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PSC Galois Extensions of Hilbertian Fields

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(Revised Version)

Abstract. We prove the following result:

Theorem. *Let K be a countable Hilbertian field, S a finite set of local primes of K, and $e \geq 0$ an integer. Then, for almost all $\boldsymbol{\sigma} \in G(K)^e$, the field $K_s[\boldsymbol{\sigma}] \cap K_{\mathrm{tot},S}$ is PSC.*

Here a **local prime** is an equivalent class $\mathfrak{p}$ of absolute values of K whose completion is a local field, $\hat{K}_{\mathfrak{p}}$. Then $K_{\mathfrak{p}} = K_s \cap \hat{K}_{\mathfrak{p}}$ and $K_{\mathrm{tot},S} = \bigcap_{\mathfrak{p}\in S} \bigcap_{\sigma\in G(K)} K_{\mathfrak{p}}^{\sigma}$. $G(K)$ stands for the absolute Galois group of K. For each $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_e) \in G(K)^e$ we denote the fixed field of $\sigma_1, \ldots, \sigma_e$ in K_s by $K_s(\boldsymbol{\sigma})$. The maximal Galois extension of K in $K_s(\boldsymbol{\sigma})$ is $K_s[\boldsymbol{\sigma}]$. Finally "almost all" means "for all but a set of Haar measure zero".

Introduction

The main goal of our work is to prove the following result:

Theorem A. *Let K be a countable separably Hilbertian field, let S be a finite set of local primes of K, and let e be a nonnegative integer. Then, for almost all $\boldsymbol{\sigma} \in G(K)^e$, the field $K_s[\boldsymbol{\sigma}] \cap K_{\mathrm{tot},S}$ is PSC.*

0.1. Notation in Theorem A

For an arbitrary field K we denote the separable (resp., algebraic) closure of K by K_s (resp., $\tilde{K}$). Let $G(K) = \mathcal{G}(K_s/K)$ be the absolute Galois group of K. If $\sigma_1, \ldots, \sigma_e \in G(K)$, then $K_s(\boldsymbol{\sigma})$ is the fixed field of $\sigma_1, \ldots, \sigma_e$ in K_s. The field $K_s[\boldsymbol{\sigma}]$ is the maximal Galois extension of K which is contained in $K_s(\boldsymbol{\sigma})$. In particular, if $e = 0$, then $K_s[\boldsymbol{\sigma}] = K_s(\boldsymbol{\sigma}) = K_s$.

The absolute Galois group of K is compact with respect to the Krull topology. So, for each e, $G(K)^e$ is a probability space with respect to the normalized Haar measure

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[FJ2, Chap. 16]. Accordingly, the clause "for almost all $\boldsymbol{\sigma} \in G(K)^e$" means "for all $\boldsymbol{\sigma} \in G(K)^e$ but for a set of measure 0".

Suppose now that K is a countable separably Hilbertian field [FJ2, Chap. 11]. By [JR1, Prop. 3.1], the field $K_s(\boldsymbol{\sigma})$ is PAC over K, for almost all $\boldsymbol{\sigma} \in G(K)^e$. In general, for a field M and a subset A of M, we say that M is **PAC over** A if for each dominant separable rational function $\varphi\colon V \to \mathbb{A}^r$ of absolutely irreducible varieties of dimension r over M there exists $\mathbf{x} \in V(M)$ such that $\varphi(\mathbf{x}) \in A^r$. A weaker possible property of M is to be **PAC**. This means that each absolutely irreducible variety over M has an M-rational point. In other words, M is PAC over M. By [Ja4, Lemma 1.2], $K_s[\boldsymbol{\sigma}]$ is PAC for almost all $\boldsymbol{\sigma} \in G(K)^e$. However, we don't know if $K_s[\boldsymbol{\sigma}]$ is PAC over K for almost all $\boldsymbol{\sigma} \in G(K)^e$.

A **local prime** of K is an equivalence class $\mathfrak{p}$ of absolute values such that the completion $\hat{K}_\mathfrak{p}$ is a **local field**. Thus, $\hat{K}_\mathfrak{p}$ is a locally compact field with respect to the $\mathfrak{p}$-adic topology. The field $K_\mathfrak{p} = K_s \cap \hat{K}_\mathfrak{p}$ is then well defined up to a K-isomorphism.

Suppose that S is a finite set of local primes of K. Let

$$K_{\mathrm{tot},S} = \bigcap_{\mathfrak{p}\in S} \bigcap_{\sigma\in G(K)} K_\mathfrak{p}^\sigma$$

be the field of **totally S-adic numbers**. It is the largest Galois extension of K in which each $\mathfrak{p} \in S$ totally splits. An algebraic extension M of K is **PSC** if each absolutely irreducible variety V over M which has a simple $M_\mathfrak{q}$-rational point for each $\mathfrak{p} \in S$ and for each extension $\mathfrak{q}$ of $\mathfrak{p}$ to M also has an M-rational point. This concludes the explanation of the terminology of Theorem A.

0.2. Historical remarks

In [Po1], Pop proves that if $\mathrm{char}(K) = 0$, then $K_{\mathrm{tot},S}$ is PSC. If $\mathrm{char}(K) > 0$, [Po1] replaces $K_{\mathrm{tot},S}$ by its maximal purely inseparable extension. In the case that K is a global field, Moret-Bailly [MoB, Thm. 1.3] and later Green, Pop, and Roquette [GPR, p. 47] sharpen this result. They fix a local prime $\mathfrak{p}_0$ of K not in S. Then they prove that if V is an absolutely irreducible affine variety over K, and if for each local prime $\mathfrak{p} \neq \mathfrak{p}_0$ of K there exists $\mathbf{x}_\mathfrak{p} \in V(K_\mathfrak{p})$ such that $|\mathbf{x}|_\mathfrak{p} \leq 1$, and, in addition, $\mathbf{x}_\mathfrak{p}$ is simple if $\mathfrak{p} \in S$, then there exists $\mathbf{x} \in V(K_{\mathrm{tot},S})$ such that $|\mathbf{x}|_\mathfrak{p} \leq 1$ for all $\mathfrak{p} \neq \mathfrak{p}_0$. This is **Rumely's local-global principle** for $K_{\mathrm{tot},S}$.

[JR2, Thm. 1.5] proves that if K is a number field and M is an algebraic extension of K which is PAC over the ring of integers of K and S consists of ultrametric primes, then $M \cap K_{\mathrm{tot},S}$ satisfies Rumely's local-global principle (where instead of excluding one prime $\mathfrak{p}_0$, one has to exclude all metric primes). This implies that M is PSC [JR2, Remark 8.3(a)]. In particular, $\tilde{K}(\boldsymbol{\sigma}) \cap K_{\mathrm{tot},S}$ is PSC for almost all $\boldsymbol{\sigma} \in G(K)^e$. If, however, K is a function field of one variable over a finite field, then [JR2, Thm. 1.5] proves Rumely's local global principle, like [Po1], only for the maximal purely inseparable extension of $M \cap K_{\mathrm{tot},S}$ and not for $M \cap K_{\mathrm{tot},S}$ itself. We explain the reason for this failure below.

Since we do not know whether $K_s[\boldsymbol{\sigma}]$ is PAC over K for almost all $\boldsymbol{\sigma} \in G(K)$, we can not use the results of [JR2] in order to prove Theorem A, even in the case where K is a number field. So, we must supply a direct proof for it.

