a_n) \in M^n$ and $\gamma > 0$. Then there is $\mathbf{x} \in M^n$ with $|\mathbf{x} - \mathbf{a}|_\mathfrak{p} < \gamma$ for each $\mathfrak{p} \in \tilde{\mathcal{T}}$, and $|\mathbf{x}|_\mathfrak{q} \le 1$, $|f_i(\mathbf{x})|_\mathfrak{q} = 1$ for each $\mathfrak{q} \in \tilde{\mathcal{U}}$, $i = 1, \ldots, m$.

Theorem A generalizes a result of Skolem [Sko] from 1934: Suppose $g \in \mathbb{Z}[X]$ is a primitive polynomial. Then, there is an algebraic integer x such that $g(x)$ is an algebraic unit.

Indeed, by Theorem A applied to $K = \mathbb{Q}$, $\mathcal{V} =$ the set of all prime numbers, $\mathcal{S} = \emptyset$, and $e = 0$, $\tilde{\mathbb{Q}}$ is an $\mathcal{S}$-Skolem field with respect to $\mathcal{V}$. The result follows then from the case $\mathcal{T} = \emptyset$, $m = n = 1$, and $f_1 = g$.

In 1963 Dade [Dad] reproved Skolem's result. Cantor and Roquette [CaR] proved in 1982 that $K_{\mathrm{tot},\mathcal{S}}$ is an $\mathcal{S}$-Skolem field with respect to $\mathcal{V}$ when K is a number field and $\mathbb{P}_\infty \not\subseteq \mathcal{V}$. A weaker version of Theorem A appears in a work of the present authors [JR2] from 1995. In that version, $\mathcal{V}$ contains only finite primes. Moreover, it proves only that for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ the maximal purely inseparable extension of $K_s(\boldsymbol{\sigma}) \cap K_{\mathrm{tot},\mathcal{S}}$ is $\mathcal{S}$-Skolem with respect to $\mathcal{V}$.

Theorem A improves the main result of [JR2] in three ways:

1. *Including infinite primes.* The set $\mathcal{V}$ may contain now infinite primes. Note: $\mathcal{V}$ may contain all infinite primes. Here we have to use that each Hilbert subset of a number field K contains elements which strongly approximate finitely many elements of K.

2. *Omitting the perfectness assumption.* When $\mathrm{char}(K) > 0$, the $\mathcal{S}$-Skolem fields we find are now separable over K. We do not need to make them perfect. This was an essential difficulty in [JR2]. Here we have overcome it by exploiting compactness arguments in a more careful way then in [JR2] (See Part B of the proof of Lemma 3.2.) We are indebted to Moret-Bailly for his help at this point.

3. *Constructing smaller $\mathcal{S}$-Skolem fields.* In addition to the fields $K_s(\boldsymbol{\sigma}) \cap K_{\mathrm{tot},\mathcal{S}}$, Theorem A says that almost all fields $K_s[\boldsymbol{\sigma}] \cap K_{\mathrm{tot},\mathcal{S}}$ are $\mathcal{S}$-Skolem with respect to $\mathcal{V}$.

Consider a nonempty finite subset $\mathcal{T}$ of $\mathcal{V}$ containing $\mathcal{S}$ and put $\mathcal{U} = (\mathcal{V} \smallsetminus \mathcal{T}) \cap \mathbb{P}_0$. Let $\mathcal{O}_{K,\mathcal{U}} = \{a \in K \mid |a|_\mathfrak{p} \le 1 \text{ for each } \mathfrak{p} \in \mathcal{U}\}$. The main step in the proof of Theorem A (Lemma 3.2) starts from a monic polynomial $f \in \mathcal{O}_{K,\mathcal{U}}[X]$ which factors into a product of distinct monic irreducible polynomials over K. It constructs a monic polynomial $h_0 \in O_{K,\mathcal{U}}[X]$ of degree $d > \deg(f)$, relatively prime to f, with d distinct roots in $K_{\mathrm{tot},\mathcal{S}}$ which are $\mathcal{T}$-close to 0. It also constructs a nonzero element $m \in \mathcal{O}_{K,\mathcal{U}}$,

$\mathcal{T}$-close to 0, such that the polynomial $maf(X) + h_0(X)$ has d distinct roots in $K_{\mathrm{tot},\mathcal{S}}$ which are $\mathcal{T}$-close to 0 for each $a \in \mathcal{O}_{K,\mathcal{U}}$. Consider the absolutely irreducible polynomial $\tilde{h}(T,X) = mf(X)T + h_0(X)$. We know that almost all fields $K_s(\boldsymbol{\sigma})$ are "PAC over $O_{K,\mathcal{U}}$" (Propositions 1.3 and 1.5). This gives $a \in O_{K,\mathcal{V}}$ and $x \in K_s(\boldsymbol{\sigma})$ with $\tilde{h}(a,x) = 0$. Let $h(X) = maf(X) + h_0(X)$. As x is a root of h, it follows from the construction that $x \in K_{\mathrm{tot},\mathcal{S}}$ and is $\mathcal{T}$-close to 0. So, $x \in K_s(\boldsymbol{\sigma}) \cap K_{\mathrm{tot},\mathcal{S}}$. As $h \in \mathcal{O}_{K,\mathcal{U}}[X]$ is monic, x is $\mathfrak{q}$-integral for each $\mathfrak{q} \in \tilde{\mathcal{U}}$. Finally, since $\gcd(f,h) = 1$, there are $g, r \in \mathcal{O}_{K,\mathcal{U}}[X]$ such that $gh + rf = 1$. Therefore $r(x)f(x) = 1$. So, since x is $\mathfrak{q}$-integral, $f(x)$ is a $\mathfrak{q}$-unit for each $\mathfrak{q} \in \tilde{\mathcal{U}}$.

For almost all fields $K_s[\boldsymbol{\sigma}]$ we know only a weaker property than being PAC over $O_{K,\mathcal{U}}$. However, $K_s[\boldsymbol{\sigma}]$ is Galois over K. Moreover, the Appendix due to Wulf-Dieter Geyer, allows us to construct h_0 as above such that $\tilde{h}(T,X)$ is stable with respect to X. That is, $\mathrm{Gal}(\tilde{h}(T,X), K(T)) \cong \mathrm{Gal}(\tilde{h}(T,X), \tilde{K}(T))$. With the weaker property of $K_s[\boldsymbol{\sigma}]$, this suffices to prove the existence of $a \in \mathcal{O}_{K,\mathcal{U}}$ and $x \in K_s[\boldsymbol{\sigma}] \cap K_{\mathrm{tot},\mathcal{S}}$ as in the preceding paragraph.

Based on [JR2], the work [JR3] of the authors proves Rumely's local-global principle for the maximal purely inseparable extensions of $K_s(\boldsymbol{\sigma}) \cap K_{\mathrm{tot},\mathcal{S}}$ for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$. It is our hope, building on the present work, to prove the local-global principle for almost all fields $M = K_s[\boldsymbol{\sigma}] \cap K_{\mathrm{tot},\mathcal{S}}$. This will imply the local-global principle for each extension of M in $K_{\mathrm{tot},\mathcal{S}}$. In particular, this will imply that each such extension is an $\mathcal{S}$-Skolem field with respect to $\mathcal{V}$.

1. Weakly P$\mathcal{S}$C fields over holomorphy domains

This section makes adjustments to [JR2, §1] in order to include infinite primes.

Recall: A field M is **pseudo algebraically closed (PAC)** if every absolutely irreducible variety over M has an M-rational point. If $\mathcal{O}$ is a subset of M, then M may have a stronger property:

Definition 1.1: [JR1, Def. 1.1]. Let $\mathcal{O}$ be a subset of a field M. We say M is **PAC over** $\mathcal{O}$ if this holds: For every absolutely irreducible variety V of dimension $r \geq 0$ and for each dominating separable rational map $\varphi\colon V \to \mathbb{A}^r$ over M there is $\mathbf{a} \in V(M)$ with $\varphi(\mathbf{a}) \in \mathcal{O}^r$. ∎

Examples arise from "Hilbertian subsets" of fields:

Definition 1.2: Let $\mathcal{O}$ be a subset of a field K. We say $\mathcal{O}$ is K**-Hilbertian** if $H \cap \mathcal{O} \neq \emptyset$ for each separable Hilbert subset H of K. (See [FrJ, Chap. 11] for the definition of a separable Hilbert subset.) Note: If $\mathcal{O} \subseteq \mathcal{O}' \subseteq K$ and $\mathcal{O}$ is K-Hilbertian, then so is $\mathcal{O}'$. ∎

Let K be a field. We denote the separable closure of K by K_s, the algebraic closure by $\tilde{K}$, and the absolute Galois group, $\mathrm{Gal}(K_s/K)$, by $\mathrm{Gal}(K)$. As $\mathrm{Gal}(K)$ is a compact topological group, it is equipped with a Haar measure. Recall that if $\sigma_1, \ldots, \sigma_e \in \mathrm{Gal}(K)$, then $K_s(\boldsymbol{\sigma})$ is the fixed field in K_s of $\sigma_1, \ldots, \sigma_e$. In the following we use the clause "for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$" with respect to the Haar measure of $\mathrm{Gal}(K)^e$. Also, we denote the maximal Galois extension of K inside $K_s(\boldsymbol{\sigma})$ by $K_s[\boldsymbol{\sigma}]$.

PROPOSITION 1.3: *Let $\mathcal{O}$ be a subset of a countable field K. Suppose $\mathcal{O}$ is K-Hilbertian. Let e be a positive integer. Then, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, the field $K_s(\boldsymbol{\sigma})$ is PAC over $\mathcal{O}$.*

Proof: [JR1, Prop. 3.1] proves the proposition when $\mathcal{O}$ is a subring of K with quotient field K. The proof applies verbatim to the general case. ∎

Data 1.4: We fix the following data for the rest of this work:

(a) K is a global field and $\mathcal{O}_K$ is its ring of integers [FrJ, §5.2].

(b) $\mathbb{P} = \mathbb{P}_K$ is the set of all primes (finite and infinite) of K. A **finite** (resp. **infinite**) **prime** of a field E is an equivalence class of non-archimedean (resp. archimedean) absolute values of E. We denote the set of all finite primes of K by $\mathbb{P}_0$ and the set of all infinite primes of K by $\mathbb{P}_\infty$. Thus, $\mathbb{P}_\infty = \emptyset$ and $\mathbb{P}_0 = \mathbb{P}$ when $\mathrm{char}(K) > 0$. For each $\mathfrak{p} \in \mathbb{P}$ we choose an absolute value $|\ |_\mathfrak{p}$ which belongs to $\mathfrak{p}$.

(c) $\mathcal{V}$ is a proper subset of $\mathbb{P}$.

(d) Let L be an algebraic extension of K and $\mathcal{R}$ a subset of $\mathbb{P}$.

$\mathcal{R}_0 = \mathcal{R} \cap \mathbb{P}_0$ and $\mathcal{R}_\infty = \mathcal{R} \cap \mathbb{P}_\infty$.

$\mathcal{R}_L$ is the set of primes of L which lie over primes in $\mathcal{R}$. For $L = \tilde{K}$ we set $\tilde{\mathcal{R}} = \mathcal{R}_{\tilde{K}}$. If $\mathfrak{q} \in \mathbb{P}_L$ lies over $\mathfrak{p} \in \mathbb{P}$, we write $\mathfrak{q}|\mathfrak{p}$ and $\mathfrak{p} = \mathfrak{q}|_K$. We denote the unique absolute value which represents $\mathfrak{q}$ and extends $|\ |_\mathfrak{p}$ by $|\ |_\mathfrak{q}$.

If L is a normal extension of K, then $\mathrm{Aut}(L/K)$ acts on $\mathcal{R}_L$ according to

$$|x|_{\mathfrak{p}^\sigma} = |x^{\sigma^{-1}}|_\mathfrak{p}, \text{ for } \mathfrak{p} \in \mathcal{R}_L \text{ and } x \in L.$$

We may choose a subset $\mathcal{R}_0$ of $\mathcal{R}_L$ which contains exactly one extension of each prime in $\mathcal{R}$. Then, for each $\mathfrak{q} \in \mathcal{R}_L$ there are $\mathfrak{p} \in \mathcal{R}_0$ and $\sigma \in \mathrm{Aut}(L/K)$ with $\mathfrak{q} = \mathfrak{p}^\sigma$. We say that $\mathcal{R}_0$ **represents** $\mathcal{R}_L$ **over** K.

$\mathcal{O}_{L,\mathcal{R}}$ is the $\mathcal{R}$**-holomorphy domain** $\{x \in L \mid\ |x|_\mathfrak{q} \le 1 \text{ for each } \mathfrak{q} \in \mathcal{R}_L\}$ of L. It is closed under multiplication. If $\mathcal{R} \subseteq \mathbb{P}_0$, then $\mathcal{O}_{L,\mathcal{R}}$ is a ring.

$\mathcal{M}_{L,\mathcal{R}} = \{x \in L \mid\ |x|_\mathfrak{q} < 1 \text{ for each } \mathfrak{q} \in \mathcal{R}_L\}$. If $\mathcal{R} \subseteq \mathbb{P}_0$, it is an ideal of $\mathcal{O}_{L,\mathcal{R}}$.

For $\mathbf{a} = (a_1, \ldots, a_n) \in L^n$, $|\mathbf{a}|_\mathcal{R} = \max\limits_{\mathfrak{q} \in \mathcal{R}_{K(\mathbf{a})}} \max\limits_{1 \le i \le n} |a_i|_\mathfrak{q} = \max\limits_{\tilde{\mathfrak{q}} \in \tilde{\mathcal{R}}} \max\limits_{1 \le i \le n} |a_i|_{\tilde{\mathfrak{q}}}$.

For $f(X) = \sum_{i=0}^n a_i X^i \in L[X]$, $|f|_\mathcal{R} = |(a_0, \ldots, a_n)|_\mathcal{R}$. ∎

The set $\mathcal{V}$ satisfies the **strong approximation theorem**: Let $\mathcal{T}$ be a finite subset of $\mathcal{V}$. For each $\mathfrak{p} \in \mathcal{T}$ consider an element $a_\mathfrak{p}$ of K and let ε be a positive real number. Then there exists $x \in \mathcal{O}_{K,\mathcal{V} \smallsetminus \mathcal{T}}$ such that $|x - a_\mathfrak{p}|_\mathfrak{p} < \varepsilon$ for each $\mathfrak{p} \in \mathcal{T}$ [CaF, p. 67]. The proof of the proposition below is an adjustment of the proof of [FrJ, Thm. 12.7].