It turns out that the only property of almost all fields $K_s(\boldsymbol{\sigma})$ that we use in order to prove Theorem A is that they are PAC over K. In other words, we prove the following result:

Theorem B. *Let S be a finite set of local primes of a field K. Let M be a field which is PAC over K. Denote the maximal Galois extension of K which is contained in $M \cap K_{\mathrm{tot},S}$ by N. Then N is PSC.*

0.3. Ingredients of the proof

Intersecting an arbitrary variety by a suitable hyperplane reduces the check for the PSC property of a field to that of curves. Indeed, in order to prove that a field N between K and $K_{\mathrm{tot},S}$ is PSC it suffices to prove that each curve C over N is birationally equivalent over N to a curve Γ with the following property:

PSC Condition: If $\Gamma_{\mathrm{simp}}(K_\mathfrak{p}) \neq \emptyset$ for each $\mathfrak{p} \in S$, then $\Gamma(N)$ is infinite.

If the function field of C over K is conservative, then C is birationally equivalent to a smooth projective curve Γ. In this case, the proof of the PSC condition for Γ relies on a combination of the theorem about the existence of a stabilizing element for regular extensions of transcendence degree 1 [FJ1, p. 654 and GeJ, p. 336, Thm. F], and Rumely's existence theorem with rationality conditions [Po1, Thms. 1.1 and 3.1]:

Proposition C. *Let S be a finite set of local primes of a field K. Consider a smooth projective curve Γ over K and let F be the function field of Γ. For $i = 1, 2$ and for each $\mathfrak{p} \in S$ let $\mathcal{U}_{i,\mathfrak{p}}$ be a nonempty $\mathfrak{p}$-open subset of $\Gamma(K_\mathfrak{p})$. Write $\mathcal{U}_i = \bigcap_{\mathfrak{p}\in S} \bigcap_{\sigma \in G(K)} \mathcal{U}_{i,\mathfrak{p}}^\sigma$, $i = 1, 2$. Let $d_0 \in \mathbb{N}$. Then F/K has a separating transcendental element t such that*

(a) *all geometric zeros of t belong to $\mathcal{U}_1$ and each of them has multiplicity 1; and*

(b) *all geometric poles of t belong to $\mathcal{U}_2$ and each of them has multiplicity 1.*

(c) *Moreover, let $\hat{F}$ be the Galois closure of $F/K(t)$. Then $\hat{F}/K$ is a regular extension and $\mathcal{G}(\hat{F}/K(t)) \cong S_d$ with $d > d_0$.*

Here is the version of Rumely's existence theorem which we use in the proof of Proposition C. It incorporates a theorem about the continuity of geometric zeros of a curve.

Proposition D. *Let S be a finite set of local primes of a field K. Consider a smooth projective curve Γ over K and let F be the function field of Γ. Let $\mathfrak{a}_0$ be a positive divisor of F/K. For each $\mathfrak{p} \in S$ let $\mathcal{U}_\mathfrak{p}$ be a nonempty $\mathfrak{p}$-open subset of $\Gamma(K_\mathfrak{p})$ and let $\mathcal{U} = \bigcap_{\mathfrak{p}\in S} \bigcap_{\sigma\in G(K)} \mathcal{U}_\mathfrak{p}^\sigma$.*

Then there exists a positive integer k_0 such that for each multiple k of k_0, for $\mathfrak{a} = k\mathfrak{a}_0$, and for each basis $y_0, y_1, \ldots, y_n$ of $\mathcal{L}_K(\mathfrak{a})$ there exists a nonempty intersection $\mathcal{A} \subseteq K^{n+1}$ of $\mathfrak{p}$-open sets, $\mathfrak{p} \in S$, such that for each $\mathbf{a} \in \mathcal{A}$, $t = \sum_{i=0}^n a_i y_i$ is a **Rumely element** *with respect to $S, \mathfrak{a}, \mathcal{U}$. This means that* $\mathrm{div}(t) = \mathbf{p}_1 + \cdots + \mathbf{p}_d - \mathfrak{a}$, *where $\mathbf{p}_1, \ldots, \mathbf{p}_d$ are distinct points of $\Gamma(\tilde{K})$ which belong to* $\mathcal{U} \smallsetminus \mathrm{Supp}(\mathfrak{a})$.

To achieve t of Proposition C, suppose that Γ is embedded in $\mathbb{P}^n$. We project Γ from a point $\mathbf{o}_n \in \mathbb{P}^n(K)$ down to $\mathbb{P}^{n-1}$, and then from a point $\mathbf{o}_{n-1} \in \mathbb{P}^{n-1}$ down to $\mathbb{P}^{n-2}$ until we reach a plane node curve, which we further project to $\mathbb{P}^1$. This is done by choosing a rational function $t = t_1/t_2$ such that t_i is a Rumely element with respect to $S, \mathfrak{a}, \mathcal{U}_i$, $i = 1, 2$. The poles of t_1 and t_2 cancel each other and we have $\mathrm{div}(t) = \mathrm{div}_0(t_1) - \mathrm{div}_0(t_2)$.

The proof of Proposition D takes advantage of the Jacobian variety J of the smooth curve Γ and the compactness of $J(\hat{K}_\mathfrak{p})$.

In addition to Proposition D, we use the following classical result:

Proposition E. *Let Γ be a smooth projective curve over K and let F be the function field of Γ over K. Then there exists a positive integer d_0 such that each divisor $\mathfrak{a}$ of F/K of degree at least d_0 is very ample. That is, each basis of $\mathcal{L}(\mathfrak{a})$ defines an embedding of Γ into the appropriate projective space.*

If the function field F of C is not conservative (so, $\mathrm{char}(K) > 0$), then C is not birationally equivalent to a smooth curve over K any more. This forces us to use heavier tools in order to sharpen and generalize Propositions C, D, and E.

First of all we extend the classical proof of Proposition E to an arbitrary projective curve Γ over K. The main ingredient in this generalization is the Riemann-Roch theorem for a semilocal ring O of the function field F of Γ. The striking phenomena here is that, unlike the genus of Γ which may decrease under purely inseparable extensions of the base field K, the genus of O does not change under arbitrary change of the base field K.

The generalized Riemann-Roch theorem is due to Rosenlicht [Ro1] who uses it in [Ro2] to construct a generalized Jacobian variety J of Γ. This variety is not an abelian variety, as it is in the case where Γ is smooth. It is rather an extension of Γ by a linear algebraic group Λ. In particular, for $\mathfrak{p} \in S$, $J(\hat{K}_\mathfrak{p})$ need not be compact. This presents a severe obstruction to the proof of Rumely's existence theorem. Nevertheless, in the case where Γ is K-normal and $\hat{K}_\mathfrak{p}/K$ is separable, Green, Pop, and Roquette [GPR] overcome this difficulty by replacing $\hat{K}_\mathfrak{p}$ with a purely inseparable extension $\hat{L}$ and observing that the group $\Lambda(\hat{L})$ is annihilated by a power of $\mathrm{char}(K)$. As a result, one may find a Rumely element as in Proposition D. Then it is possible to find an element t that satisfies Conditions (a) and (b) of Proposition C.

This is not good enough, since t has to satisfy Condition (c) of Proposition C as well. Fortunately, Neumann [Neu] establishes this condition in the case where, in our terminology, Γ is a special cusp curve. It is a lucky coincidence that the generalized Jacobian variety of such a curve is an extension of an abelian variety by a linear group Λ such that $\Lambda(\hat{K}_\mathfrak{p})$ is annihilated by a power of $\mathrm{char}(K)$. So, we are able to use the method of [GPR] to prove the analog of Proposition C for special cusp curves.

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1. Pseudo S-adically closed fields

Consider a field K and an equivalence class $\mathfrak{p}$ of absolute values of K (see [CaF, Chap. II] or [Art, Chap. 1] for the definition and the basic properties of absolute values). Choose a completion, $\hat{K}_\mathfrak{p}$, of K at $\mathfrak{p}$. Choose a representative $|\ |_\mathfrak{p}$ for $\mathfrak{p}$ and extend it to $\hat{K}_\mathfrak{p}$ in the unique possible way. We call the induced topology on K and on $\hat{K}_\mathfrak{p}$ the **$\mathfrak{p}$-adic topology**. A basic open neighborhood of $a \in \hat{K}_\mathfrak{p}$ in the $\mathfrak{p}$-adic topology is the set $\{x \in \hat{K}_\mathfrak{p} \mid |x-a|_\mathfrak{p} < \varepsilon\}$ for some positive real number ε.

We say that $\mathfrak{p}$ is a **local prime** of K if $\hat{K}_\mathfrak{p}$ is locally compact in the $\mathfrak{p}$-adic topology. If $|\ |_\mathfrak{p}$ is **ultra-metric** (i.e., $|x+y|_\mathfrak{p} \le \max(|x|_\mathfrak{p}, |y|_\mathfrak{p})$ for all $x, y \in K$), then $\hat{K}_\mathfrak{p}$ is either a finite extension of $\mathbb{Q}_\mathfrak{p}$ for a certain prime number p, or $\hat{K}_\mathfrak{p}$ is a formal power series $\mathbb{F}_q((t))$ for some power q of a prime number [Kow]. If $|\ |_\mathfrak{p}$ is **metric** (i.e., can be chosen to satisfy $|x+y|_\mathfrak{p} \le |x|_\mathfrak{p} + |y|_\mathfrak{p}$ for all $x, y \in K$ and $|n| = n$ for all $n \in \mathbb{N}$), then $\hat{K}_\mathfrak{p} = \mathbb{R}$ (in this case we call $\mathfrak{p}$ **real**) or $\hat{K}_\mathfrak{p} = \mathbb{C}$ (in which case we call $\mathfrak{p}$ **complex**).

We assume that all completions of K as well as their algebraic closures are embedded in a universal field Ω which is algebraically closed. We call $K_\mathfrak{p} = K_s \cap \hat{K}_\mathfrak{p}$ the **$\mathfrak{p}$-closure** of K at $\mathfrak{p}$. In the ultra-metric case $K_\mathfrak{p}$ is the Henselization of K with respect to $\mathfrak{p}$. In the metric case $K_\mathfrak{p}$ is either a real closure of K with respect to the $\mathfrak{p}$-ordering or $K_\mathfrak{p} = \tilde{K}$. In each case $K_\mathfrak{p}$ depends on the embeddings of $\tilde{K}$ and $\hat{K}_\mathfrak{p}$ in Ω only up to K-conjugation.

Let S be a set of local primes of K. Then

$$K_{\mathrm{tot},S} = \bigcap_{\mathfrak{p} \in S} \bigcap_{\sigma \in G(K)} K_\mathfrak{p}^\sigma$$

is the field of **totally S-adic** elements of $\tilde{K}$. By definition, $K_{\mathrm{tot},S}$ is the maximal Galois extension of K in which for each $\mathfrak{p} \in S$ and for each prime $\mathfrak{P}$ of $K_{\mathrm{tot},S}$ above $\mathfrak{p}$, we have $(\widehat{K_{\mathrm{tot},S}})_\mathfrak{P} = \hat{K}_\mathfrak{p}$.

Remark 1.1. The S-topology. For each algebraic extension L of K we denote the set of all extensions of the primes in S to L by S_L. A **basic S-open subset** of $\tilde{K}$ is a set of the form

$$\mathcal{U} = \bigcap_{\mathfrak{p} \in S_L} \bigcap_{\tilde{\mathfrak{p}} | \mathfrak{p}} \{x \in \tilde{K} \mid |x - a_\mathfrak{p}|_{\tilde{\mathfrak{p}}} < \varepsilon_\mathfrak{p}\},$$

where L is a finite extension of K, for each $\mathfrak{p} \in S_L$ the element $a_\mathfrak{p}$ belongs to L, $\varepsilon_\mathfrak{p} > 0$, and $\tilde{\mathfrak{p}}$ ranges over all extensions of $\mathfrak{p}$ to $\tilde{K}$. By the weak approximation theorem [CaF, p. 48], $\mathcal{U}$ contains elements of L, in particular $\mathcal{U}$ is nonempty.

Each nonempty finite intersection of basic S-open subsets of $\tilde{K}$ contains a basic S-open subset of $\tilde{K}$. Hence, the unions of basic S-open subsets of $\tilde{K}$ form a topology on $\tilde{K}$ which we call the **S-topology**.

For each positive integer n, equip $\tilde{K}^n$ with the product topology. It is a Hausdorff topology. Each polynomial in $\tilde{K}[X_1, \ldots, X_n]$ induces an S-continuous map $\tilde{K}^n \to \tilde{K}$. So, if A is an affine $\tilde{K}$-algebraic set, then $A(\tilde{K})$ is closed. Hence, the S-topology naturally extends to a topology of the sets $A(\tilde{K})$ for each quasi-projective set A over $\tilde{K}$.

For each algebraic extension L of K the S-topology of $A(\tilde{K})$ induces a topology on $A(L)$. Thus, we call a subset of $A(L)$, **S-open**, if it is the intersection of an S-open subset of $A(\tilde{K})$ with $A(L)$. □

If $L \subseteq K_{\mathrm{tot},S}$, $\mathfrak{p} \in S$, and $\mathfrak{q}$ is an extension of $\mathfrak{p}$ to L, then each extension of $\mathfrak{q}$ to $K_{\mathrm{tot},S}$ is conjugate over K to the prime of $K_{\mathrm{tot},S}$ which is induced by $K_{\mathfrak{p}}$. It follows that $\mathfrak{q}$ is a local prime of L and each closure $L_{\mathfrak{q}}$ of L at $\mathfrak{q}$ is isomorphic to $K_{\mathfrak{p}}$ over K.

Remark 1.2. Local primes.

(a) If K is a number field, then each absolute value of K defines a local prime $\mathfrak{p}$ of K. In case $\mathfrak{p}$ is a metric prime, one also says that $\mathfrak{p}$ is **infinite** or **archimedean**. If $\mathfrak{p}$ is a finite prime (in our terminology, ultra-metric prime) and p is the prime number under $\mathfrak{p}$, then $\hat{K}_{\mathfrak{p}}$ is a finite extension of the field $\hat{\mathbb{Q}}_p$ of p-adic numbers.

(b) If K is a finite extension of $\mathbb{F}_p(t)$, then each absolute value of K defines an ultra-metric prime $\mathfrak{p}$. The residue field $\bar{K}_{\mathfrak{p}}$ is $\mathbb{F}_{p^m}$, for some positive integer m, and $\hat{K}_{\mathfrak{p}} = \mathbb{F}_{p^m}((x))$, where x is a prime element of K at $\mathfrak{p}$ [Che., p. 46].

(c) Let S be a set of rational primes. For each $p \in S$ the field $\hat{\mathbb{Q}}_p$ has cardinality $2^{\aleph_0}$ and hence also transcendence degree $2^{\aleph_0}$ over $\mathbb{Q}$. Choose a transcendence base X_p for $\hat{\mathbb{Q}}_p/\mathbb{Q}$. Let T be a set of cardinality $\leq 2^{\aleph_0}$ and let $K = \mathbb{Q}(T)$. Then we may embed T into X_p and extend this embedding to an embedding of K into $\hat{\mathbb{Q}}_p$. It induces a local prime $\mathfrak{p}$ on K such that $\hat{K}_{\mathfrak{p}} = \hat{\mathbb{Q}}_p$. Another embedding into $\hat{\mathbb{Q}}_p$ may induce another local prime on K. In this way we get sets of local primes of K which may have large cardinalities.