PROPOSITION 1.5: *The subset $\mathcal{O}_{K,\mathbb{P}_0 \cap \mathcal{V}} \cap \mathcal{M}_{K,\mathbb{P}_\infty \cap \mathcal{V}}$ of $\mathcal{O}_{K,\mathcal{V}}$ is K-Hilbertian.*

Proof: Assume without loss that $\mathcal{V}$ is cofinite in $\mathbb{P}$ and let $\mathbb{P}_\infty \cap \mathcal{V} = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$. Let H be a separable Hilbert subset of K. ¿From [FrJ, Lemma 12.1] there exist absolutely irreducible polynomials $h_1, \ldots, h_m \in \mathcal{O}_K[T, X]$, monic and separable in X, with $\deg_X(h_i) > 1$, $i = 1, \ldots, m$, such that $\bigcap_{i=1}^m H'_K(h_i) \subseteq H$, where $H'_K(h_i) = \{a \in K \mid h_i(a,b) \neq 0 \text{ for each } b \in K\}$. Apply [FrJ, Lemma 12.6] to find distinct $\mathfrak{p}_1, \ldots, \mathfrak{p}_m \in \mathbb{P}_0 \cap \mathcal{V}$ and elements $a_1, \ldots, a_m \in \mathcal{O}_K$ such that $x \in H'_K(h_i)$ for each $x \in K$ with $|x - a_i|_{\mathfrak{p}_i} < 1$, $i = 1, \ldots, m$. Let $\mathcal{T} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m, \mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$. Then use the strong approximation theorem to find $x \in \mathcal{O}_{K,\mathcal{V} \smallsetminus \mathcal{T}}$ such that $|x - a_i|_{\mathfrak{p}_i} < 1$, $i = 1, \ldots, m$, and $|x|_{\mathfrak{q}_j} < 1$, $j = 1, \ldots, n$. Then $x \in H$ and $|x|_{\mathfrak{p}_i} \leq 1$, $i = 1, \ldots, m$. Hence, $x \in H \cap \mathcal{O}_{K,\mathbb{P}_0 \cap \mathcal{V}} \cap \mathcal{M}_{K,\mathbb{P}_\infty \cap \mathcal{V}}$, as desired. ∎

Combine Proposition 1.3 with Proposition 1.5:

Corollary 1.6: *Let e be a positive integer. Then, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, the field $K_s(\boldsymbol{\sigma})$ is PAC over $\mathcal{O}_{K,\mathbb{P}_0 \cap \mathcal{V}} \cap \mathcal{M}_{K,\mathbb{P}_\infty \cap \mathcal{V}}$.*

Data 1.7: We add the following data to Data 1.4 and fix it for the rest of this work:

(a) Let $\mathfrak{p} \in \mathbb{P}$.

$\tilde{\mathfrak{p}}$ is a fixed extension of $\mathfrak{p}$ to a prime of $\tilde{K}$. If $\tilde{\mathfrak{q}} \in \tilde{\mathbb{P}}$ and $\tilde{\mathfrak{q}}|\mathfrak{p}$, then there is $\sigma \in \mathrm{Gal}(K)$ such that $\tilde{\mathfrak{q}} = \tilde{\mathfrak{p}}^\sigma$.

$\hat{K}_\mathfrak{p}$ is the completion of K at $\mathfrak{p}$ inside the completion of $\tilde{K}$ at $\tilde{\mathfrak{p}}$. Then $|\ |_\mathfrak{p}$ uniquely extends to an absolute value $|\ |_\mathfrak{p}$ of $\hat{K}_\mathfrak{p}$ and then uniquely to an absolute value of $\tilde{K}\hat{K}_\mathfrak{p}$. The restriction of the latter to $\tilde{K}$ coincide with $|\ |_{\tilde{\mathfrak{p}}}$.

If $\mathfrak{p} \in \mathbb{P}_\infty$, then either $\hat{K}_\mathfrak{p} \cong \mathbb{R}$ or $\hat{K}_\mathfrak{p} \cong \mathbb{C}$; in the former case $\mathfrak{p}$ is **real**, in the latter case $\mathfrak{p}$ is **complex**.

$K_\mathfrak{p} = K_s \cap \hat{K}_\mathfrak{p}$. It is well defined up to a K-isomorphism. If $\mathfrak{p} \in \mathbb{P}_0$, then $K_\mathfrak{p}$ is an Henselian closure of K at $\mathfrak{p}$. As $\hat{K}_\mathfrak{p}/K_\mathfrak{p}$ is a separable extension [Ja1, Lemma 2.2], so is $\hat{K}_\mathfrak{p}/K$.

$K_{\mathrm{t}\mathfrak{p}} = \bigcap_{\sigma \in \mathrm{Gal}(K)} K_\mathfrak{p}^\sigma$.

(b) $\mathcal{S}$ is a finite subset of $\mathcal{V}$.

(c) $N = K_{\mathrm{tot},\mathcal{S}} = \bigcap_{\mathfrak{p} \in \mathcal{S}} K_{\mathrm{t}\mathfrak{p}}$. This is the maximal Galois extension of K in which each $\mathfrak{p} \in \mathcal{S}$ totally splits. If $\mathcal{S} = \emptyset$, we let $N = K_s$.

Note: If L is a subextension of N/K, then $L_{\mathrm{tot},\mathcal{S}_L} = N$. ∎

We call a polynomial $f \in \tilde{K}[X]$ of degree n **separable** if it has n distinct roots. We call f N-**admissible** if in addition it is monic and all its roots are in N.

PROPOSITION 1.8: *Let $\tilde{\mathfrak{q}} \in \tilde{\mathbb{P}}$ and let E be a separable algebraic extension of $K_{\mathfrak{p}}^{\sigma}$, where $\mathfrak{p} = \tilde{\mathfrak{q}}|_K$ and $\tilde{\mathfrak{q}} = \tilde{\mathfrak{p}}^{\sigma}$ for $\sigma \in \mathrm{Gal}(K)$. Let $f \in K[X]$ be a monic separable polynomial of degree n and let $x_1, \ldots, x_n$ be its distinct roots. Then, for each $\varepsilon > 0$ there is $\delta > 0$ such that if g is a monic polynomial in $E[X]$ of degree n and $|g - f|_{\tilde{\mathfrak{q}}} < \delta$, then g is separable and its roots can be enumerated as $y_1, \ldots, y_n$ with $|y_i - x_i|_{\tilde{\mathfrak{q}}} < \varepsilon$ and $E(x_i) = E(y_i)$.*

Proof: If $\tilde{\mathfrak{q}} \in \tilde{\mathbb{P}}_0$, the proposition follows from a combination of the theorem about the continuity of roots of polynomials and Krasner's lemma [Ja2, Prop. 12.3]. If $\tilde{\mathfrak{q}} \in \tilde{\mathbb{P}}_\infty$, the proposition follows from Sturm's theorem for real roots and from the theorem about the continuity of roots of polynomials for complex roots. ∎

The following lemma replaces [JR2, Lemma 1.2].

LEMMA 1.9: *Let $\mathcal{T}$ be a nonempty finite subset of $\mathbb{P}$ which contains $\mathcal{S}$. Let $f \in K[X]$ be an N-admissible polynomial of degree n and let $x_1, \ldots, x_n \in N$ be its distinct roots. Then, for each $\varepsilon > 0$ there is $\delta > 0$ with the following property: If*

(1a) *$g \in N[X]$ is a monic polynomial of degree n with $|g - f|_{\mathcal{T}} < \delta$,*

then g is N-admissible and for each $\tilde{\mathfrak{q}} \in \tilde{\mathcal{T}}$

(1b) *the roots of g can be enumerated as $y_1, \ldots, y_n$ with $|y_i - x_i|_{\tilde{\mathfrak{q}}} < \varepsilon$.*

If, in addition, there is $a \in K$ with $|x_i - a|_{\mathfrak{p}} < \varepsilon$ for each $\mathfrak{p} \in \mathcal{T}$, $i = 1, \ldots, n$, then we can choose δ such that (1a) implies $|y_i - a|_{\mathcal{T}} < \varepsilon$, $i = 1, \ldots, n$.

Proof: For each $\mathfrak{p} \in \mathcal{T}$ Proposition 1.8 gives $\delta_{\mathfrak{p}} > 0$ such that if g is a monic polynomial in $N[X]$ of degree n and $|g - f|_{\tilde{\mathfrak{p}}} < \delta_{\mathfrak{p}}$, then g has n distinct roots $y_1, \ldots, y_n$ with $|y_i - x_i|_{\tilde{\mathfrak{p}}} < \varepsilon$ and $K_{\mathfrak{p}}N(x_i) = K_{\mathfrak{p}}N(y_i)$. In particular, for each $\mathfrak{p} \in \mathcal{S}$, $y_1, \ldots, y_n \in K_{\mathfrak{p}}$.

Let $\delta = \max_{\mathfrak{p} \in \mathcal{T}} \delta_{\mathfrak{p}}$ and consider a polynomial g as in (1a). For each $\tilde{\mathfrak{q}} \in \tilde{\mathcal{T}}$ there exist $\mathfrak{p} \in \mathcal{T}$ and $\sigma \in \mathrm{Gal}(K)$ such that $\tilde{\mathfrak{q}} = \tilde{\mathfrak{p}}^{\sigma}$. Since N/K is Galois, $g^{\sigma^{-1}} \in N[X]$. Also, $|g^{\sigma^{-1}} - f|_{\tilde{\mathfrak{p}}} = |g - f|_{\tilde{\mathfrak{q}}} < \delta$. Hence (1b) holds for $\tilde{\mathfrak{p}}$ and $g^{\sigma^{-1}}$, and the roots of $g^{\sigma^{-1}}$ are in $K_{\mathfrak{p}}$ if $\mathfrak{p} \in \mathcal{S}$. Moreover, σ permutes the roots of f and maps the roots of $g^{\sigma^{-1}}$ onto the roots of g. So (1b) holds for $\tilde{\mathfrak{q}}$ and g, and the roots of g are in $K_{\mathfrak{p}}^{\sigma}$ if $\mathfrak{p} \in \mathcal{S}$. It follows that all roots of g belong to N, so g is N-admissible.

If, in addition, there is $a \in K$ with $|x_i - a|_{\mathfrak{p}} < \varepsilon$ for each $\mathfrak{p} \in \mathcal{T}$, $i = 1, \ldots, n$, then, by (1b), we can choose δ such that (1a) implies $|y_i - a|_{\mathcal{T}} < \varepsilon$, $i = 1, \ldots, n$. ∎

Definition 1.10: Let M be a subextension of N/K and $\mathcal{O}$ a subset of M.

(a) We say M is **pseudo $\mathcal{S}$-adically closed (P$\mathcal{S}$C)** if every absolutely irreducible variety over M has an M-rational point provided it has a simple $K_{\mathfrak{p}}$-rational point for each $\mathfrak{p} \in \mathcal{S}$.

(b) A polynomial $h \in N[T, X]$ is N**-admissible with respect to** X if h is monic in X and $h(0, X)$ is N-admissible.

(c) An absolutely irreducible polynomial $h \in K[T, X]$, separable with respect to X, is K**-stable with respect to** X if $\mathrm{Gal}(h(T, X), K(T)) \cong \mathrm{Gal}(h(T, X), \tilde{K}(T))$ [FrJ, §15.3].

(d) We say M is **weakly** P$\mathcal{S}$C **over** $\mathcal{O}$ if for each absolutely irreducible N-admissible polynomial $h \in M[T, X]$ with respect to X and for each $g \in M[T]$ with $g(0) \neq 0$, there exists $(a, b) \in \mathcal{O} \times M$ such that $h(a, b) = 0$ and $g(a) \neq 0$.

(e) We say M is **weakly K-stably** P$\mathcal{S}$C **over** $\mathcal{O}$ if for each absolutely irreducible N-admissible and K-stable polynomial $h \in K[T, X]$ with respect to X and for each $g \in K[T]$ with $g(0) \neq 0$, there exists $(a, b) \in \mathcal{O} \times M$ such that $h(a, b) = 0$ and $g(a) \neq 0$. ∎

Remark 1.11: (a) If $\mathcal{S} = \emptyset$, then $N = K_s$. So, M is weakly P$\mathcal{S}$C over $\mathcal{O}$ if and only if M is PAC over $\mathcal{O}$ [JR1, Lemma 1.3].

(b) If M is weakly P$\mathcal{S}$C over $\mathcal{O}$, then it is also weakly K-stably P$\mathcal{S}$C over $\mathcal{O}$.

(c) If $\mathcal{O} \subseteq \mathcal{O}' \subseteq M$ and M is weakly P$\mathcal{S}$C over $\mathcal{O}$ (resp. weakly K-stably P$\mathcal{S}$C over $\mathcal{O}$), then it is also weakly P$\mathcal{S}$C over $\mathcal{O}'$ (resp. weakly K-stably P$\mathcal{S}$C over $\mathcal{O}'$).

(d) Suppose $\mathfrak{p}$ is a complex prime of K and let $\mathcal{S}' = \mathcal{S} \cup \{\mathfrak{p}\}$. Then $K_{\mathrm{t}\mathfrak{p}} = \tilde{K}$ and therefore $N = K_{\mathrm{tot},\mathcal{S}} = K_{\mathrm{tot},\mathcal{S}'}$. Thus M is weakly P$\mathcal{S}$C over $\mathcal{O}$ if and only if M is weakly P$\mathcal{S}'$C over $\mathcal{O}$.

(e) If $\mathcal{S}$ does not contain complex primes and M is a P$\mathcal{S}$C field, then M is weakly P$\mathcal{S}$C over $\mathcal{O}_{M,\mathcal{S}}$ [Raz, Prop. 3.3]. ∎

The following lemma replaces [JR2, Lemma 1.4].

LEMMA 1.12: *Let M_0 be an algebraic extension of K, $M = M_0 \cap N$, and e a positive integer. Let $\mathcal{O}$ be a subset of $\mathcal{O}_{M,\mathcal{S}}$ such that $\mathcal{O}_{K,\mathcal{V}} \cdot \mathcal{O} \subseteq \mathcal{O}$ and M_0 is PAC over $\mathcal{O}$.*

Then:

(a) *M is weakly* P$\mathcal{S}$C *over $\mathcal{O}$. In particular, $K_{\mathrm{tot},\mathcal{S}}$ is weakly* P$\mathcal{S}$C *over $\mathcal{O}_{K,\mathcal{V}}$ and $K_s(\boldsymbol{\sigma}) \cap K_{\mathrm{tot},\mathcal{S}}$ is weakly* P$\mathcal{S}$C *over $\mathcal{O}_{K,\mathcal{V}}$ for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$.*

(b) *Let M' be the maximal Galois extension of K inside M. Then M' is weakly K-stably* P$\mathcal{S}$C *over $\mathcal{O}_{K,\mathcal{V}}$. In particular, $K_s[\boldsymbol{\sigma}] \cap K_{\mathrm{tot},\mathcal{S}}$ is weakly K-stably* P$\mathcal{S}$C *over $\mathcal{O}_{K,\mathcal{V}}$ for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$.*

Proof: Let $\mathcal{T}$ be a nonempty finite subset of $\mathcal{V}$ containing $\mathcal{S}$.