(d) If K is as in (b), then $\hat{K}_{\mathfrak{p}}$ is a separable extension of K [Ja3, Lemma 2.2]. We give an example where the latter conclusion does not hold.

Consider the field $\mathbb{F}_p((t))$ of formal power series in t over $\mathbb{F}_p$. There are $2^{\aleph_0}$ elements of the form $\sum_{n=0}^{\infty} a_n t^{pn}$ with $a_n \in \mathbb{F}_p$. Since the algebraic closure of $\mathbb{F}_p(t)$ is countable, we may choose x of that form which is not algebraic over $\mathbb{F}_p(t)$. Let $K = \mathbb{F}_p(t,x)$ and let $\mathfrak{p}$ be the local prime of K that the t-adic valuation of $\mathbb{F}_p((t))$ induces. Then $\hat{K}_{\mathfrak{p}} = \mathbb{F}_p((t))$.

Now observe that $x^{1/p} = \sum_{n=0}^{\infty} a_n t^n$ is in $\hat{K}_{\mathfrak{p}}$ while $K(x^{1/p})$ is a purely inseparable extension of $K(x)$ of degree p. Conclude that $\hat{K}_{\mathfrak{p}}/K$ is not a separable extension. □

Definition 1.3. PSC fields. Let S be a set of local primes of a field K and let N be an algebraic extension of K. We say that N is **pseudo S-adically closed**, and abbreviate it by PSC, if every variety V over N satisfies the following local global principle for N:

(2.1) If $V_{\mathrm{simp}}(N_{\mathfrak{q}}) \neq \emptyset$ for each $\mathfrak{q} \in S_N$, then $V(N) \neq \emptyset$.

Here, and in the sequel, whenever we speak about a **variety over** K, we mean a separated scheme V of finite type over K such that $V \times_K \tilde{K}$ is reduced and irreducible. In particular, V is nonempty. In the terminology of Weil's Foundation [Wei], V is a nonvoid abstract variety defined over K (in particular V is absolutely irreducible). Likewise a **curve over** K is a variety over K of dimension 1. As usual, V_{simp} denotes the Zariski-open subset of V of all simple points of V. Also, $V(N)$ is the set of all N-rational points of V. Usually, we take V affine and embedded in $\mathbb{A}^n$ or projective and embedded in $\mathbb{P}^n$ for some positive integer n. Then we view each N-rational point of V as an n-tuple of elements of N or as an equivalence class of nonzero $(n+1)$-tuples of N modulo multiplication by elements of N. $\square$

Remark 1.4. Comparison with other definitions. Consider a field K, a set S of local primes of K, and an algebraic extension N of K.

(a) By definition, N is PSC if and only if N is PS_NC.

(b) Suppose that V is a variety over K with a simple $K_\mathfrak{p}$-rational point for each $\mathfrak{p} \in S$. Then it has a simple $K_\mathfrak{p}^\sigma$-rational point for each $\mathfrak{p} \in S$ and each $\sigma \in G(K)$.

(c) Let $\mathcal{K}$ be a family of field extensions of K. We say that K is P$\mathcal{K}$C if every variety V over K with a simple $\bar{K}$-rational point for each $\bar{K} \in \mathcal{K}$ has a K-rational point [Ja2, §7]. In particular, N is PSC if and only if N is $P\mathcal{N}C$ with $\mathcal{N}$ being the set $\{N_\mathfrak{p}^\sigma \mid \mathfrak{p} \in S_N,\ \sigma \in G(N)\}$.

(d) In particular, suppose that S is the empty set. Then for N to be PSC means that every variety V over N has an N-rational point. So, N is PSC if and only if N is PAC.

(e) Let S_0 be the set obtained from S by removing all complex local primes. Since each variety has simple $\tilde{K}$-rational points, N is PSC if and only if N is PS_0C.

(f) One says that N is **PRC** if N is $P\mathcal{N}C$ with respect to the family $\mathcal{N}$ of all real closures of N. Suppose that S consists of all real metric absolute values of K. Then, the corresponding completions are $\mathbb{R}$. Hence, if in this case N is PSC, then it is also PRC.

The converse is not true. Indeed, take $K = N$ to be a real closed field which can not be embedded in $\mathbb{R}$ (e.g., take K to be a nonprincipal ultrapower of $\mathbb{R}$). Then K is PRC but not PAC [FJ2, Thm. 10.12]. Since S is empty, this means, by (d), that K is not PSC.

(g) Consider a prime number p. Take now S to be the set of all local primes $\mathfrak{p}$ of a field K such that $\hat{K}_\mathfrak{p} = \hat{\mathbb{Q}}_p$. For such $\mathfrak{p}$, $\mathrm{char}(K) = 0$, the residue field, $\bar{K}_\mathfrak{p}$, of $K_\mathfrak{p}$ is $\mathbb{F}_p$, and $v_\mathfrak{p}(p)$ is the smallest positive integer of $v_\mathfrak{p}(K_\mathfrak{p}^\times)$. In other words $v_\mathfrak{p}$ is a **p-adic valuation** of $K_\mathfrak{p}$. Also, $[E : K_\mathfrak{p}] = [\bar{E}_\mathfrak{p} : \bar{K}_\mathfrak{p}](v_\mathfrak{p}(E^\times) : v_\mathfrak{p}(K_\mathfrak{p}^\times))$ for each finite extension E of $K_\mathfrak{p}$ [Ja3, Lemma 2.2(a)]. Thus if $[E : K_\mathfrak{p}] > 1$, then either the residue field of E is larger than $\mathbb{F}_p$, or $v_\mathfrak{p}(p)$ is not the smallest positive element of $v_\mathfrak{p}(E^\times)$. So, the unique extension of v_p to E is not a p-adic valuation. Conclude that $K_\mathfrak{p}$ is a p-adic closure of K.

It follows that if N is a subextension of $K_{\mathrm{tot},S}/K$ and N is PSC, then N is also PpC. That is, N is $P\mathcal{N}C$ with respect to the set $\mathcal{N}$ of all p-adic closures of N.

The converse is not true. Suppose, for example, that K is a p-adically closed field such that the value group of the unique p-adic valuation v of K [HJ1, Prop. 6.3] is not isomorphic to $\mathbb{Z}$ (e.g., K is a nonprincipal ultrapower of $\hat{\mathbb{Q}}_p$). Then K is PpC and $G(K) \cong G(\hat{\mathbb{Q}}_p)$ [HJ1, Cor. 6.6]. In particular $\mathrm{cd}(G(K)) = 2$ [Rib, p. 281] and therefore $G(K)$ is not projective. Hence, K is not PAC [FJ2, Thm. 10.17]. Since S is empty, this implies that K is not PSC.

(h) Let $\mathfrak{p}$ be a local prime of K. If $\hat{K}_\mathfrak{p}/K_\mathfrak{p}$ is a separable extension, then it is also a regular extension. It follows from [Ja3, Lemma 2.3] that $K_\mathfrak{p}$ is existentially closed in $\hat{K}_\mathfrak{p}$. In particular, if a variety V over $K_\mathfrak{p}$ has a simple $\hat{K}_\mathfrak{p}$-rational point, then V

also has a simple $K_\mathfrak{p}$-rational point. If $\hat{K}_\mathfrak{p}/K_\mathfrak{p}$ is not separable, then $\mathrm{char}(K) > 0$. Nevertheless, $K_\mathfrak{p}$ is Henselian. So, the implication "$V_{\mathrm{simp}}(\hat{K}_\mathfrak{p}) \neq \emptyset \Longrightarrow V_{\mathrm{simp}}(K_\mathfrak{p}) \neq \emptyset$" is still valid.

Indeed, the function field F of V over $K_\mathfrak{p}$ is a regular extension of $K_\mathfrak{p}$. Let $\mathbf{t} = (t_1, \ldots, t_r)$ be a separating transcendence basis for $F/K_\mathfrak{p}$, let y be a primitive element for $F/K_\mathfrak{p}(\mathbf{t})$, and let $f \in K_\mathfrak{p}[\mathbf{t}, Y]$ be a monic irreducible polynomial such that $f(\mathbf{t}, y) = 0$. Then $\frac{\partial f}{\partial Y}(\mathbf{t}, y) \neq 0$. The equation $f(\mathbf{T}, \mathbf{Y}) = \mathbf{0}$ defines a variety W over $K_\mathfrak{p}$ which is birationally equivalent to V. Thus V_{simp} has a nonempty Zariski-open subset V_0, W has a nonempty Zariski-open subset W_0 on which $\frac{\partial f}{\partial Y}$ does not vanish, and there exists an isomorphism $\varphi\colon V_0 \to W_0$ which is defined over $K_\mathfrak{p}$. As $V_{\mathrm{simp}}(\hat{K}_\mathfrak{p}) \neq 0$, the density theorem 8.2(b) gives a point $\hat{\mathbf{p}} \in V_0(\hat{K}_\mathfrak{p})$. Then $(\hat{\mathbf{a}}, \hat{b}) = \varphi(\hat{\mathbf{p}}) \in W_0(\hat{K}_\mathfrak{p})$. In particular, $f(\hat{\mathbf{a}}, \hat{b}) = 0$ and $\frac{\partial f}{\partial Y}(\hat{\mathbf{a}}, \hat{b}) \neq 0$. Since $K_\mathfrak{p}$ is $\mathfrak{p}$-dense in $\hat{K}_\mathfrak{p}$, we may approximate $\hat{\mathbf{a}}$ by $\mathbf{a} \in K_\mathfrak{p}^r$. Then, we may use the Henselian property of $\hat{K}_\mathfrak{p}$ in order to find $b \in \hat{K}_\mathfrak{p}$ such that $f(\mathbf{a}, b) = 0$ and $\frac{\partial f}{\partial Y}(\mathbf{a}, b) \neq 0$. Thus, b is separable over $K_\mathfrak{p}$ and therefore belongs to $K_\mathfrak{p}$. It follows that $(\mathbf{a}, b) \in W_0(K_\mathfrak{p})$. The point $\mathbf{p} = \varphi^{-1}(\mathbf{a}, b)$ belongs then to $V_{\mathrm{simp}}(K_\mathfrak{p})$.

Note that indeed, the assumption "$\hat{K}_\mathfrak{p}/K$ is separable" does not appear in any of the theorems of this work.

(i) Suppose that S is a finite set of local primes of K, let $N = K_{\mathrm{tot},S}$, and let N_{ins} be the maximal purely inseparable extension of N. The main theorem of [Po1] states:

(2.2) If V is a $\tilde{K}$-normal variety over N_{ins} (i.e., $V \times_{K_{\mathrm{ins}}} \tilde{K}$ is normal) and V has a simple $\hat{N}_{\mathrm{ins},\mathfrak{q}}$-rational point for each $\mathfrak{q} \in S_{N_{\mathrm{ins}}}$, then V has an N_{ins}-rational point.

One knows that every variety V over N_{ins} is birationally equivalent over N_{ins} to a $\tilde{K}$-normal variety. Moreover, under this equivalence, simple $\hat{N}_{\mathrm{ins},\mathfrak{q}}$-rational points are mapped onto simple $\hat{N}_{\mathrm{ins},\mathfrak{q}}$-rational points. Hence, by (2), N_{ins} is PSC.

(j) In [GPR, §1.4], Green, Pop, and Roquette consider a global field K and a finite set of local primes S of K. Let $N = K_{\mathrm{tot},S}$ and let O_N be the ring of all elements $a \in N$ which are $\mathfrak{q}$-integral for each $\mathfrak{q} \in S_N$. For each $\mathfrak{p} \in S$ let $\hat{O}_\mathfrak{p}$ be the ring of integers of $\hat{K}_\mathfrak{p}$. The Main Theorem of [GPR] implies that each affine variety W over K satisfies the following local-global principle:

(2.3) If $W_{\mathrm{simp}}(\hat{O}_\mathfrak{p}) \neq \emptyset$ for each $\mathfrak{p} \in S$, then $W(O_N) \neq \emptyset$.

We claim that this result implies that N is PSC. Indeed, suppose that V is a variety over N with a point $\mathbf{a}_\mathfrak{q} \in V_{\mathrm{simp}}(N_\mathfrak{q})$, for each $\mathfrak{q} \in S_N$. Then V is already defined over a finite subextension K' of N/K. For each $\mathfrak{p} \in S_{K'}$, let $O_\mathfrak{p}$ be the ring of $\mathfrak{p}$-integers of $K'_\mathfrak{p}$. Further, choose $\mathfrak{q} \in S_N$ over $\mathfrak{p}$ and let $\mathbf{a}_\mathfrak{p} = \mathbf{a}_\mathfrak{q}$. By Proposition 8.2(b), we may assume that V is affine and is embedded in $\mathbb{A}^n$. Next choose $c \in K^\times$ such that $c\mathbf{a}_\mathfrak{p} \in O_\mathfrak{p}^n$ for each $\mathfrak{p} \in S_{K'}$. Multiplication with c gives an automorphism of $\mathbb{A}^n$ which maps V onto an affine variety W over K' and $\mathbf{a}_\mathfrak{p}$ onto the point $c\mathbf{a}_\mathfrak{p} \in W_{\mathrm{simp}}(O_\mathfrak{p})$. By (2.3), applied to K' instead of to K, $W(O_N) \neq \emptyset$. Hence, applying c^{-1}, we get that $V(N) \neq \emptyset$. Conclude that N is PSC.

(k) Suppose again that K is a global field and that S is a finite set of ultra-metric local primes of K. [JR2, Thm. 1.5 and Cor. 1.9] proves that for almost all $\boldsymbol{\sigma} \in G(K)^e$, each affine variety W which is defined over $N = (K_s(\boldsymbol{\sigma}) \cap K_{\mathrm{tot},S})_{\mathrm{ins}}$ satisfies (2.3). As in (j), this implies that N is PSC. $\square$

Once a variety V over a PSC field N has a simple N-rational point, it has many N-rational points:

Proposition 1.5. *Let S be a set of local primes of a field K and let N be a subextension of $K_{\mathrm{tot},S}/K$. Suppose that N is PSC. Let V be a variety over K such that $V_{\mathrm{simp}}(K_{\mathfrak{p}}) \neq \emptyset$ for each $\mathfrak{p} \in S$. Then $V(N)$ is Zariski-dense in V. In particular, $V_{\mathrm{simp}}(N)$ is not empty.*

Proof. Let W be a nonempty Zariski-open subset of V. In particular, W itself is a variety over K. By Proposition 8.2(b), $W_{\mathrm{simp}}(K_{\mathfrak{p}}) \neq \emptyset$ for each $\mathfrak{p} \in S$. Hence, $W(N) \neq \emptyset$. $\square$

The following lemma reduces the check for PSC to curves, indeed to only one curve out of each birationally equivalence class.