Proof of (a): Let $h \in M[T,X]$ be an absolutely irreducible N-admissible polynomial with respect to X and let $g \in M[T]$ with $g(0) \neq 0$. Let L be a finite subextension of M/K which contains the coefficients of h. Lemma 1.9, applied to $L, \mathcal{S}_L, \mathcal{T}_L$ instead of to $K, \mathcal{S}, \mathcal{T}$, gives $\delta > 0$ such that if $k \in N[X]$ is a monic polynomial of the same degree as $h(0,X)$ with $|k(X) - h(0,X)|_{\mathcal{T}} < \delta$, then k is N-admissible. Also, there exists $\gamma > 0$ such that if $a \in N$ satisfies $|a|_{\mathcal{T}} < \gamma$, then $|h(a,X) - h(0,X)|_{\mathcal{T}} < \delta$. Use the strong approximation theorem to find $0 \neq m \in \mathcal{O}_{K,\mathcal{V}}$ such that $|m|_{\mathcal{T}} < \gamma$. Let $T' = \frac{1}{m}T$. Since M_0 is PAC over $\mathcal{O}$, the absolutely irreducible polynomial $h(mT', X)$ has a zero (c,b) in $\mathcal{O} \times M_0$ such that $g(mc) \neq 0$ [JR1, Lemma 1.3]. Hence, $a = mc \in \mathcal{O}_{K,\mathcal{V}} \cdot \mathcal{O} \subseteq \mathcal{O}$ satisfies $h(a,b) = 0$ and $g(a) \neq 0$. Check that $|h(a,X) - h(0,X)|_{\mathcal{T}} < \delta$. Hence, all roots of $h(a,X)$ belong to N. In particular $b \in M_0 \cap N = M$. Conclude that M is weakly P$\mathcal{S}$C over $\mathcal{O}$.

Proof of (b): Let $h \in K[T,X]$ be an absolutely irreducible N-admissible and K-stable polynomial with respect to X and let $g \in K[T]$ with $g(0) \neq 0$. Choose a transcendental element t over K and an element x such that $h(t,x) = 0$. Set $F = K(t,x)$ and let $\hat{F}$ be the Galois hull of $F/K(t)$. Choose a primitive element y for $\hat{F}/K(t)$. By assumption, $\hat{F}/K$ is a regular extension. Since M_0 is PAC over $\mathcal{O}$, there exist $a \in \mathcal{O}$, $b \in \tilde{K}$, and $c \in M_0$ such that (a,b,c) is a K-specialization of (t,x,y), $g(a) \neq 0$, and a is $\mathcal{T}$-close enough to 0 to get $h(a,X)$ is $\mathcal{T}$-closed enough to $h(0,X)$ so that $b \in N$. Then $K(b) \subseteq K(c)$ and $K(c)$ is a Galois extension of K which is contained in M_0. Then $b \in K(c) \cap N \subseteq M_0 \cap N = M$. Since $K(c) \cap N$ is a Galois of K, it follows that $b \in M'$.

∎

Remark: If M is a subextension of N/K which is weakly P$\mathcal{S}$C over $\mathcal{O} = \mathcal{O}_{K,\mathbb{P}_0\cap\mathcal{V}} \cap \mathcal{M}_{K,\mathbb{P}_\infty\cap\mathcal{V}}$, we can prove directly that the maximal Galois extension M' of K inside M is weakly K-stably P$\mathcal{S}$C over $\mathcal{O}$. ∎

Example 1.13: Let f, r, g be nonzero polynomials in $N[X]$ such that f is N-admissible. Suppose $\gcd(r, f) = 1$, $\deg(r) < \deg(f)$, and $g(0) \neq 0$. Let $0 \neq m \in K$. Then $h(T, X) = mr(X)T + f(X)$ is an absolutely irreducible polynomial which is monic in X. Since $h(0, X) = f(X)$ is N-admissible, h is N-admissible with respect to X.

Let M be a subextension of N/K and let $\mathcal{O}$ be a subset of M.

(a) Suppose M is weakly P$\mathcal{S}$C over $\mathcal{O}$ and $f, r, g \in M[X]$. Then there exists $(a, b) \in \mathcal{O} \times M$ such that $mr(b)a + f(b) = 0$ and $g(a) \neq 0$.

(b) Suppose M is weakly K-stably P$\mathcal{S}$C over $\mathcal{O}$, $f, r, g \in K[X]$, and $\frac{f}{r}$ is a Morse function (Definition 4.1). Then $-\frac{1}{m}\frac{f}{r}$ is also a Morse function. Therefore, by Proposition 4.2, $h(T, X)$ is K-stable with respect to X. Hence there exists $(a, b) \in \mathcal{O} \times M$ such that $mr(b)a + f(b) = 0$ and $g(a) \neq 0$. ∎

The next lemma replaces [JR2, Lemma 1.8]:

LEMMA 1.14 (Quasi uniform approximation): *Let M be a subextension of N/K which is weakly* P$\mathcal{S}$C *over $\mathcal{O}_{M,\mathcal{V}}$ and let $\mathcal{T}$ be a finite subset of $\mathcal{V}$ which contains $\mathcal{S}$. Let $x \in N$ and $\varepsilon > 0$. Then M has a finite subset B such that for each $\tilde{\mathfrak{q}} \in \tilde{\mathcal{T}}$ there is $b \in B$ with $|b - x|_{\tilde{\mathfrak{q}}} < \varepsilon$.*

Proof: Assume without loss that $x \neq 0$ and $\mathcal{T} \neq \emptyset$. Since N/M is Galois, $\mathrm{irr}(x, M)$ is an N-admissible polynomial which has x as a root. Hence, it suffices to prove the following statement about N-admissible polynomials $h \in M[X]$:

(2) There exists a finite set $B_h \subset M$ such that for each root z of h and for each $\tilde{\mathfrak{q}} \in \tilde{\mathcal{T}}$ there is $b \in B_h$ with $|b - z|_{\tilde{\mathfrak{q}}} < \varepsilon$.

The case $\deg(h) = 1$ being trivial we assume that $d = \deg(h) \geq 2$ and proceed by induction on d. Let L be a finite extension of K in M which contains the coefficients of h. Note that $L_{\mathrm{tot},\mathcal{S}_L} = K_{\mathrm{tot},\mathcal{S}} = N$. Hence, by Lemma 1.9 applied to $L, \mathcal{S}_L, \mathcal{T}_L$ instead of to $K, \mathcal{S}, \mathcal{T}$, there is $\delta > 0$ with the following property:

(3) Every monic polynomial $h_1 \in N[X]$ of degree d which satisfies $|h_1 - h|_{\mathcal{T}} < \delta$ is N-admissible and for each $\tilde{\mathfrak{q}} \in \tilde{\mathcal{T}}$ and each root z of h there is a root y of h_1 with $|y - z|_{\tilde{\mathfrak{q}}} < \frac{\varepsilon}{2}$.

Choose $0 \neq m \in K$ such that $|m|_{\mathcal{T}} < \delta$. Since M is weakly P$\mathcal{S}$C over $\mathcal{O}_{M,\mathcal{V}}$, Example 1.13 (a) (applied to $h, 1$ instead of to f, r) gives $a \in \mathcal{O}_{M,\mathcal{V}}$ and $c \in M$ with $ma + h(c) = 0$. It follows that the monic polynomial $h_1(X) = ma + h(X) \in M[X]$ of degree d satisfies $h_1(c) = 0$ and $|h_1 - h|_{\mathcal{T}} < \delta$. Hence, h_1 satisfies the conclusion of (3).

In particular $g(X) = \frac{h_1(X)}{X-c} \in M[X]$ is an N-admissible polynomial of degree $d-1$. By the induction hypothesis, there exists a finite subset $B_g \subseteq M$ such that for each root y of g and for each $\tilde{\mathfrak{q}} \in \tilde{\mathcal{T}}$ there is $b \in B_g$ with $|b - y|_{\tilde{\mathfrak{q}}} < \frac{\varepsilon}{2}$.

Let $B_h = B_g \cup \{c\}$ and consider $\tilde{\mathfrak{q}} \in \tilde{\mathcal{T}}$. Let z be a root of h. By (3) there exists a root y of h_1 such that $|y - z|_{\tilde{\mathfrak{q}}} < \frac{\varepsilon}{2}$. So, $y = c$ or y is a root of g. In the later case there exists $b \in B_g$ such that $|b - y|_{\tilde{\mathfrak{q}}} < \frac{\varepsilon}{2}$ and therefore $|b - z|_{\tilde{\mathfrak{q}}} < \varepsilon$. In both cases the induction is complete. ∎

Finally, we replace [JR2, Prop. 1.9]:

PROPOSITION 1.15: *Let M be a subextension of N/K which is weakly K-stably* P$\mathcal{S}$C *over $\mathcal{O}_{M,\mathcal{V}}$. Let $\mathfrak{p}$ be a prime in $\mathcal{V} \smallsetminus \mathcal{S}$ and $\tilde{\mathfrak{q}}$ an extension of $\mathfrak{p}$ to $\tilde{K}$. Suppose $\tilde{\mathfrak{q}} = \tilde{\mathfrak{p}}^\sigma$ for $\sigma \in \mathrm{Gal}(K)$. Then $K_{\mathfrak{p}}^\sigma M = K_s$ and M is $\tilde{\mathfrak{q}}$-dense in $\tilde{K}$.*

Proof: Consider $0 \neq x \in K_s$. Let $h_{\mathfrak{p}} = \mathrm{irr}(x, K)$ and $n = \deg(h_{\mathfrak{p}})$. Let $\mathcal{T}$ be a nonempty finite subset of $\mathcal{V} \smallsetminus \{\mathfrak{p}\}$ containing $\mathcal{S}$. Choose n distinct elements $a_1, \ldots, a_n \in K$ and let $h_{\mathcal{T}}(X) = \prod_{i=1}^n (X - a_i)$. By Lemma 1.9 and Proposition 1.8, there exists $\delta > 0$ such that

(4a) if $h \in N[X]$ is a monic polynomial of degree n and $|h - h_{\mathcal{T}}|_{\mathcal{T}} < \delta$, then h has n distinct roots in N, and

(4b) if $h \in M[X]$ is a monic polynomial of degree n and $|h - h_{\mathfrak{p}}|_{\tilde{\mathfrak{q}}} < \delta$, then we can enumerate the roots of $h_{\mathfrak{p}}$ as $x_1, \ldots, x_n$ and the roots of h as $x'_1, \ldots, x'_n$ such that $K_{\mathfrak{p}}^\sigma M(x'_i) = K_{\mathfrak{p}}^\sigma M(x_i)$.

If $\mathrm{char}(K) | n$, let $f(X) = X$. Otherwise, let $f(X) = 1$. Use Proposition 4.3 and the assumption $\mathfrak{p} \notin \mathcal{S}$ with the weak approximation theorem to find a monic polynomial

$h \in K[X]$ such that $|h - h_{\mathcal{T}}|_{\mathcal{T}} < \delta$, $\delta_1 = |h - h_{\mathfrak{p}}|_{\mathfrak{p}} < \delta$, and $\frac{h}{f}$ is a Morse function. By (4a), h is N-admissible.

To prove $K_{\mathfrak{p}}^{\sigma} M = K_s$ we may assume $K(x)/K$ is Galois. Choose $0 \neq m \in K$ with $|m|_{\mathfrak{p}} < \delta - \delta_1$. Then Example 1.13 (b) supplies $c \in \mathcal{O}_{M,\mathcal{V}}$ and $b \in M$ with $mcf(b) + h(b) = 0$. So, b is a root of the monic polynomial $h_1(X) = mcf(X) + h(X) \in M[X]$. By (4b), $K_{\mathfrak{p}}^{\sigma} M(b) = K_{\mathfrak{p}}^{\sigma} M(x')$ for some root x' of $h_{\mathfrak{p}}$. It follows that $K_{\mathfrak{p}}^{\sigma}(x') \subseteq K_{\mathfrak{p}}^{\sigma} M$. But $K(x) = K(x')$. Hence $x \in K_{\mathfrak{p}}^{\sigma} M$. Conclude that $K_{\mathfrak{p}}^{\sigma} M = K_s$.

In particular, M is $\tilde{\mathfrak{q}}$-dense in K_s. As K_s is $\tilde{\mathfrak{q}}$-dense in $\tilde{K}$ [JR1, Lemma 9.1], M is $\tilde{\mathfrak{q}}$-dense in $\tilde{K}$. ∎

2. A compactness lemma

We keep Data 1.4 and 1.7 in force. The main results of this note depend on the assumption that K is a global field. Here we use the finiteness of the class number of K and Dirichlet's unit theorem to prove that a certain group is compact.

For each finite subset $\mathcal{R}$ of $\mathbb{P}$ regard $\hat{K}_{\mathcal{R}}^{\times} = \prod_{\mathfrak{p}\in\mathcal{R}} \hat{K}_{\mathfrak{p}}^{\times}$ as a topological group with the product topology. For $\mathcal{R} \subseteq \mathcal{V}$, identify $\mathcal{O}_{K,\mathcal{V}_0\smallsetminus\mathcal{R}}^{\times}$ with its image in $\hat{K}_{\mathcal{R}}^{\times}$ under the diagonal embedding. Lemma 2.3 below replaces [JR2, Lemma 2.2]. It states that the group $\hat{K}_{\mathcal{R}}^{\times}/\mathcal{O}_{K,\mathcal{V}_0\smallsetminus\mathcal{R}}^{\times}$ is compact. Its proof follows the proof of [CaR, Lemma 4.4] which proves the same result only in the case $\mathbb{P}_\infty \not\subseteq \mathcal{V}$.

Here we follow the terminology of [HeR] for compact groups. Thus, a topological group G is **compact** if every covering of G by open sets has a finite subcovering ([Bou, §9.1] uses the terminology **quasi-compact** instead).

Suppose N is a normal subgroup of a topological group G which is not necessarily closed. The following rules hold:

(1a) The quotient map $G \to G/N$ is continuous and open [HeR, Chap. II, (5.16) and (5.17)].

(1b) If G is compact, so is G/H [HeR, Chap. II, (5.22)].

(1c) If H and G/H are compact, so is G [HeR, Chap. II (5.25)].

We use the following lemma from linear algebra in the proof of Lemma 2.3.

Lemma 2.2: *Let E be a field, $a_1,\ldots,a_n$ nonzero elements of E, and $\mathbf{x}_i = (x_{i1},\ldots,x_{in})$, $i = 1,\ldots,n-1$, linearly independent vectors in E^n. Suppose $\sum_{j=1}^n a_j x_{ij} = 0$, $i = 1,\ldots,n-1$. Let J be a proper subset of $\{1,\ldots,n\}$. Put $m = |J|$ and $\mathbf{x}'_i = (x_{ij})_{j\in J}$, $i = 1,\ldots,n-1$. Then $\mathbf{x}'_1,\ldots,\mathbf{x}'_{n-1}$ span a subspace of E^m of dimension m.*

Proof: Assume without loss $n \notin J$. By assumption, the rows of the $(n-1)\times n$ matrix $A = (x_{ij})_{1\le i\le n-1,\, 1\le j\le n}$ are linearly independent. Hence, $n-1$ of the columns of A are linearly independent. By assumption, the nth column of A is a linear combination of the first ones. Hence, the first $n-1$ columns are linearly independent. Conclude: $\mathrm{rank}\big((x_{ij})_{1\le i\le n-1,\, j\in J}\big) = m$. So, m of the vectors $\mathbf{x}'_1,\ldots,\mathbf{x}'_{n-1}$ are linearly independent. ∎

Lemma 2.3: *Let $\mathcal{T}$ be a nonempty finite subset of $\mathcal{V}$. Then the group $\hat{K}_{\mathcal{T}}^{\times}/\mathcal{O}_{K,\mathcal{V}_0 \smallsetminus \mathcal{T}}^{\times}$ is compact.*

Proof: Choose $\mathfrak{p}_0 \in \mathbb{P} \smallsetminus \mathcal{V}$. Then, $\mathcal{V}_0 \smallsetminus \mathcal{T} \subseteq (\mathbb{P}_0 \smallsetminus \{\mathfrak{p}_0\}) \smallsetminus \mathcal{T}$. So, $\mathcal{O}_{K,(\mathbb{P}_0 \smallsetminus \{\mathfrak{p}_0\}) \smallsetminus \mathcal{T}}^{\times} \leq \mathcal{O}_{K,\mathcal{V}_0 \smallsetminus \mathcal{T}}^{\times} \leq \hat{K}_{\mathcal{T}}^{\times}$. Thus, $\hat{K}_{\mathcal{T}}/O_{K,\mathcal{V}_0 \smallsetminus \mathcal{T}}^{\times}$ is a quotient of $\hat{K}_{\mathcal{T}}^{\times}/O_{K,(\mathbb{P}_0 \smallsetminus \{\mathfrak{p}_0\}) \smallsetminus \mathcal{T}}^{\times}$. Using (1b), this allows us to assume $\mathcal{V} = \mathbb{P} \smallsetminus \{\mathfrak{p}_0\}$. The rest of the proof naturally breaks up into several parts.