Lemma 1.6. *Let S be a finite set of local primes of a field K and let N be a subextension of $K_{\mathrm{tot},S}/K$. Suppose that each curve C over K is birationally equivalent to a curve Γ over K that satisfies the*

PSC Condition: *If $\Gamma_{\mathrm{simp}}(K_{\mathfrak{p}}) \neq \emptyset$ for each $\mathfrak{p} \in S$, then $\Gamma(N)$ is Zariski-dense in Γ.*

Then N is PSC.

Proof. The proof splits into three parts.

Part A: *Varieties over K.* We prove that if W is an arbitrary affine variety over K and $W_{\mathrm{simp}}(K_{\mathfrak{p}}) \neq \emptyset$ for each $\mathfrak{p} \in S$, then $W(N) \neq \emptyset$.

Indeed, for each $\mathfrak{p} \in S$ choose $\mathbf{a}_{\mathfrak{p}} \in W_{\mathrm{simp}}(K_{\mathfrak{p}})$. Since all $K_{\mathfrak{p}}$ are separable over K, there exists an affine curve C on W over K which goes through each of the points $\mathbf{a}_{\mathfrak{p}}$ [JR2, Lemma 10.1]. By assumption, there exists a curve Γ which is birationally equivalent to C over K and which satisfies the PSC-Condition. Then there exist nonempty affine Zariski-open subsets Γ_0 and C_0 of Γ_{simp} and C_{simp} respectively, and there exists an isomorphism $\Gamma_0 \to C_0$ over K. By the density theorem 8.2(b), we may assume that for each $\mathfrak{p} \in S$ we have $\mathbf{a}_{\mathfrak{p}} \in C_0(K_{\mathfrak{p}})$. Hence, $\Gamma_0(K_{\mathfrak{p}}) \neq \emptyset$. So, by the P$S$C-condition, $\Gamma_0(N) \neq \emptyset$. Hence $C(N) \neq \emptyset$. So, $W(N) \neq \emptyset$, as claimed.

Part B: *Descent.* Let U be an affine variety over N with a simple $N_{\mathfrak{q}}$-rational point for each $\mathfrak{q} \in S_N$. We prove that $U(N) \neq \emptyset$.

Indeed, U is already defined over a subextension K' of N/K of a finite degree d. Choose $\sigma_1, \ldots, \sigma_d \in G(K)$ whose restrictions to K' are the d K-embeddings of K' into K_s. Weil's descent gives an affine variety W over K and a K'-isomorphism $\varphi\colon W \to \prod_{i=1}^d \sigma_i U$ [FJ2, Prop. 9.34]. Let $\mathfrak{p} \in S$ and let $1 \leq i \leq d$. Then $N \subseteq K_{\mathrm{tot},S} \subseteq \sigma_i^{-1} K_{\mathfrak{p}}$, and therefore $\sigma_i^{-1} K_{\mathfrak{p}}$ is $N_{\mathfrak{q}}$ for some $\mathfrak{q} \in S_N$. Choose $\mathbf{a}_i \in U_{\mathrm{simp}}(\sigma_i^{-1} K_{\mathfrak{p}})$. Then $\sigma_i \mathbf{a}_i \in (\sigma_i U)_{\mathrm{simp}}(K_{\mathfrak{p}})$. Hence, $\varphi^{-1}(\sigma_1 \mathbf{a}_1, \ldots, \sigma_d \mathbf{a}_d) \in W_{\mathrm{simp}}(K_{\mathfrak{p}})$. By Part A, $W(N) \neq \emptyset$. Hence, $U(N) \neq \emptyset$.

Part C: *Conclusion of the proof.* Consider an arbitrary variety V over N such that $V_{\mathrm{simp}}(N_{\mathfrak{q}}) \neq \emptyset$ for each $\mathfrak{q} \in S_N$. Let U be an open affine subset of V. By Proposition 8.2(b), $U_{\mathrm{simp}}(N_{\mathfrak{q}}) \neq \emptyset$ for each $\mathfrak{q} \in S_N$. Hence, by Part B, $U(N) \neq \emptyset$ and therefore $V(N) \neq \emptyset$. Conclude that N is PSC. □

The local-global theorem for varieties that a P$\mathcal{K}$C field satisfies gives a local-global theorem for the corresponding Brauer groups.

Proposition 1.7. *Let K be a field and let $\mathcal{K}$ be a family of separable algebraic field extensions of K such that K is P$\mathcal{K}$C. Then*

(a) *the natural map $\iota\colon \mathrm{Br}(K) \to \prod_{\bar{K}\in\mathcal{K}} \mathrm{Br}(\bar{K})$ is injective.*

(b) *If $G(\bar{K})$ is projective for each $\bar{K} \in \mathcal{K}$, then $G(K)$ is projective.*

Proof of (a). Starting from a central simple K-algebra A, one may use the properties of the Brauer-Severi variety associated with A (as stated in [Jac, §§3.5–3.8]) to prove (a). We prefer however to use the notion of the reduced norm in order to give a more elementary proof of (a).

We are using $\mathrm{Br}(K)$ to denote the Brauer group of K. Each nontrivial element of $\mathrm{Br}(K)$ may be represented by a central division algebra D over K such that $\dim_K D = n^2$ and $n > 1$. Assume that for each $\bar{K} \in \mathcal{K}$ the algebra $D \otimes_K \bar{K}$ represents the trivial element of $\mathrm{Br}(\bar{K})$. This means that there exists a $\bar{K}$-algebra isomorphism $\alpha\colon D \otimes_K \bar{K} \to M_n(\bar{K})$. Choose a K-basis $\{e_{ij} \mid 1 \leq i,j \leq n\}$ for D and let $E_{ij} = \alpha(e_{ij})$. Then $\{E_{ij} \mid 1 \leq i,j \leq n\}$ is a $\bar{K}$-basis for $M_n(\bar{K})$. Write each $a \in D$ as $\sum_{i,j=1}^n a_{ij}e_{ij}$ with $a_{ij} \in K$. The matrix $\mathbf{a} = (a_{ij})_{1\leq i,j\leq n} \in M_n(K)$ then satisfies $\alpha(a) = \sum_{i,j=1}^n a_{ij}E_{ij} = (\lambda_{kl}(\mathbf{a}))_{1\leq k,l\leq n}$, where λ_{kl} are linearly independent linear forms over $\bar{K}$ in the n^2 variables X_{ij}. Let $\mathbf{X} = (X_{ij})_{1\leq i,j\leq n}$. By a theorem of Skolem and Noether [Deu, p. 43, Satz 5], each automorphism of $M_n(\bar{K})$ is inner. This implies that $p(\mathbf{X}) = \det(\lambda_{kl}(\mathbf{X}))$ is a polynomial of degree n over K which is independent of $\bar{K}$ and α. The element $\mathrm{red.norm}(a) = p(\mathbf{a}) = \det(\alpha(a)) = \det(\lambda_{kl}(\mathbf{a}))$ is the **reduced norm** of a. It satisfies the formula $\mathrm{red.norm}(ab) = \mathrm{red.norm}(a)\cdot\mathrm{red.norm}(b)$ for all $b \in D$. In particular, if $a \in D$ and $a \neq 0$, then there exists $a' \in D$ such that $aa' = 1$. So, $\mathrm{red.norm}(a) \neq 0$.

The change of variables $Y_{kl} = \lambda_{kl}(\mathbf{X})$ maps $p(\mathbf{X})$ onto $\det(\mathbf{Y})$, which is an absolutely irreducible polynomial. Hence, $p(\mathbf{X})$ is also absolutely irreducible. Let $\mathbf{b}$ be the matrix in $M_n(\bar{K})$ which has a zero entries everywhere except in the first $n-1$ places along the diagonal matrix. Then $\mathbf{b}$ is a simple zero of $\det(\mathbf{Y}) = 0$. It follows that also $p(\mathbf{X})$ has a simple $\bar{K}$-rational zero. Since K is P$\mathcal{K}$C, there exists $\mathbf{a} \in M_n(K)$ which is a simple nontrivial zero of $\mathbf{p}(\mathbf{X}) = 0$. Then $a = \sum_{i,j=1}^n a_{ij}e_{ij}$ is a nonzero element of D whose reduced norm is 0. This contradiction proves that the map ι is injective.

Proof of (b). Let K' be a finite separable extension of K. Let $\mathcal{K}' = \{\bar{K}K' \mid \bar{K} \in \mathcal{K}\}$. By [Ja2, Lemma 7.2], K' is P$\mathcal{K}'$C. Since $G(\bar{K})$ is projective, $\mathrm{Br}(\bar{K}K') = 0$ [Rib, p. 261, Cor. 3.7]. Hence, by (a), $\mathrm{Br}(K') = 0$. It follows from [Rib, p. 261] that $G(K)$ is projective. □

Remark 1.8. Part (b) of Proposition 1.7 generalizes the second part of Theorem 3.2 of [Po2], which is proved by different methods. □

2. Ample divisors of curves

The theory of very ample divisors for smooth curves over algebraically closed fields is well documented in various text books. Here we prove the existence of very ample divisors for singular curves over arbitrary fields.

2.1. Points on curves versus prime divisors of function fields

Let K be a field. Consider the m-dimensional projective space $\mathbb{P}^m = \mathbb{P}^m_K$ over a field K. For each field extension L of K the points of $\mathbb{P}^m(L)$ are the equivalence classes $(p_0{:}p_1{:}\cdots{:}p_m)$ of $(m+1)$-tuples of elements of L, not all zero, modulo multiplication by elements of $L^\times$. We write $K(p_0{:}\cdots{:}p_m)$ for the field $K(\frac{p_0}{p_j}, \frac{p_1}{p_j}, \ldots, \frac{p_m}{p_j})$, where j is chosen such that $p_j \neq 0$.

Consider a projective curve $\Gamma \subset \mathbb{P}^m$ over K (which is, by our convention, absolutely irreducible). Let F be the function field of Γ over K. Denote the genus of F/K by g. Denote the set of all prime divisors of F/K by $\mathrm{PrimDiv}(F/K)$. For each $P \in \mathrm{PrimDiv}(F/K)$ let v_P be the normalized discrete valuation associated with P. Also, let O_P (resp., M_P) be the valuation ring (resp., its maximal ideal) of P. Then choose a place $\varphi_P{:}\, F \to \tilde{K} \cup \{\infty\}$ that represents P. Note that φ_P is determined by P only up to conjugation over K. For each $f \in O_P$ write $\varphi_P(f)$ also as $f(P)$.

Now choose a generic point $\mathbf{x} = (x_0{:}x_1{:}\cdots{:}x_m)$ for Γ over K with homogeneous coordinates $x_0, x_1, \ldots, x_m$ in F such that $K(x_0{:}\cdots{:}x_m) = F$. For each prime divisor P of F/K choose $y \in F^\times$ such that $\frac{x_i}{y} \in O_P$ for each i and $\frac{x_j}{y} \notin M_P$ for at least one j (e.g., $y = x_j$ with $v_P(x_j) = \min(v_P(x_0), \ldots, v_P(x_m))$. Then the prime ideal $K[\frac{x_0}{y}, \ldots, \frac{x_m}{y}] \cap M_P$ of $K[\frac{x_0}{y}, \ldots, \frac{x_m}{y}]$ gives a closed point $\mathbf{p}$ of Γ called the **center** of P at Γ. The point $(\frac{x_0}{y}(P){:}\cdots{:}\frac{x_m}{y}(P))$ of $\Gamma(\tilde{K})$, with homogeneous coordinates $\frac{x_i}{y}(P)$, lies over $\mathbf{p}$ and we sometimes abuse notation and call it also the **center** of P at Γ (although it is determined by P only up to K-conjugation). We will also say that P **lies over** $\mathbf{p}$.

Conversely, let $\mathbf{p}$ be a closed point of Γ. Denote the local ring of $\mathbf{p}$ in F by $O_{\Gamma,\mathbf{p}}$ and let $M_{\Gamma,\mathbf{p}}$ be its maximal ideal. There are finitely many K-homomorphisms $\varphi_\mathbf{p}{:}\, O_{\Gamma,\mathbf{p}} \to \tilde{K}$ with kernel $M_{\Gamma,\mathbf{p}}$, all conjugate over K. Nevertheless, we abuse notation and for $f \in O_{\Gamma,\mathbf{p}}$ we write $\varphi_\mathbf{p}(f)$ also as $f(\mathbf{p})$ remembering that $f(\mathbf{p})$ is determined by $\mathbf{p}$ only up to K-conjugation. Next, extend $\varphi_\mathbf{p}$ to a K-place $\varphi{:}\, F \to \tilde{K} \cup \{\infty\}$. Let P be the corresponding prime divisor of F/K that φ determines. Then P lies over $\mathbf{p}$. That is, $O_{\Gamma,\mathbf{p}} \subseteq O_P$ and $O_{\Gamma,\mathbf{p}} \cap M_P = M_{\Gamma,\mathbf{p}}$.

There are only finitely many prime divisors P of F/K that lie over each point $\mathbf{p}$ of

Γ. If $\mathbf{p}$ is simple[1], then $O_{\Gamma,\mathbf{p}}$ is a valuation ring and P is unique[2]. So, we identify $\mathbf{p}$ in this case with P and write $v_{\mathbf{p}}$ instead of v_P.

2.2. Semilocal rings

Let O be a subring of F which contains K and with quotient field F. We say O is a **similocal ring** of F/K if O has only finitely many maximal ideals, say $Q_1, \dots, Q_n$. In this case $O = \bigcap_{i=1}^n O_{Q_i}$.

Conversely, let $O_1, \dots, O_n$ be local rings of F that contain K. Write $O = \bigcap_{i=1}^n O_i$ and suppose F is the quotient field of O. Denote the maximal ideal of O_i by M_i, $i = 1, \dots, n$. Like every ring between K and F, the ring O is Noetherian. Indeed, let A be a nonzero ideal of O and choose $b \in A$, $b \neq 0$. Then $\dim_K O/Ob < \infty$ [Ro1, Thm. 1]. As A/Ob is a subspace of O/Ob, it has a finite basis, say $a_1 + Ob, \dots, a_m + Ob$. It follows that $A = Oa_1 + \cdots + Oa_m + Ob$ is finitely generated. Conclude that O is Noetherian. Since the transcendence degree of F/K is 1, the dimension of O is at most 1. That is, each nonzero prime ideal of O is maximal.

For each i between 1 and n consider the prime ideal $P_i = M_i \cap O$ of O, which, by what we have just said, is maximal (but it is possible that $P_i = P_j$ for $i \neq j$). Then, $O \subseteq \bigcap_{i=1}^n O_{P_i} \subseteq \bigcap_{i=1}^n O_i = O$. So, $O = \bigcap_{i \in I} O_{P_i}$. Hence, by [Bou, p. 93, Cor.], the P_i's are all maximal ideals of O. Conclude that O is a semilocal ring of F/K.