Part A: *Reduction to compactness of divisor classes.* For each $\mathfrak{p} \in \mathcal{V}$ let $U_{\mathfrak{p}} = \{\alpha \in \hat{K}_{\mathfrak{p}}^{\times} \mid |\alpha|_{\mathfrak{p}} = 1\}$ be the group of units in $\hat{K}_{\mathfrak{p}}$. If $\mathfrak{p}$ is infinite and real, $U_{\mathfrak{p}} = \{\pm 1\}$; if $\mathfrak{p}$ is complex, $U_{\mathfrak{p}}$ is the multiplicative group of the unit circle. When $\mathfrak{p}$ is finite, $U_{\mathfrak{p}}$ is profinite. In any case, $U_{\mathfrak{p}}$ is compact.

For each finite subset $\mathcal{R}$ of $\mathcal{V}$ let $U_{\mathcal{R}} = \prod_{\mathfrak{p} \in \mathcal{R}} U_{\mathfrak{p}}$. By the preceding paragraph $U_{\mathcal{R}}$ is compact. Consider the **$\mathcal{R}$-divisor class** $C_{\mathcal{R}} = \hat{K}_{\mathcal{R}}/U_{\mathcal{R}}\mathcal{O}_{K,\mathcal{V}_0 \smallsetminus \mathcal{R}}^{\times}$. It appears as the quotient of the right upper groups in the diagram

$$\begin{tikzcd}
 & \hat{K}_{\mathcal{R}}^{\times} \arrow[d, dash] \\
U_{\mathcal{R}} \arrow[r, dash] \arrow[d, dash] & U_{\mathcal{R}}\mathcal{O}_{K,\mathcal{V}_0 \smallsetminus \mathcal{R}}^{\times} \arrow[d, dash] \\
U_{\mathcal{R}} \cap \mathcal{O}_{K,\mathcal{V}_0 \smallsetminus \mathcal{R}}^{\times} \arrow[r, dash] & \mathcal{O}_{K,\mathcal{V}_0 \smallsetminus \mathcal{R}}^{\times} .
\end{tikzcd}$$

For $\mathcal{R} = \mathcal{T}$, the diagram gives a short exact sequence:

$$1 \longrightarrow U_{\mathcal{T}}/(U_{\mathcal{T}} \cap \mathcal{O}_{K,\mathcal{V}_0 \smallsetminus \mathcal{T}}^{\times}) \longrightarrow \hat{K}_{\mathcal{T}}^{\times}/O_{K,\mathcal{V}_0 \smallsetminus \mathcal{T}}^{\times} \longrightarrow \mathcal{C}_{\mathcal{T}} \longrightarrow 1.$$

By the preceding paragraph and (1b), the second term is compact. So, in order to prove that the middle term is compact, it suffices, by (1c), to prove that $C_{\mathcal{T}}$ is compact.

Part B: *Separation of the finite and the infinite parts of $C_{\mathcal{T}}$.* Consider the epimorphism $p\colon \hat{K}_{\mathcal{T}}^{\times} \to \hat{K}_{\mathcal{T}_0}^{\times}$ given by $p\big((\alpha_{\mathfrak{p}})_{\mathfrak{p} \in \mathcal{T}}\big) = (\alpha_{\mathfrak{p}})_{\mathfrak{p} \in \mathcal{T}_0}$. Observe: $\mathcal{V}_0 \smallsetminus \mathcal{T} = \mathcal{V}_0 \smallsetminus \mathcal{T}_0$. So, p maps $U_{\mathcal{T}}$ onto $U_{\mathcal{T}_0}$ and $\mathcal{O}_{K,\mathcal{V}_0 \smallsetminus \mathcal{T}}^{\times}$, considered as a subgroup of $\hat{K}_{\mathcal{T}}^{\times}$, onto $\mathcal{O}_{K,\mathcal{V}_0 \smallsetminus \mathcal{T}_0}^{\times}$, considered as a subgroup of $\hat{K}_{\mathcal{T}_0}^{\times}$. Let $\bar{p}\colon C_{\mathcal{T}} \to C_{\mathcal{T}_0}$ be the epimorphism that p induces.

Now consider the injection $i\colon \hat{K}_{\mathcal{T}_\infty} \to \hat{K}_{\mathcal{T}}^\times$ given by

$$i\big((\alpha_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{T}_\infty}\big) = \big((1)_{\mathfrak{p}\in\mathcal{T}_0}, (\alpha_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{T}_\infty}\big).$$

It maps $U_{\mathcal{T}_\infty}$ into $U_{\mathcal{T}}$. For $a \in \mathcal{O}_{K,\mathcal{V}_0\smallsetminus\mathcal{T}_\infty}^\times$ we have

$$i(a) = \big((1)_{\mathfrak{p}\in\mathcal{T}_0}, (a)_{\mathfrak{p}\in\mathcal{T}_\infty}\big) = \big((a^{-1})_{\mathfrak{p}\in\mathcal{T}_0}, (1)_{\mathfrak{p}\in\mathcal{T}_\infty}\big)a \in U_{\mathcal{T}}\mathcal{O}_{K,\mathcal{V}_0\smallsetminus\mathcal{T}}^\times .$$

So, i induces a homomorphism $\bar{\imath}\colon C_{\mathcal{T}_\infty} \to C_{\mathcal{T}}$.

Suppose $\alpha = (\alpha_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{T}_\infty} \in \hat{K}_{\mathcal{T}_\infty}^\times$ and $i(\alpha) \in U_{\mathcal{T}}\mathcal{O}_{K,\mathcal{V}_0\smallsetminus\mathcal{T}}^\times$. Then there are $\mu_{\mathfrak{p}} \in U_{\mathfrak{p}}$, $\mathfrak{p} \in \mathcal{T}$, and $a \in O_{K,\mathcal{V}_0\smallsetminus\mathcal{T}}^\times$ with $1 = \mu_{\mathfrak{p}}a$ for $\mathfrak{p} \in \mathcal{T}_0$ and $\alpha_{\mathfrak{p}} = \mu_{\mathfrak{p}}a$ for $\mathfrak{p} \in \mathcal{T}_\infty$. In particular, $a \in \mathcal{O}_{K,\mathcal{V}_0\smallsetminus\mathcal{T}_\infty}^\times$ and $\alpha = \big((\mu_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{T}_\infty}\big)a \in U_{\mathcal{T}_\infty}\mathcal{O}_{\mathcal{V}_0\smallsetminus\mathcal{T}_\infty}^\times$. So, $\bar{\imath}$ is injective.

By definition, $\mathrm{Im}(\bar{\imath}) \subseteq \mathrm{Ker}(\bar{p})$. Conversely, suppose $\alpha = \big((\alpha_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{T}}\big) \in \hat{K}_{\mathcal{T}}^\times$ and $p(\alpha) \in U_{\mathcal{T}_0}\mathcal{O}_{K,\mathcal{V}_0\smallsetminus\mathcal{T}_0}^\times$. Then there are $\mu_{\mathfrak{p}} \in U_{\mathfrak{p}}$, $\mathfrak{p} \in \mathcal{T}_0$, and $a \in \mathcal{O}_{K,\mathcal{V}_0\smallsetminus\mathcal{T}_0}^\times$ with $\alpha_{\mathfrak{p}} = \mu_{\mathfrak{p}}a$, $\mathfrak{p} \in \mathcal{T}_0$. So, $\alpha = i\big((\alpha_{\mathfrak{p}}a^{-1})_{\mathfrak{p}\in\mathcal{T}_\infty}\big) \cdot \big((\mu_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{T}_0}, (1)_{\mathfrak{p}\in\mathcal{T}_\infty}\big)a \in i(K_{\mathcal{T}_\infty}^\times)U_{\mathcal{T}}\mathcal{O}_{K,\mathcal{V}_0\smallsetminus\mathcal{T}}^\times$. Conclude: $\mathrm{Ker}(\bar{p}) = \mathrm{Im}(\bar{\imath})$.

We have therefore established a short exact sequence $1 \to C_{\mathcal{T}_\infty} \xrightarrow{\bar{\imath}} C_{\mathcal{T}} \xrightarrow{\bar{p}} C_{\mathcal{T}_0} \to 1$. By (1c) is suffices to prove that each of the groups $C_{\mathcal{T}_0}$ and $C_{\mathcal{T}_\infty}$ is compact. We do this in Parts C and D.

Part C: *$C_{\mathcal{T}}$ is finite when $\mathcal{T} = \mathcal{T}_0$.* To prove this, let D be the group of divisors of K and P the group of principal divisors. For each $\mathfrak{p} \in \mathbb{P}_0$ let $\mathrm{ord}_{\mathfrak{p}}$ be the normalized valuation associated with $\mathfrak{p}$. Define a homomorphism d from $\hat{K}_{\mathcal{T}}^\times$ into the group D of divisors of K by $d\big((\alpha_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{T}}\big) = \sum_{\mathfrak{p}\in\mathcal{T}} \mathrm{ord}_{\mathfrak{p}}(\alpha_{\mathfrak{p}})\mathfrak{p}$. If $(\alpha_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{T}} \in d^{-1}(P)$, then there is $a \in K^\times$ with $\sum_{\mathfrak{p}\in\mathcal{T}} \mathrm{ord}_{\mathfrak{p}}(\alpha_{\mathfrak{p}})\mathfrak{p} = \sum_{\mathfrak{p}\in\mathbb{P}_0} \mathrm{ord}_{\mathfrak{p}}(a)\mathfrak{p}$. So, $a \in \mathcal{O}_{K,\mathbb{P}_0\smallsetminus\mathcal{T}}^\times$. Then $d^{-1}(P) = U_{\mathcal{T}}\mathcal{O}_{K,\mathbb{P}_0\smallsetminus\mathcal{T}}^\times \le U_{\mathcal{T}}\mathcal{O}_{K,\mathcal{V}_0\smallsetminus\mathcal{T}}^\times$. So, d induces an embedding of $\hat{K}_{\mathcal{T}}^\times/U_{\mathcal{T}}\mathcal{O}_{K,\mathbb{P}_0\smallsetminus\mathcal{T}}^\times$ into D/P. The latter is a finite group [CaF, p. 71]. So, the former is a finite group. It follows that $C_{\mathcal{T}} = \hat{K}_{\mathcal{T}}^\times/U_{\mathcal{T}}\mathcal{O}_{K,\mathcal{V}_0\smallsetminus\mathcal{T}}^\times$ is a finite group.

Part D: *$C_{\mathcal{T}}$ is compact when $\mathcal{T} = \mathcal{T}_\infty$.* Indeed, let $s = |\mathbb{P}_\infty|$ and $t = |\mathcal{T}|$. Consider a subset $\mathcal{R}$ of $\mathbb{P}_\infty \cup \{\mathfrak{p}_0\}$ and let $r = |\mathcal{R}|$. Define a homomorphism $\lambda_{\mathcal{R}}\colon \hat{K}_{\mathcal{R}}^\times \to \mathbb{R}^r$ by $\lambda_{\mathcal{R}}\big((\alpha_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{R}}\big) = \big(\log|\alpha_{\mathfrak{p}}|_{\mathfrak{p}}\big)_{\mathfrak{p}\in\mathcal{R}}$. Then $\mathrm{Ker}(\lambda_{\mathcal{R}}) = U_{\mathcal{R}}$ and $\lambda_{\mathcal{R}}(\hat{K}_{\mathcal{R}}^\times) = \mathbb{R}^r$ if $\mathcal{R} \subseteq \mathbb{P}_\infty$. Hence $C_{\mathcal{R}} \cong \mathbb{R}^r/\lambda_{\mathcal{R}}(\mathcal{O}_{K,\mathcal{V}_0\smallsetminus\mathcal{R}}^\times)$ if $\mathcal{R} \subseteq \mathbb{P}_\infty$. There are two cases to consider.

Case D1: $\mathfrak{p}_0 \in \mathbb{P}_\infty$. Then $\mathcal{V}_0 \smallsetminus \mathcal{T} = \mathbb{P}_0$ and $\mathcal{T} \subset \mathbb{P}_\infty$. By Dirichlet's unit theorem [CaF, p. 72, Thm], $\lambda_{\mathbb{P}_\infty}(\mathcal{O}_{K,\mathbb{P}_0}^\times)$ is a lattice of rank $s-1$ in the hyperplane $H = \{(x_\mathfrak{p})_{\mathfrak{p}\in\mathbb{P}_\infty} \mid \sum_{\mathfrak{p}\in\mathbb{P}_\infty} n_\mathfrak{p} x_\mathfrak{p} = 0\}$ of $\mathbb{R}^s$. Here $n_\mathfrak{p}$ are positive integers chosen for the product formula to hold in K. In particular, $\lambda_{\mathbb{P}_\infty}(\mathcal{O}_{K,\mathbb{P}_0}^\times)$ contains $s-1$ linearly independent vectors $(x_{i,\mathfrak{p}})_{\mathfrak{p}\in\mathbb{P}_\infty}$, $i = 1,\ldots,s-1$. Their projections on the $\mathcal{T}$-coordinates, namely $(x_{i,\mathfrak{p}})_{\mathfrak{p}\in\mathcal{T}}$, $i = 1,\ldots,s-1$, contain t linearly independent vectors over $\mathbb{R}$ (Lemma 2.2). The latter vectors belong to $\lambda_\mathcal{T}(\mathcal{O}_{K,\mathbb{P}_0}^\times)$. Hence, $C_\mathcal{T} \cong \mathbb{R}^t/\lambda_\mathcal{T}(\mathcal{O}_{K,\mathbb{P}_0}^\times)$ is a quotient of $(\mathbb{R}/\mathbb{Z})^t$ which is a compact group. Conclude from (1b): $C_\mathcal{T}$ is compact.

Case D2: $\mathfrak{p}_0 \in \mathbb{P}_0$. In this case $\mathcal{V}_0 \smallsetminus \mathcal{T} = \mathbb{P}_0 \smallsetminus \{\mathfrak{p}_0\}$ and $\mathcal{T} \subseteq \mathbb{P}_\infty$. Note that $\mathbb{P} \smallsetminus (\mathbb{P}_0 \smallsetminus \{\mathfrak{p}_0\}) = \mathbb{P}_\infty \cup \{\mathfrak{p}_0\}$ has $s+1$ elements. So, by Dirichlet's unit theorem, $\lambda_{\mathbb{P}_\infty\cup\{\mathfrak{p}_0\}}(\mathcal{O}_{K,\mathbb{P}_0\smallsetminus\{\mathfrak{p}_0\}}^\times)$ is a lattice of rank s in the hyperplane

$$H = \big\{(x_\mathfrak{p})_{\mathfrak{p}\in\mathbb{P}_\infty\cup\{\mathfrak{p}_0\}} \mid \sum_{\mathfrak{p}\in\mathbb{P}_\infty\cup\{\mathfrak{p}_0\}} n_\mathfrak{p} x_\mathfrak{p} = 0\big\}$$

of $\mathbb{R}^{s+1}$. In particular, $\lambda_{\mathbb{P}_\infty\cup\{\mathfrak{p}_0\}}\big(\mathcal{O}_{K,\mathbb{P}_0\smallsetminus\{\mathfrak{p}_0\}}^\times\big)$ contains s linearly independent vectors $(x_{i,\mathfrak{p}})_{\mathfrak{p}\in\mathbb{P}_\infty\cup\{\mathfrak{p}_0\}}$, $i = 1,\ldots,s$. Their projection on the $\mathcal{T}$-coordinates contain t linearly independent vectors (Lemma 2.2). The latter belong to $\lambda_\mathcal{T}(\mathcal{O}_{K,\mathbb{P}_0\smallsetminus\{\mathfrak{p}_0\}}^\times)$. Hence, $C_\mathcal{T} \cong \mathbb{R}^t/\lambda_\mathcal{T}(\mathcal{O}_{K,\mathbb{P}_0\smallsetminus\{\mathfrak{p}_0\}}^\times)$ is a quotient of $(\mathbb{R}/\mathbb{Z})^t$. Conclude: $C_\mathcal{T}$ is compact. ∎

The use of compactness of the above group will be through the following lemma.

Lemma 2.4: *Let U be a nonempty open subset of a compact group G. Then there exist $u_1,\ldots,u_n \in U$ such that $1 = u_1\cdots u_n$.*

Proof: Choose $u \in U$. Then $V = u^{-1}U$ is an open neighborhood of 1 such that $uV \subseteq U$. If necessary, replace V by $V \cap V^{-1}$ to assume that V is closed under taking inverse. As G is compact, the sequence $u, u^2, u^3, \ldots$ contains a subsequence $u^{k_1}, u^{k_2}, u^{k_3}, \ldots$ which converges to an element u' of G. Hence, $u^{k_2-k_1}, u^{k_3-k_2}, \ldots$ converge to 1. In particular, there exists an integer $n \ge 2$ such that $v = u^n \in V$. By assumption, $v^{-1} \in V$ and therefore $uv^{-1} \in U$. So, $1 = u^n v^{-1} = u\cdots u(uv^{-1})$ is the desired presentation of 1. ∎

3. Skolem density problems

This section makes adjustments to [JR2, §3,4] in order to include infinite primes and omit the assumption on perfectness.

Data and Assumption 3.1: We fix the following data and assumptions for the rest of this work:

(1a) $\mathcal{V}$ is an infinite proper subset of $\mathbb{P}$.

(1b) $\mathcal{T}$ is a finite subset of $\mathcal{V}$ which contains $\mathcal{S}$.

(1c) $\mathcal{U} = \mathcal{V}_0 \smallsetminus \mathcal{T}$.

(1d) M is a subextension of N/K which is either weakly P$\mathcal{S}$C over $\mathcal{O}_{M,\mathcal{V}}$ or weakly K-stable P$\mathcal{S}$C over $\mathcal{O}_{M,\mathcal{V}}$ and Galois over K. ∎

Lemmas 3.2 and 3.4 below strengthen [JR2, Lemma 3.2]. We are gratefull to Moret-Bailly for his help in the proof of Lemma 3.2.

LEMMA 3.2: *Assume $\mathcal{T} \neq \emptyset$. Consider a monic polynomial $f \in K[X]$ and a positive real number ε. Suppose that*

(2a) $f \in \mathcal{O}_{K,\mathcal{U}}[X]$;

(2b) $f = f_1 \cdots f_r$, *where $f_1, \ldots, f_r \in \mathcal{O}_{K,\mathcal{U}}[X]$ are distinct monic irreducible polynomials over K; and*

(2c) $\mathcal{O}_{K,\mathcal{U}}[c] = \mathcal{O}_{K(c),\mathcal{U}}$ *for each root c of $f(X)$.*

Then there are a monic polynomial $h_0 \in \mathcal{O}_{K,\mathcal{U}}[X]$ of degree $d > \deg(f)$, relatively prime to f, and $\gamma > 0$ such that if a monic polynomial $h \in \mathcal{O}_{N,\mathcal{U}}[X]$ of degree d, relatively prime to f, satisfies $|h - h_0|_{\mathcal{T}} < \gamma$, then each root x of h satisfies:

(3a) $f(x) \in (\mathcal{O}_{\tilde{K},\mathcal{U}})^\times$;

(3b) $|x|_{\mathcal{T}} < \varepsilon$; *and*

(3c) x *is simple and belongs to N.*

Moreover, there exists $h \in \mathcal{O}_{M,\mathcal{U}}[X]$ as above with a root in M.

Proof: Replace $f(X)$ with $f(X)(X - c)$ for some $c \in \mathcal{O}_{K,\mathcal{U}}$ different from the roots of f, if necessary, to assume that $\mathrm{char}(K) \nmid \deg(f)$ if $\mathrm{char}(K) > 0$.

The proof is rather long. So, it may help the reader to know in advance what its main features are.

Part A: *Outline of the proof.* First we use Lemma 2.3 in order to prove that the group $\prod_{\mathfrak{p}\in\mathcal{T}} \big(\hat{K}_\mathfrak{p}[X]/f(X)\hat{K}_\mathfrak{p}[X]\big)^\times$ modulo $\big(\mathcal{O}_{K,\mathcal{U}}[X]/f(X)\mathcal{O}_{K,\mathcal{U}}[X]\big)^\times$ is compact. Then we use Lemma 2.4 to find a large positive integer d, a polynomial $l_0 \in \mathcal{O}_{K,\mathcal{U}}[X]$ relatively prime to f, and for each $\mathfrak{p} \in \mathcal{T}$ a monic polynomial $h_\mathfrak{p} \in \hat{K}_\mathfrak{p}[X]$ of degree d with d distinct zeros in $\hat{K}_\mathfrak{p}$ close to 0 such that $\frac{h_\mathfrak{p}}{f}$ is a Morse function and $h_\mathfrak{p} \equiv l_0 \mod f \cdot \hat{K}_\mathfrak{p}[X]$. Next we apply the strong approximation theorem to find a monic polynomial $h_0 \in \mathcal{O}_{K,\mathcal{U}}[X]$ of degree d, relatively prime to f, which is $\mathfrak{p}$-close to $h_\mathfrak{p}$ for each $\mathfrak{p} \in \mathcal{T}$. Suppose $h \in \mathcal{O}_{N,\mathcal{U}}[X]$ is a monic polynomial of degree d which is relatively prime to f and $\mathcal{T}$-close to h_0. Then each root x of h satisfies (3). Using Example 1.13, we find $h \in \mathcal{O}_{M,\mathcal{U}}[X]$ with a root in M.

Part B: *A compact group.* For each integral domain R which contains $\mathcal{O}_{K,\mathcal{U}}$ let

$$H(R) = \big(R[X]/f(X)R[X]\big)^\times \cong \prod_{i=1}^{r} \big(R[X]/f_i(X)R[X]\big)^\times.$$

Let x_i be a root of f_i, $i = 1, \ldots, r$. Then, by (2c),

$$H(\mathcal{O}_{K,\mathcal{U}}) \cong \prod_{i=1}^{r} \mathcal{O}_{K,\mathcal{U}}[x_i]^\times = \prod_{i=1}^{r} \mathcal{O}^\times_{K(x_i),\mathcal{U}}. \tag{4}$$

Now let $\mathfrak{p} \in \mathcal{T}$. In order to compute $H(\hat{K}_\mathfrak{p})$ we decompose each f_i into its irreducible monic components over $\hat{K}_\mathfrak{p}$:

$$f_i = f_{\mathfrak{p},i,1} \cdots f_{\mathfrak{p},i,r_{i,\mathfrak{p}}},$$

where $r_{i,\mathfrak{p}}$ is the number of primes of $K(x_i)$ which lie over $\mathfrak{p}$ [CaF, p. 58, Cor.]. Let x_{ij} be a root of $f_{\mathfrak{p},i,j}$, $j = 1, \ldots, r_{i,\mathfrak{p}}$. Since $\hat{K}_\mathfrak{p}/K$ is separable (Data 1.7(a)), $f_{\mathfrak{p},i,1}, \ldots, f_{\mathfrak{p},i,r_{i,\mathfrak{p}}}$ are distinct and therefore

$$\big(\hat{K}_\mathfrak{p}[X]/f_i(X)\hat{K}_\mathfrak{p}[X]\big)^\times \cong \prod_{j=1}^{r_{i,\mathfrak{p}}} \hat{K}_\mathfrak{p}(x_{ij})^\times \cong \prod_{\substack{\mathfrak{q}\in\mathbb{P}_{K(x_i)} \\ \mathfrak{q}|\mathfrak{p}}} \widehat{K(x_i)}_\mathfrak{q}^\times.$$

Thus

$$H(\hat{K}_\mathfrak{p}) \cong \prod_{i=1}^{r} \prod_{\substack{\mathfrak{q}\in\mathbb{P}_{K(x_i)} \\ \mathfrak{q}|\mathfrak{p}}} \widehat{K(x_i)}_\mathfrak{q}^\times.$$

So,

$$\prod_{\mathfrak{p}\in\mathcal{T}} H(\hat{K}_{\mathfrak{p}}) \cong \prod_{i=1}^{r} \prod_{\mathfrak{q}\in\mathcal{T}_{K(x_i)}} \widehat{K(x_i)}_{\mathfrak{q}}^{\times}. \tag{5}$$

Identify $H(\mathcal{O}_{K,\mathcal{U}})$ with its image in $\prod_{\mathfrak{p}\in\mathcal{T}} H(\hat{K}_{\mathfrak{p}})$ under the injection given by $g(X) + f(X)\mathcal{O}_{K,\mathcal{U}}[X] \mapsto \big(g(X) + f(X)\hat{K}_{\mathfrak{p}}[X]\big)_{\mathfrak{p}\in\mathcal{T}}$ (note that $f \in \mathcal{O}_{K,\mathcal{U}}[X]$ is monic). Then, by (4), (5), and Lemma 2.3,

$$\Big(\prod_{\mathfrak{p}\in\mathcal{T}} H(\hat{K}_{\mathfrak{p}})\Big)/H(\mathcal{O}_{K,\mathcal{U}}) \cong \prod_{i=1}^{r} \Big(\Big(\prod_{\mathfrak{q}\in\mathcal{T}_{K(x_i)}} \widehat{K(x_i)}_{\mathfrak{q}}^{\times}\Big)/\mathcal{O}_{K(x_i),\mathcal{V}_0\smallsetminus\mathcal{T}}^{\times}\Big) \tag{6}$$

is compact.

Part C: *An open subset of the compact group.* Let D be the set of all positive integers d with $d > \deg(f)$ and $\mathrm{char}(K)|d$ if $\mathrm{char}(K) > 0$. Thus, if $\mathrm{char}(K) > 0$, then $d \not\equiv \deg(f)$ modulo $\mathrm{char}(K)$ for each $d \in D$. For each $d \in D$ and each $\mathfrak{p} \in \mathcal{T}$, let $\Omega_{\mathfrak{p},d}$ be the set of all polynomials h in $\hat{K}_{\mathfrak{p}}[X]$ of the form $h(X) = \prod_{i=1}^{d}(X - a_{i,\mathfrak{p}})$ with

(7a) $a_{1,\mathfrak{p}}, \dots, a_{d,\mathfrak{p}}$ are in $\hat{K}_{\mathfrak{p}}$, mutually distinct, and none of which is a root of f. In particular, h is relatively prime to f;

(7b) $|a_{i,\mathfrak{p}}|_{\mathfrak{p}} < \varepsilon$, $i = 1, \dots, d$; and

(7c) $\frac{h}{f}$ is a Morse function (Definition 4.1).

Also, let

$$W_{\mathfrak{p},d} = \{h(X) + f(X)\hat{K}_{\mathfrak{p}}[X] \mid h \in \Omega_{\mathfrak{p},d}\}.$$

By (7) and Proposition 4.3, $W_{\mathfrak{p},d}$ is a nonempty open subset of $H(\hat{K}_{\mathfrak{p}})$. Hence,

$$W_d = \prod_{\mathfrak{p}\in\mathcal{T}} W_{\mathfrak{p},d}$$

is a nonempty open subset of $\prod_{\mathfrak{p}\in\mathcal{T}} H(\hat{K}_{\mathfrak{p}})$. So,

(8) the image of W_d in $\Big(\prod_{\mathfrak{p}\in\mathcal{T}} H(\hat{K}_{\mathfrak{p}})\Big)/H(\mathcal{O}_{K,\mathcal{U}})$ is nonempty and open.

Claim: *$W_{\mathfrak{p},d}W_{\mathfrak{p},d'} \subseteq W_{\mathfrak{p},d+d'}$ for all $d, d' \in D$.* Indeed, let $h \in \Omega_{\mathfrak{p},d}$ and $h' \in \Omega_{\mathfrak{p},d'}$. Since they are monic, we can write them uniquely as $h = kf + l$ and $h' = k'f + l'$, where

k and k' are monic of degrees $d-\deg(f)$, $d'-\deg(f)$, respectively, and l, l' have degrees $< \deg(f)$ and are invertible modulo f. If h and h' are not coprime, choose $k_1 \in \hat{K}_{\mathfrak{p}}[X]$ monic of degree $d-\deg(f)$ such that $h_1 = k_1 f + l$ is in $\Omega_{\mathfrak{p},d}$ and is relatively prime to h'. Then

$$\begin{aligned}\Big(h(X) + f(X)\hat{K}_{\mathfrak{p}}[X]\Big)\Big(h'(X) + f(X)\hat{K}_{\mathfrak{p}}[X]\Big) &= l(X)l'(X) + f(X)\hat{K}_{\mathfrak{p}}[X] \\ &= h_1(X)h'(X) + f(X)\hat{K}_{\mathfrak{p}}[X]\end{aligned}$$

is in $W_{\mathfrak{p},d+d'}$.

It follows that in the group $\prod_{\mathfrak{p}\in\mathcal{T}} H(\hat{K}_{\mathfrak{p}})$ we have

$$W_d W_{d'} \subseteq W_{d+d'}. \tag{9}$$

Choose some $d_0 \in D$. By (6), (8), and Lemma 2.4, there exists a positive integer n with

$$W_{d_0}^n \cap H(\mathcal{O}_{K,\mathcal{U}}) \neq \emptyset. \tag{10}$$

Let $d = nd_0$. By (9), $W_{d_0}^n \subseteq W_d$. Hence, by (10),

$$W_d \cap H(\mathcal{O}_{K,\mathcal{U}}) \neq \emptyset. \tag{11}$$

Part D: *Construction of h_0 and γ.* By (11), there is $l_0 \in \mathcal{O}_{K,\mathcal{U}}[X]$ of degree $< \deg(f)$ which is invertible modulo f in $\mathcal{O}_{K,\mathcal{U}}[X]$ and, for each $\mathfrak{p} \in \mathcal{T}$, there exists $h_{\mathfrak{p}} \in \Omega_{\mathfrak{p},d}$ such that $h_{\mathfrak{p}} \equiv l_0 \mod f$ in $\hat{K}_{\mathfrak{p}}[X]$. Write each $h_{\mathfrak{p}}$ as $h_{\mathfrak{p}} = k_{\mathfrak{p}} f + l_0$ with $k_{\mathfrak{p}} \in \hat{K}_{\mathfrak{p}}[X]$ monic of degree $e = d - \deg(f)$. For each $\mathfrak{p} \in \mathcal{T}$, let $U_{\mathfrak{p}}$ be the set of all $\mathbf{b} = (b_1, \ldots, b_e)$ in $\hat{K}_{\mathfrak{p}}^e$ such that the polynomial $\big(X^e + \sum_{i=0}^{e-1} b_i X^i\big) f(X) + l_0(X)$ belongs to $\Omega_{\mathfrak{p},d}$. Since $k_{\mathfrak{p}} f + l_0 \in \Omega_{\mathfrak{p},d}$, $U_{\mathfrak{p}}$ is nonempty.

Now use the strong approximation theorem to find $\mathbf{c} = (c_1, \ldots, c_e) \in (\mathcal{O}_{K,\mathcal{U}})^e \cap \bigcap_{\mathfrak{p}\in\mathcal{T}} U_{\mathfrak{p}}$. Let $k_0(X) = X^e + \sum_{i=0}^{e-1} c_i X^i \in \mathcal{O}_{K,\mathcal{U}}[X]$ and let $h_0 = k_0 f + l_0$. Then $h_0 \in \Omega_{\mathfrak{p},d}$ for each $\mathfrak{p} \in \mathcal{T}$. That is, $\frac{h_0}{f}$ is a Morse function, $\gcd(f, h_0) = 1$, each root x of h_0 is simple and belongs to $K_{\mathfrak{p}}$, and $|x|_{\mathfrak{p}} < \varepsilon$ for each $\mathfrak{p} \in \mathcal{T}$. Finally, use Lemma 1.9 to find $\gamma > 0$ such that if $h \in N[X]$ is monic of degree $d = \deg(h_0)$ and $|h - h_0|_{\mathcal{T}} < \gamma$, then each root y of h is simple and belongs to N, and $|y|_{\mathcal{T}} < \varepsilon$.

Part E: *Conclusion of the proof.* Let $h \in \mathcal{O}_{N,\mathcal{U}}[X]$ be a monic polynomial of degree $d = \deg(h_0)$, relatively prime to f, with $|h - h_0|_{\mathcal{T}} < \gamma$. Let x be a root of h. Then x is simple and belongs to N, and $|x|_{\mathcal{T}} < \varepsilon$. Thus, (3b) and (3c) are satisfied.

Since $h \in \mathcal{O}_{N,\mathcal{U}}[X]$ is monic, $x \in \mathcal{O}_{N,\mathcal{U}}$. As $\gcd(f,h) = 1$, there are $g, r \in \mathcal{O}_{N,\mathcal{U}}[X]$ such that $gh + rf = 1$. Since $h(x) = 0$, $r(x)f(x) = 1$. But $r(x), f(x) \in \mathcal{O}_{N,\mathcal{U}}$, so $f(x) \in (\mathcal{O}_{N,\mathcal{U}})^\times$. Thus $f(x) \in (\mathcal{O}_{\tilde{K},\mathcal{U}})^\times$ and (3a) holds.

Finally $h_0 \in \mathcal{O}_{K,\mathcal{U}}[X]$ is N-admissible, $\gcd(f,h_0) = 1$, $\deg(f) < d = \deg(h_0)$, and $\frac{h_0}{f}$ is a Morse function. Choose, by the strong approximation theorem, $0 \neq m \in \mathcal{O}_{K,\mathcal{U}}$ such that $|m|_{\mathcal{T}} < \gamma$. Apply Example 1.13 (b) to find $(a, x_1) \in \mathcal{O}_{M,\mathcal{V}} \times M$ with $mf(x_1)a + h_0(x_1) = 0$. Let $h(X) = maf(X) + h_0(X)$. Then $h \in \mathcal{O}_{M,\mathcal{U}}[X]$ is a monic polynomial of degree d, relatively prime to f, and $|h - h_0|_{\mathcal{T}} < \gamma$. So, its root $x_1 \in M$ satisfies (3). ∎

We denote the maximal purely inseparable extension of a field E by E_{ins}.

Remark 3.3: Lemma 3.2 generalizes [JR2, Lemma 3.2]. In the latter lemma $f \in K[X]$ is separable. That is f decomposes into distinct linear factors over $\tilde{K}$. For nonseparable $f \in K[X]$, [JR2, Thm. 4.3, Case A1] replaces f by its separable kernel $f' \in K_{\mathrm{ins}}[X]$ and K by a finite purely inseparable extension K'. This construction works because M is assumed in [JR2] to be perfect. Here, M being separable over K need not be perfect. So, we decompose f into irreducible factors over $\hat{K}_{\mathfrak{p}}$ for each $\mathfrak{p} \in \mathcal{T}$. As $\hat{K}_{\mathfrak{p}}/K$ is separable, these factors are distinct. This makes the arguments in Part B of the proof of Lemma 3.2 work.

Cantor and Roquette assume throughout their work [CaR] that K is a number field. They note in [CaR, Rem. 1.7] that their proofs work also when K is a function field of one variable over a finite field. However, at the beginning of the proof of [CaR, Lemma 5.2] they write “we may assume that $f(X)$ is free from multiple roots”, where $f \in K[X]$ is a non-constant polynomial. To make this assumption they replace f by its separable kernel. As in the preceding paragraph, this forces a purely inseparable extension of K. So, when $\mathrm{char}(K) > 0$, the proof of [CaR] holds only for $K_{\mathrm{tot},\mathcal{S},\mathrm{ins}}$ and not for $K_{\mathrm{tot},\mathcal{S}}$.

[GPR] proves Rumely's local-global principle for $K_{\mathrm{tot},\mathcal{S}}$ when K is an arbitrary global field. However, the proof of [GPR, Thm. 4.1] applies [CaR, Lemma 5.2] to a polynomial $a_0(X)$ instead of $f(X)$ which need not be separable. So the proof of Rumely's local-global principle that [GPR] gives seems to hold only for $K_{\mathrm{tot},\mathcal{S},\mathrm{ins}}$ but not for $K_{\mathrm{tot},\mathcal{S}}$. ∎

LEMMA 3.4: *Consider a monic polynomial $f \in \mathcal{O}_{K,\mathcal{U}}[X]$, an element $a \in M$, and a positive real number γ. Then there exists $x \in \mathcal{O}_{M,\mathcal{U}}$ with $|x-a|_\mathcal{T} < \gamma$ and $f(x) \in (\mathcal{O}_{M,\mathcal{U}})^\times$.*

Proof: Let L be the Galois closure of $K(a)/K$. Use the strong approximation theorem to find $b \in \mathcal{O}_{K(a),\mathcal{U}}$ such that $\gamma_1 = |b-a|_\mathcal{T} < \gamma$ and $f(b) \neq 0$. For each root c of $f(X)$ let $\mathfrak{f}_c$ be the conductor of $\mathcal{O}_{K,\mathcal{U}}[c-b]$ in its integral closure $\mathcal{O}_{K(c-b),\mathcal{U}}$ [ZaS, p. 269]. It is a nonzero ideal of $\mathcal{O}_{K,\mathcal{U}}[c-b]$, so $\mathfrak{f}_c \cap \mathcal{O}_{K,\mathcal{U}}$ is a nonzero ideal of $\mathcal{O}_{K,\mathcal{U}}$. By (1), $\mathcal{U}_L$ is infinite. So, we can choose $\mathfrak{p}_0 \in \mathcal{U}_L$ such that $\{z \in \mathcal{O}_{K,\mathcal{U}} \mid |z|_{\mathfrak{p}_0} < 1\} \not\supseteq \mathfrak{f}_c \cap \mathcal{O}_{K,\mathcal{U}}$ for each root c of $f(X)$ and $|f(b^\sigma)|_{\mathfrak{p}_0} = 1$ for each $\sigma \in \mathrm{Gal}(L/K)$. Let $\mathcal{R} = \{\mathfrak{p}_0|_K\}$, $\mathcal{T}' = \mathcal{T} \cup \mathcal{R}$, and $\mathcal{U}' = \mathcal{U} \smallsetminus \mathcal{R}$. Then $|f(b)|_\mathfrak{p} = 1$ for each $\mathfrak{p} \in \mathcal{R}_L$. Moreover, $\mathcal{O}_{K,\mathcal{U}'}[c-b] = \mathcal{O}_{K(c-b),\mathcal{U}'}$ for each root c of $f(X)$ [ZaS, Ch. V §5, Lemma, p. 269].

Let $g(Y) = \prod_{\sigma \in \mathrm{Gal}(L/K)} f(Y+b^\sigma) \in \mathcal{O}_{K,\mathcal{U}}[Y]$. Write $g(Y) = g_1(Y)^{\alpha_1} \cdots g_r(Y)^{\alpha_r}$, with $g_1, \ldots, g_r \in \mathcal{O}_{K,\mathcal{U}}[Y]$ distinct monic irreducible polynomials over K. Put $\tilde{g} = g_1 \cdots g_r$. Suppose $\sigma \in \mathrm{Gal}(L/K)$ and c is a root of $f(X)$. Then $\mathcal{O}_{K,\mathcal{U}'}[c^{\sigma^{-1}} - b] = \mathcal{O}_{K(c^{\sigma^{-1}}-b),\mathcal{U}'}$. Hence, $\mathcal{O}_{K,\mathcal{U}'}[c-b^\sigma] = \mathcal{O}_{K(c-b^\sigma),\mathcal{U}'}$. So each root d of $g(Y)$ (hence of $\tilde{g}(Y)$) satisfies $\mathcal{O}_{K,\mathcal{U}'}[d] = \mathcal{O}_{K(d),\mathcal{U}'}$.

Since $\mathcal{T}' \neq \emptyset$, Lemma 3.2 with $\mathcal{T}', \mathcal{U}', \tilde{g}$, $\delta = \min\{\gamma-\gamma_1, 1\}$ replacing $\mathcal{T}, \mathcal{U}, f$, ε gives $y \in \mathcal{O}_{M,\mathcal{U}'}$ with $\tilde{g}(y) \in (\mathcal{O}_{M,\mathcal{U}'})^\times$ and $|y|_{\mathcal{T}'} < \delta$. Let $x = y+b$. Then

$$|x-a|_\mathcal{T} \le |x-b|_\mathcal{T} + |b-a|_\mathcal{T} = |y|_\mathcal{T} + |b-a|_\mathcal{T} < (\gamma-\gamma_1) + \gamma_1 = \gamma,$$

By the choice of δ, $|y|_\mathcal{R} < 1$. So, $y \in \mathcal{O}_{M,\mathcal{U}}$ and $|f(x)|_\mathfrak{p} = |f(b)|_\mathfrak{p} = 1$ for each $\mathfrak{p} \in \mathcal{R}_M$. As $b \in \mathcal{O}_{M,\mathcal{U}}$, we have $x = y+b \in \mathcal{O}_{M,\mathcal{U}}$.

¿From $g_1(y), \ldots, g_r(y) \in \mathcal{O}_{M,\mathcal{U}'}$ and $g_1(y) \cdots g_r(y) = \tilde{g}(y) \in (\mathcal{O}_{M,\mathcal{U}'})^\times$ follows that $g_i(y) \in (\mathcal{O}_{M,\mathcal{U}'})^\times$, $i = 1, \ldots, r$. Hence $g(y) = g_1(y)^{\alpha_1} \cdots g_r(y)^{\alpha_r} \in (\mathcal{O}_{M,\mathcal{U}'})^\times$. Now each $f(y+b^\sigma)$ is in $\mathcal{O}_{M,\mathcal{U}'}$ and $g(y) = \prod_{\sigma\in\mathrm{Gal}(L/K)} f(y+b^\sigma)$. So, $f(y+b^\sigma) \in (\mathcal{O}_{M,\mathcal{U}'})^\times$ for each $\sigma \in \mathrm{Gal}(L/K)$. In particular, $f(x) = f(y+b) \in (\mathcal{O}_{M,\mathcal{U}'})^\times$. Hence, since $|f(x)|_\mathfrak{p} = 1$ for each $\mathfrak{p} \in \mathcal{R}_M$, $f(x) \in (\mathcal{O}_{M,\mathcal{U}})^\times$, as desired. ∎

COROLLARY 3.5: *Let $\mathcal{R}$ be a finite subset of $\mathcal{U}$, $a \in \tilde{K}$, and $\gamma > 0$. Then there is $x \in \tilde{K}$ with $|x-a|_\mathfrak{p} < 1$ for each $\mathfrak{p} \in \tilde{\mathcal{R}}$ and $|x|_\mathfrak{q} = 1$ for each $\mathfrak{q} \in \tilde{\mathcal{U}} \smallsetminus \tilde{\mathcal{R}}$.*

Proof: When $\mathrm{char}(K) = 0$ apply Lemma 3.4 with $\emptyset, \mathcal{R}, \mathcal{U} \smallsetminus \mathcal{R}, X, \tilde{K}$ replacing $\mathcal{S}, \mathcal{T}, \mathcal{U}, f(X), M$ to achieve x.

Suppose $\mathrm{char}(K) > 0$. Take a power q of $\mathrm{char}(K)$ with $a^q \in K_s$. Lemma 3.4 with $\emptyset, \mathcal{R}, \mathcal{U} \smallsetminus \mathcal{R}, X^q, K_s$ replacing $\mathcal{S}, \mathcal{T}, \mathcal{U}, f(X), M$ gives $y \in K_s$ with $|y - a^q|_\mathfrak{p} < 1$ for each $\mathfrak{p} \in \mathcal{R}_{K_s}$ and $|y|_\mathfrak{q} = 1$ for each $\mathfrak{q} \in \mathcal{U}_{K_s} \smallsetminus \mathcal{R}_{K_s}$. Then $x = y^{1/q}$ satisfies $|x-a|_\mathfrak{p} < 1$ for each $\mathfrak{p} \in \tilde{\mathcal{R}}$ and $|x|_\mathfrak{q} = 1$ for each $\mathfrak{q} \in \tilde{\mathcal{U}} \smallsetminus \tilde{\mathcal{R}}$. ∎

In Case A1 of the proof of Theorem 3.7, it becomes necessary to enlarge $\mathcal{T}$ (thus shrinking $\mathcal{U}$). Lemma 3.6 takes care of this enlargement.

LEMMA 3.6: *Let f be a polynomial in $K[X]$ with $|f|_\mathfrak{p} = 1$ for each $\mathfrak{p} \in \mathcal{U}$. Consider a finite subset $\mathcal{R}$ of $\mathcal{U}$, an element $a \in M$, and $\gamma > 0$. Let $\mathcal{T}' = \mathcal{T} \cup \mathcal{R}$ and $\mathcal{U}' = \mathcal{U} \smallsetminus \mathcal{R} = \mathcal{V}_0 \smallsetminus \mathcal{T}'$. Suppose*

(12) *for each $a' \in M$ and each $\gamma' > 0$ there is $x' \in \mathcal{O}_{M,\mathcal{U}'}$ with $|x'-a'|_{\mathcal{T}'} < \gamma'$ and $f(x') \in (\mathcal{O}_{M,\mathcal{U}'})^\times$.*

Then

(13) *there exists $x \in \mathcal{O}_{M,\mathcal{U}}$ with $|x-a|_\mathcal{T} < \gamma$ and $f(x) \in (\mathcal{O}_{M,\mathcal{U}})^\times$.*

Proof:

CLAIM A: *There exists a finite subextension L of M/K which contains a such that for each $\mathfrak{q} \in \mathcal{R}_N$ there exists $b_\mathfrak{q} \in L$ with $|b_\mathfrak{q}|_\mathfrak{q} \le 1$ and $|f(b_\mathfrak{q})|_\mathfrak{q} = 1$.*

Choose a finite set $\mathcal{R}_0$ that represents $\mathcal{R}_N$ over K. Let $\mathfrak{p} \in \mathcal{R}_0$ and let $\mathfrak{p}_0 = \mathfrak{p}|_K$. The Henselian closure $M_\mathfrak{p}$ of M with respect to $\mathfrak{p}|_M$ is K_s (Proposition 1.15). Hence the corresponding residue field $\bar{M}_\mathfrak{p}$ is infinite. By assumption, the reduced polynomial $\bar{f} \in \bar{K}_{\mathfrak{p}_0}[X]$ is nonzero. Hence, there is $x_\mathfrak{p} \in M$ with $|x_\mathfrak{p}|_\mathfrak{p} \le 1$, $\bar{f}(\bar{x}_\mathfrak{p}) \ne 0$, and $|f(x_\mathfrak{p})|_\mathfrak{p} = 1$. Since N/K is Galois, the finite subset $C_\mathfrak{p} = \{x_\mathfrak{p}^\sigma \mid \sigma \in \mathrm{Gal}(N/K)\}$ is contained in N. If M is weakly P$\mathcal{S}$C over $\mathcal{O}_{M,\mathcal{V}}$, then Lemma 1.14 gives a finite subset $B_\mathfrak{p}$ of M such that for each $c \in C_\mathfrak{p}$ and for each $\mathfrak{q} \in \mathcal{R}_N$ which lies over $\mathfrak{p}_0$ there exists $b \in B_\mathfrak{p}$ with $|b-c|_\mathfrak{q} < 1$. If M/K is Galois, set $B_\mathfrak{p} = C_\mathfrak{p}$.

Choose a finite extension L of K in M which contains a and $B_\mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{R}_0$. For each $\mathfrak{q} \in \mathcal{R}_N$ there exists $\mathfrak{p} \in \mathcal{R}_0$ and $\sigma \in \mathrm{Gal}(N/K)$ with $\mathfrak{q} = \mathfrak{p}^\sigma$. Choose $b_\mathfrak{q} \in B_\mathfrak{p}$ with $|b_\mathfrak{q} - x_\mathfrak{p}^\sigma|_\mathfrak{q} < 1$. Then $|x_\mathfrak{p}^\sigma|_\mathfrak{q} = |x_\mathfrak{p}|_\mathfrak{p} \le 1$ and therefore $|b_\mathfrak{q}|_\mathfrak{q} \le 1$. Hence, since the coefficients of f are $\mathfrak{q}$-integral, $|f(b_\mathfrak{q}) - f(x_\mathfrak{p}^\sigma)|_\mathfrak{q} < 1$. Also, $|f(x_\mathfrak{p}^\sigma)|_\mathfrak{q} = |f^\sigma(x_\mathfrak{p}^\sigma)|_{\mathfrak{p}^\sigma} = |f(x_\mathfrak{p})|_\mathfrak{p} = 1$. Hence $|f(b_\mathfrak{q})|_\mathfrak{q} = 1$, as claimed.

Claim B: *There exists $y \in \mathcal{O}_{L,\mathcal{R}}$ with $|y-a|_\mathcal{T} < \frac{\gamma}{2}$ and $|f(y)|_\mathcal{R} = 1$.* Indeed, choose a finite set $\mathcal{R}_1$ which represents $\mathcal{R}_N$ over L and choose a finite set $\mathcal{T}_1$ which represents $\mathcal{T}_N$ over L. For each $\mathfrak{p} \in \mathcal{R}_1$ Claim A gives $b_\mathfrak{p} \in L$ such that $|b_\mathfrak{p}|_\mathfrak{p} \le 1$ and $|f(b_\mathfrak{p})|_\mathfrak{p} = 1$. The weak approximation theorem gives $y \in L$ such that

$$\begin{aligned} |y-a|_\mathfrak{p} &< \frac{\gamma}{2} && \text{for each } \mathfrak{p} \in \mathcal{T}_1 \text{ and} \\ |y-b_\mathfrak{p}|_\mathfrak{p} &< 1 && \text{for each } \mathfrak{p} \in \mathcal{R}_1. \end{aligned}$$

Now let $\mathfrak{q} \in \mathcal{T}_N$. Then there exists $\sigma \in \mathrm{Gal}(N/L)$ and $\mathfrak{p} \in \mathcal{T}_1$ such that $\mathfrak{q} = \mathfrak{p}^\sigma$. Since $a, y \in L$, we have $|y-a|_\mathfrak{q} = |y-a|_\mathfrak{p} < \frac{\gamma}{2}$. If $\mathfrak{q} \in \mathcal{R}_N$, there exists $\sigma \in \mathrm{Gal}(N/L)$ and $\mathfrak{p} \in \mathcal{R}_1$ such that $\mathfrak{q} = \mathfrak{p}^\sigma$. As in Claim A, $|y|_\mathfrak{q} \le 1$ and $|f(y)|_\mathfrak{q} = |f(y)|_\mathfrak{p} = |f(b_\mathfrak{p})|_\mathfrak{p} = 1$, as claimed.

Conclusion of the proof: By assumption, there exists $x \in M$ with $|x-y|_{\mathcal{T}'} < \min\{\frac{\gamma}{2}, 1\}$, $|x|_\mathfrak{q} \le 1$, and $|f(x)|_\mathfrak{q} = 1$ for each $\mathfrak{q} \in \mathcal{U}'_N$. In particular, if $\mathfrak{q} \in \mathcal{T}_N$, then, by Claim B, $|x-a|_\mathfrak{q} \le |x-y|_\mathfrak{q} + |y-a|_\mathfrak{q} < \gamma$. If $\mathfrak{q} \in \mathcal{R}_N$, then $|x-y|_\mathfrak{q} < 1$ and $\mathfrak{q}$ is finite. Since y and the coefficients of f are $\mathfrak{q}$-integral, $|f(x) - f(y)|_\mathfrak{q} < 1$. Hence $|x|_\mathfrak{q} = |y|_\mathfrak{q} \le 1$ and $|f(x)|_\mathfrak{q} = |f(y)|_\mathfrak{q} = 1$. Conclude that $|x|_\mathfrak{q} \le 1$ and $|f(x)|_\mathfrak{q} = 1$ for each $\mathfrak{q} \in \mathcal{U}_N$, as desired. ∎

A data for an $(\mathcal{S}, \mathcal{V})$-**Skolem density problem** for an algebraic extension M' of K consists of a quadruple $(\mathcal{T}', \mathbf{f}, \mathbf{a}, \gamma)$ in which

(14a) $\mathcal{T}'$ is a finite subset of $\mathcal{V}$ containing $\mathcal{S}$;

(14b) $\mathbf{f} = (f_1, \ldots, f_m)$ and $f_i \in \tilde{K}[X_1, \ldots, X_n]$ is $\mathfrak{p}$-**primitive**, i.e. $|f_i|_\mathfrak{p} = 1$, for each $\mathfrak{p} \in \tilde{\mathcal{V}}_0 \smallsetminus \tilde{\mathcal{T}}'$, $i = 1, \ldots, m$;

(14c) a point $\mathbf{a} = (a_1, \ldots, a_n) \in M'^n$; and

(14d) a positive real number γ.

A **solution** is a point $\mathbf{x} \in (\mathcal{O}_{M',\mathcal{V}_0 \smallsetminus \mathcal{T}'})^n$ with $|\mathbf{x}-\mathbf{a}|_{\mathcal{T}'} < \gamma$ and $f(\mathbf{x}) \in \mathcal{O}^{\times}_{\tilde{K},\mathcal{V}_0 \smallsetminus \mathcal{T}'}$.

M' is called **an $\mathcal{S}$-Skolem field with respect to $\mathcal{V}$** if every $(\mathcal{S}, \mathcal{V})$-Skolem density problem for M' has a solution.

THEOREM 3.7: *M is an $\mathcal{S}$-Skolem field with respect to $\mathcal{V}$.*

Proof: Let $\mathbf{f} = (f_1, \ldots, f_m)$ with $f_i \in \tilde{K}[X_1, \ldots, X_n]$ and $|f_i|_{\mathfrak{p}} = 1$ for each $\mathfrak{p} \in \tilde{\mathcal{U}}$, $i = 1, \ldots, m$, $\mathbf{a} = (a_1, \ldots, a_n) \in M^n$, and $\gamma > 0$. We prove that the $\mathcal{S}$-Skolem density problem (with respect to $\mathcal{V}$) with data $(\mathcal{T}, \mathbf{f}, \mathbf{a}, \gamma)$ has a solution. We have to find $\mathbf{x} \in (\mathcal{O}_{M,\mathcal{U}})^n$ with $|\mathbf{x} - \mathbf{a}|_{\mathcal{T}} < \gamma$ and $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}) \in (\mathcal{O}_{\tilde{K},\mathcal{U}})^{\times}$.

We prove this by induction on n. We break the proof into three parts.

PART A: *$n = 1$ and $f_1, \ldots, f_m \in K[X_1]$.* Put $a = a_1$ and $X = X_1$. There are two cases to consider.

CASE A1: *m=1.* Put $f = f_1$. Let c be the leading coefficient of f and let $\mathcal{R} = \{\mathfrak{p} \in \mathcal{U} \mid |c|_{\mathfrak{p}} < 1\}$. Then $\mathcal{R}$ is a finite subset of $\mathcal{U}$. Let $\mathcal{T}' = \mathcal{T} \cup \mathcal{R}$ and $\mathcal{U}' = \mathcal{U} \smallsetminus \mathcal{R}$. By Lemma 3.4, for each $a' \in M$ and each $\gamma' > 0$ there is $x' \in \mathcal{O}_{M,\mathcal{U}'}$ with $|x' - a'|_{\mathcal{T}'} < \gamma'$ and $f(x') \in (\mathcal{O}_{M,\mathcal{U}'})^{\times}$. Lemma 3.6 then gives $x \in M$ that solves the problem with data $(\mathcal{T}, f, a, \gamma)$.

CASE A2: *m is arbitrary.* The polynomial $f = f_1 \cdots f_m$ satisfies $|f|_{\mathfrak{p}} = 1$ for each $\mathfrak{p} \in \mathcal{U}$. By Case A1, there exists $x \in \mathcal{O}_{M,\mathcal{U}}$ such that $|x - a|_{\mathcal{T}} < \gamma$ and $f(x) \in (\mathcal{O}_{\tilde{K},\mathcal{U}})^{\times}$. As $f_1(x), \ldots, f_m(x)$ are in $\mathcal{O}_{\tilde{K},\mathcal{U}}$ and their product is in $(\mathcal{O}_{\tilde{K},\mathcal{U}})^{\times}$, each of them is in $(\mathcal{O}_{\tilde{K},\mathcal{U}})^{\times}$. Hence $x \in M$ solves the problem with data $(\mathcal{T}, \mathbf{f}, a, \gamma)$.

PART B: *The general case for $n = 1$.* Let K' be a finite extension of K which contains the coefficients of $f_1, \ldots, f_m$. The norm $g_i(X) = N_{K'/K} f_i(X)$ is a product $g_i(X) = \prod f_{ij}(X)$ of polynomials f_{ij} which are conjugate to f_i over K. In particular $|f_{ij}|_{\mathfrak{p}} = 1$ and therefore $|g_i|_{\mathfrak{p}} = 1$ for each $\mathfrak{p} \in \tilde{\mathcal{U}}$. Apply Part A to the problem with data $(\mathcal{T}, \mathbf{g}, a, \gamma)$, where $\mathbf{g} = (g_1, \ldots, g_m)$, to get $x \in M$ that solves it. That is $x \in \mathcal{O}_{M,\mathcal{U}}$, $|\mathbf{x} - \mathbf{a}|_{\mathcal{T}} < \gamma$, and $g_1(x), \ldots, g_m(x) \in (\mathcal{O}_{\tilde{K},\mathcal{U}})^{\times}$. Then for each i, the elements $f_{ij}(x)$

are in $\mathcal{O}_{\tilde{K},\mathcal{U}}$ and their product is $g_i(x) \in (\mathcal{O}_{\tilde{K},\mathcal{U}})^\times$. Thus, each $f_{ij}(x)$ is in $(\mathcal{O}_{\tilde{K},\mathcal{U}})^\times$. In particular each $f_i(x)$ is in $(\mathcal{O}_{\tilde{K},\mathcal{U}})^\times$. Conclude that x solves the problem with data $(\mathcal{T}, \mathbf{f}, a, \gamma)$.

PART C: $n > 1$. Suppose now the theorem holds for $n-1$. Let L be a finite extension of K which contains all coefficients of $f_1, \ldots, f_m$. Consider each f_i as a polynomial in $X_1, \ldots, X_{n-1}$ with coefficients in $L[X_n]$ and let $\{h_{ij}(X_n) \in L[X_n] \mid j \in J_i\}$ be the set of all nonzero coefficients of $f_i(\mathbf{X})$. Then $|h_{ij}|_{\mathcal{U}_L} \leq 1$ and $\mathcal{R}_{ij} = \{\mathfrak{p} \in \mathcal{U}_L \mid |h_{ij}|_\mathfrak{p} < 1\}$ is a finite subset of $\mathcal{U}_L$ for each i and each $j \in J_i$. For each i and j and for each $\mathfrak{p} \in \mathcal{R}_{ij}$, let $b_{i,j,\mathfrak{p}} \in L$ be a coefficient of h_{ij} with $|b_{i,j,\mathfrak{p}}|_\mathfrak{p} = |h_{ij}|_\mathfrak{p}$. Then use the weak approximation theorem to find $b_{ij} \in L$ such that $|b_{ij} - b_{i,j,\mathfrak{p}}|_\mathfrak{p} < 1$ for each $\mathfrak{p} \in \mathcal{R}_{ij}$. Find, by Corollary 3.5, $a_{ij} \in \tilde{K}$ with $|a_{ij} - b_{ij}|_\mathfrak{p} < 1$ for each $\mathfrak{p} \in \tilde{\mathcal{R}}_{ij}$ and $|a_{ij}|_\mathfrak{q} = 1$ for each $\mathfrak{q} \in \tilde{\mathcal{U}} \smallsetminus \tilde{\mathcal{R}}_{ij}$. Let $h'_{ij} = \frac{1}{a_{ij}} h_{ij}$. Then $|h'_{ij}|_\mathfrak{p} = 1$ for each $\mathfrak{p} \in \tilde{\mathcal{U}}$. By Part B, applied to the polynomials h'_{ij}, $i = 1, \ldots, m$, $j \in J_i$, there is $x_n \in \mathcal{O}_{M,\mathcal{U}}$ with $|x_n - a_n|_\mathcal{T} < \gamma$ and $h'_{ij}(x_n) \in (\mathcal{O}_{\tilde{K},\mathcal{U}})^\times$, $i = 1, \ldots, m$, $j \in J_i$.

Consider i between 1 and m and $\mathfrak{p} \in \tilde{\mathcal{U}}$. Since $|f_i|_\mathfrak{p} = 1$, there is $j \in J_i$ with $|h_{ij}|_\mathfrak{p} = 1$. Then $|a_{ij}|_\mathfrak{p} = 1$ and therefore $|h_{ij}(x_n)|_\mathfrak{p} = |a_{ij} h'_{ij}(x_n)|_\mathfrak{p} = 1$. Thus $g_i(X_1, \ldots, X_{n-1}) = f_i(X_1, \ldots, X_{n-1}, x_n)$ satisfies $|g_i|_\mathfrak{p} = 1$ for each $\mathfrak{p} \in \tilde{\mathcal{U}}$. Apply induction to the polynomials $g_1, \ldots, g_m$, to get $x_1, \ldots, x_{n-1} \in \mathcal{O}_{M,\mathcal{U}}$ such that $|x_l - a_l|_\mathcal{T} < \gamma$, $l = 1, \ldots, n-1$, and $g_i(x_1, \ldots, x_{n-1}) \in (\mathcal{O}_{\tilde{K},\mathcal{U}})^\times$.

The point $\mathbf{x} = (x_1, \ldots, x_n) \in M^n$ solves the problem with data $(\mathcal{T}, \mathbf{f}, \mathbf{a}, \gamma)$. ∎

COROLLARY 3.8: *Let e be a nonnegative integer. Then, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, both $K_s(\boldsymbol{\sigma}) \cap K_{\mathrm{tot},\mathcal{S}}$ and $K_s[\boldsymbol{\sigma}] \cap K_{\mathrm{tot},\mathcal{S}}$ are $\mathcal{S}$-Skolem fields with respect to $\mathcal{V}$.*

Proof: Combine Lemma 1.12 with Theorem 3.7. ∎

Remark 3.9: As in Corollary 3.5 we can prove that if M' is an algebraic extension of K which is an $\mathcal{S}$-Skolem field with respect to $\mathcal{V}$, then so is M'_{ins}. In particular, Theorem 3.7 and Corollary 3.8 imply the results of [JR2]. ∎

4. Appendix: Morse functions

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Let K be a field of characteristic $p \geq 0$ which we assume for simplicity to be algebraically closed.

Definition 4.1: Let $\varphi = \frac{f}{g}$ be a rational function in $K(t)$, represented as a quotient of two relatively prime polynomials f and g in $K[t]$, with $\varphi(\infty) = \infty$, i.e. $\deg f > \deg g$. Then φ is called a **Morse function**, if it satisfies the following conditions (the first two are for the elements of K which are not roots of g):

(a) The critical points of φ, i.e. the zeros of φ', are non degenerate.

(b) The critical values of φ are distinct, i.e.

$$\varphi'(\tau) = \varphi'(\eta) = 0 \quad \text{and} \quad \varphi(\tau) = \varphi(\eta) \implies \tau = \eta\,.$$

(c) If $p > 0$, then $\varphi(t)$ has no pole of order divisible by p, i.e. g has no zero of order divisible by p and $\deg f \not\equiv \deg g \mod p$. ∎

To make the definition precise we have to explain what **non degenerate** means in (a). If $p \neq 2$, it means that φ' has simple roots, i.e.

$$\varphi'(\tau) = 0 \Longrightarrow \varphi''(\tau) \neq 0\,. \tag{1}$$

If $p = 2$, this does not work since always $\varphi'' = 0$. We have to switch to the modified Hasse-Schmidt derivatives, the second one is given by the expansion (let u, t be independent variables)

$$\varphi(t+u) \equiv \varphi(t) + \varphi'(t)u + \varphi^{[2]}(t)u^2 \mod u^3\,.$$

Comparing to Taylor's formula we have

$$\varphi^{[2]}(t) = \frac{\varphi''(t)}{2} \qquad \text{if } p \neq 2;$$

but, if char$K = 2$, then $\varphi^{[2]}$ is the welcome substitute for the vanishing φ''. Moreover we have in all characteristics, for $a \in K$,

$$\varphi(a) = \varphi'(a) = 0 \neq \varphi^{[2]}(a) \Longleftrightarrow a \text{ is a root of multiplicity 2 of } \varphi\,.$$

So, instead of (1), the non degeneracy of the critical points in condition (a) for all characteristics is the condition

$$\text{(a)} \qquad \varphi'(\tau) = 0 \Longrightarrow \varphi^{[2]}(\tau) \neq 0 \,.$$

Definition 1 is made to fit the following

PROPOSITION 4.2: *Let $\varphi = \frac{f}{g} \in K(t)$ be a Morse function of degree $n = \deg f > \deg g$. Then the Galois group of the covering $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree n is the full symmetric group, i.e.*

$$\mathrm{Gal}(f(t) - xg(t), K(x)) \cong S_n \,.$$

Proof: As $f(t) - xg(t)$ is an irreducible polynomial of degree n in t, its Galois group G is a transitive subgroup of S_n. We look at the ramification of φ over the points of $K = \mathbb{A}^1(K)$: Condition (a) says that the orders of ramification are all ≤ 2, so an equation $\varphi(t) = \alpha$ for $\alpha \in K$ has at most double roots in K. Condition (b) says that no two critical points are in the same fibre, so an equation $\varphi(t) = \alpha$ for $\alpha \in K$ has at least $n-1$ roots in K. Therefore the map φ has the simplest ramification behaviour over the affine line, the ramification group over a finite point $x = \alpha$ just permutes two roots of $f(t) - xg(t)$, i.e. is generated by a transposition. At ∞ condition (c) says that the ramification at ∞ is tame. Now the affine line $\mathbb{A}^1$ has a trivial tame fundamental group, i.e. there is no unramified covering of $\mathbb{A}^1$ which is only tamely ramified at ∞. Therefore the Galois group G is generated by the ramification groups of the finite points, i.e. by transpositions. The proposition follows from the standard fact that a transitive subgroup of a finite symmetric group which is generated by transpositions cannot be proper. ∎

Remark: We proved proposition 4.2 for algebraically closed fields K. But then it holds for all fields K since S_n is the maximal Galois group of a polynomial of degree n. ∎

We are now going to construct Morse functions $\varphi = \frac{f}{g}$ where the denominator g is given as a separable polynomial.

Proposition 4.3: *Let K be an algebraically closed field, let $g \in K[X]$ be a separable polynomial of degree $d \geq 0$, let $n > d$ be an integer such that $n \not\equiv d \mod p$ if $p > 0$. Then the polynomials $f = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0 \in K[X]$ of degree n such that $\frac{f}{g}$ is a Morse function form a Zariski-open dense subset of the affine n-space with coordinates $a_0, \ldots, a_{n-1}$.*

Proof: Condition (c) of the definition of a Morse function is fulfilled since g is separable and $n \not\equiv d \mod p$. Condition (a) means that the numerator of φ', i.e.

$$D(t) = f(t)g'(t) - f'(t)g(t)$$

and the numerator of $\varphi^{[2]}$ which is $gD_2 - g'D$ with

$$D_2(t) = f(t)g^{[2]}(t) - f^{[2]}(t)g(t)$$

have no common root. ¿From $\gcd(g, D) = 1$ follows that we have to achieve $\gcd(D, D_2) = 1$. The resultant of the polynomials D and D_2 is a polynomial $R(a_0, \ldots, a_{n-1})$ in the coefficients of f, and (a) just says $R \neq 0$. Condition (b) means that the product

$$\Pi = \prod_{i \neq j} \big(f(\tau_i)g(\tau_j) - f(\tau_j)g(\tau_i)\big)$$

where τ_i are the roots of $D(t)$ does not vanish. By the theorem on symmetric functions, $\Pi = \Pi(a_1, \ldots, a_n)$ is again a polynomial in the coefficients of f. So the space of polynomials f giving Morse functions φ is given by the inequalities

$$R(a_1, \ldots, a_n) \neq 0, \qquad \Pi(a_1, \ldots, a_n) \neq 0. \tag{2}$$

Hence, it is a Zariski-open set in the affine n-space.

To show that it is dense, it suffices to show that it is not empty. We first assume $d > 0$ and handle the case $d = 0$ at the end. We will show that the following polynomial will satisfy the inequalities (2):

$$f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_2 t^2 + a_1 t + u = f_\circ(t) + u \tag{3}$$

with u transcendental over K and

$$\gcd(f_\circ', g') = 1, \tag{3a}$$

$$f_\circ^{[2]} \neq 0. \tag{3b}$$

Condition (3b) can be satisfied e.g. by the inequality $a_2 \neq 0$. Condition (3a) is for any choice of $a_{n-1}, \ldots, a_2$ satisfied for almost all $a_1 \in K$.

We prove that f/g is a Morse function in the following four steps.

Claim A: *The polynomial equation*

$$D(\tau) = f(\tau)g'(\tau) - f'(\tau)g(\tau) = (f_\circ(\tau) + u)g'(\tau) - f_\circ'(\tau)g(\tau) = 0 \tag{4}$$

for the critical points τ of φ is irreducible over $K(u)$. The degree of $D(t)$ is $n + d - 1$.

Proof of Claim A: $D(t)$ is irreducible in the polynomial ring $K[u,t]$ since it is linear in u and $\gcd(f_\circ' g, g') = 1$. The degree of $D(t)$ is $n+d-1$ since $D(t) = f(t)g'(t) - f'(t)g(t) = (d-n)b_d t^{n+d-1} + \ldots$, where b_d is the leading coefficient of g, and $d-n$ does not vanish by assumption.

Claim B: *The critical points τ of φ are transcendental over K. Moreover u is a rational function of degree $n + d - 1$ of any such τ.*

Proof of Claim B: Otherwise the equation (4) has a vanishing coefficient at u, so $g'(\tau) = 0$, so $f_\circ'(\tau)g(\tau) = 0$, from which by $\gcd(f_\circ', g') = 1$ follows that $g(\tau) = 0$. This contradicts the separability of g. The irreducible equation (4) also gives the presentation

$$u = f_\circ'(\tau) \cdot \frac{g(\tau)}{g'(\tau)} - f_\circ(\tau)\,. \tag{5}$$

Claim C: *The polynomials $D(t)$ and $D_2(t)$ have no common root.*

Proof of Claim C: If τ were a common root of $D(t)$ and $D_2(t)$, then besides (4)

$$(f_\circ(\tau) + u)g^{[2]}(\tau) = f_\circ^{[2]}(\tau)g(\tau) \tag{6}$$

would hold. This is an equation for τ over $K(u)$ of lower degree than $D(\tau) = 0$. By Claim A, it has to be trivial. This gives $g^{[2]}(\tau) = 0$ and therefore $f_\circ^{[2]}(\tau) \cdot g(\tau) = 0$. Since we assumed $f_\circ^{[2]} \neq 0$, we have from Claim B that $f_\circ^{[2]}(\tau) \neq 0$ and $g(\tau) \neq 0$. This shows that (6) is impossible.

Claim D: *The critical values of* φ *are distinct.*

Proof of Claim D: Assume we have two critical points τ and η , i.e. $D(\tau) = D(\eta) = 0$, with same φ-values, so

$$\frac{f(\tau)}{g(\tau)} = \frac{f(\eta)}{g(\eta)} =: v. \tag{7}$$

Then there is by Claim A an isomorphism σ, trivial on $K(u)$, with

$$\sigma(\tau) = \eta\,.$$

Now, by assumption, σ also fixes the quotient v in (7). So σ is the identity on the field

$$L = K(u, v) \subseteq K(\tau)\,.$$

To show $\tau = \eta$ it suffices to show

$$L = K(\tau)\,.$$

To prove this, let $r = [K(\tau) : L]$. This degree r divides the degrees in (5) and (7), namely

$$[K(\tau) : K(u)] = n + d - 1 \qquad \text{and} \qquad [K(\tau) : K(v)] = n\,.$$

Now denote the pole of τ in $K(\tau)/K$ by $\mathfrak{p}$. Consider its ramification over certain subfields. In equations (5) and (7), the degree of the numerator is larger than the degree of the denominator. Therefore $u\mathfrak{p} = v\mathfrak{p} = \infty$, so $\mathfrak{p}$ is a common pole of u and v. ¿From (5) follows that the other poles of u are the zeros of g'; from (7) follows that the other poles of v are the zeros of g. ¿From $\gcd(g, g') = 1$ follows that $\mathfrak{p}$ is the only common pole of u and v. Since K is algebraically closed, $\mathfrak{p}$ is purely ramified over L. So the degree r of the extension $K(\tau)/L$ is the ramification index of $\mathfrak{p}$ over L. The ramification index of $\mathfrak{p}$ in $K(\tau)/K(v)$ is $n - d$ by (7) and is divisible by the ramification index r in $K(\tau)/L$. From

$$r \;\text{ divides }\; \gcd(n + d - 1, n, n - d) = 1$$

follows $L = K(\tau)$ and the proof of the proposition in case $d > 0$ is complete.

It remains to consider the case $g = 1$, i.e. the construction of Morse polynomials $\varphi = f$. If $n = 1$, all polynomials f satisfy (2), so assume $n > 1$. We will show that the following polynomial will satisfy the inequalities (2):

$$(3)' \qquad f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_2t^2 + u_1t + a_0 = f_\circ(t) + u_1t$$

with u_1 transcendental over K and $f_\circ^{[2]} \neq 0$ which may be satisfied by $a_2 \neq 0$. We prove this again in four steps as in the case $d > 0$.

Claim A': *The polynomial equation*

$$(4)' \qquad f'(\tau) = f_\circ'(\tau) + u_1 = 0$$

for the critical points τ of f is irreducible of degree $n - 1$ over $K(u_1)$.

Claim B': From $(4)'$ follows that the critical points τ are transcendental over K, and u_1 is a polynomial of degree $n - 1$ in $K[\tau]$.

Claim C': *The polynomials $f' = f_\circ' + u_1$ and $f^{[2]} = f_\circ^{[2]}$ have no common root, since we assumed $f_\circ^{[2]} \neq 0$ and its roots are algebraic over K.*

Claim D': *The critical values of f are distinct.*

Proof of Claim D': Let τ and η be two roots of f' with

$$(7)' \qquad f(\tau) = f(\eta) =: v_1\,.$$

Then there is by Claim A' an isomorphism σ, trivial on $K(u_1)$ with

$$\sigma(\tau) = \eta\,.$$

Now σ also fixes v_1 in $(7)'$. So σ is the identity on the field

$$L = K(u_1, v_1) \subseteq K(\tau)\,.$$

From

$$[K(\tau) : K(u_1)] = n - 1 \qquad \text{and} \qquad [K(\tau) : K(v_1)] = n$$

follows $L = K(\tau)$, so σ fixes τ, so $\eta = \tau$. This finishes the discussion in case $d = 0$.

Now the proof of the proposition is complete. ∎

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