Let $\bar{\mathbf{S}}$ be the set of all $P \in \mathrm{PrimDiv}(F/K)$ that **lie over** O, that is $O \subseteq O_P$. By [Ro1, Thm. 3], $\bar{\mathbf{S}}$ is also the set of all prime divisors of F/K that lie over at least one of the rings O_i. Moreover, $\bar{\mathbf{S}}$ is finite [Ro1, Cor. 2]. Let $\bar{O} = \bigcap_{P \in \bar{\mathbf{S}}} O_P$. By [Lan, p. 12, Prop. 4], $\bar{O}$ is the integral closure of O in F. It is a Dedekind domain with finitely many maximal ideals (hence a principal ideal domain), namely $\bar{M}_P = \bar{O} \cap M_P$, with $P \in \bar{\mathbf{S}}$ [FJ2, Prop. 2.12]. By [Ro1, p. 170, Thm. 1] $\dim_K \bar{O}/O < \infty$. Let $C = \{a \in O \mid a\bar{O} \subseteq O\}$ be the conductor of O in $\bar{O}$. It is the largest ideal of O which is also an ideal of $\bar{O}$. In particular, $C = \prod_{P \in \bar{\mathbf{S}}} \bar{M}_P^{k_P}$ for some nonnegative integers k_P (see also [Ro1, p. 171, Cor. 2]). It follows that

(1) if $a \in O$, $x \in F$, and $v_P(x - a) \geq k_P$ for each $P \in \bar{\mathbf{S}}$, then $x \in O$.

By [ZaS, p. 269, Cor.],

(2) $k_P > 0$ if and only if the local ring of O at $O \cap M_P$ is not integrally closed.

We call $\mathfrak{c} = \sum_{P \in \bar{\mathbf{S}}} k_P P$ the **conductor divisor** of O.

For each divisor $\mathfrak{a}$ of F/K we consider the K-vector space

$$\mathcal{L}(\mathfrak{a}) = \mathcal{L}_{F/K}(\mathfrak{a}) = \{x \in F \mid \mathrm{div}(x) + \mathfrak{a} \geq 0\}.$$

---

[1]Following Weil [Wei, p. 99] and Lang [Lan, p. 198] we call $\mathbf{p}$ **simple** if it satisfies the **Jacobian criterion**. That is, we take an affine open neighborhood $\Gamma_0 \subset \mathbb{A}^m$ for $\mathbf{p}$, let $f_1, \dots, f_r$ be generators for the ideal of all polynomials in $K[X_1, \dots, X_m]$ that vanish on Γ_0 and demand that $\mathrm{rank}\left(\frac{\partial f_i}{\partial X_j}(\mathbf{p})\right) = m - 1$. If $\mathbf{p}$ is not simple, we call it **singular**. We denote the set of all singular points of Γ by Γ_{sing}.

[2]Moreover, if $\tilde{\Gamma} = \Gamma \times_K \tilde{K}$ and $\tilde{\mathbf{p}}$ is a point of $\tilde{\Gamma}$ that lies over $\mathbf{p}$, then $O_{\tilde{\Gamma},\tilde{\mathbf{p}}}$ is still a valuation ring. So, one may also say that $\mathbf{p}$ is **geometrically simple**.

A divisor $\mathfrak{a} = \sum_{i=1}^{m} a_i P_i$ of F/K is **O-smooth** if none of the P_i's is in $\bar{\mathbf{S}}$. In this case we also consider the subspace $\mathcal{L}_O(\mathfrak{a}) = \mathcal{L}_{O/K}(\mathfrak{a}) = \mathcal{L}(\mathfrak{a}) \cap O$. By (1) (with $a = 0$),

$$(3) \qquad \mathcal{L}(\mathfrak{a} - \mathfrak{c}) \subseteq \mathcal{L}_O(\mathfrak{a}).$$

Indeed, if $x \in \mathcal{L}(\mathfrak{a} - \mathfrak{c})$ and $P \in \bar{\mathbf{S}}$, then $v_P(\mathfrak{a}) = 0$ and so $v_P(x) = v_P(x) + v_P(\mathfrak{a}) \geq k_P$.

The Riemann-Roch theorem for Γ controls the dimension of $\mathcal{L}_O(\mathfrak{a})$:

(4) There exists a nonnegative integer genus(O/K) and for each O-smooth divisor $\mathfrak{a}$ of F/K there exists a nonnegative integer $i(\mathfrak{a})$ such that

(4a) $\dim(\mathcal{L}_O(\mathfrak{a})) = \deg(\mathfrak{a}) + 1 - \text{genus}(O/K) + i(\mathfrak{a})$ [Ro1, Thm. 7];

(4b) if $\deg(\mathfrak{a}) > 2g - 2 + \deg(\mathfrak{c})$, then $\dim(\mathcal{L}_O(\mathfrak{a})) = \deg(\mathfrak{a}) + 1 - \text{genus}(O/K)$ [Ro1, Thm. 9].

2.3. The map associated with a Γ-smooth divisor

Let $\mathbf{S}$ be the (finite) set of singular points of Γ. Let $\bar{\mathbf{S}}$ be the set of all prime divisors of F/K that lie over $\mathbf{S}$. Let

$$O = \bigcap_{\mathbf{p} \in \mathbf{S}} O_{\Gamma,\mathbf{p}}, \qquad \bar{O} = \bigcap_{P \in \bar{\mathbf{S}}} O_P.$$

We call O the **semilocal ring of singularities of** Γ. If Γ is smooth, then $O = F$ and genus$(F/K) = g$ is the usual genus of Γ (or of F/K) [Ro1, Thm. 8].

A **Γ-smooth divisor** is a formal sum $\mathfrak{a} = \sum_{i=1}^{n} a_i \mathbf{p}_i$, with $\mathbf{p}_i$ simple points of Γ. By our identification, it is also an O-smooth divisor. Suppose that $n = \dim(\mathcal{L}_O(\mathfrak{a})) - 1 \geq 2$. Choose a K-basis $f_0, \ldots, f_n$ for $\mathcal{L}_O(\mathfrak{a})$. Then $K(f_0 : \cdots : f_n) \subseteq F$ and therefore $\mathbf{f} = (f_0 : \cdots : f_n)$ generates an absolutely irreducible projective curve Δ in $\mathbb{P}^n$ over K (Δ is the closure of $\mathbf{f}$ in $\mathbb{P}^n$). Moreover, the map $\mathbf{x} \mapsto \mathbf{f}$ defines a rational map $\psi\colon \Gamma \to \Delta$. The map ψ is defined at a point $\mathbf{p}$ of Γ if there exists $y \in F^\times$ such that each $\frac{f_i}{y}$ belongs to $O_{\Gamma,\mathbf{p}}$ and not all of them are in $M_{\Gamma,\mathbf{p}}$. In this case $\frac{\mathbf{f}}{y}(\mathbf{p}) = \left(\frac{f_0}{y}(\mathbf{p}) : \cdots : \frac{f_n}{y}(\mathbf{p})\right)$ is a well defined point of $\Delta(\tilde{K})$ which does not depend on the choice of y but depends on the choice of the place φ over $\mathbf{p}$ up to K-conjugation. So, the point $\mathbf{q}$ of Δ which lies below $\frac{\mathbf{f}}{y}(\mathbf{p})$ is well defined and $\psi(\mathbf{p}) = \mathbf{q}$. In particular, $O_{\Delta,\mathbf{q}} \subseteq O_{\Gamma,\mathbf{p}}$. We abuse notation and write in this case also $\psi(\mathbf{p}) = \frac{\mathbf{f}}{y}(\mathbf{p})$. We add the subscript $\mathfrak{a}$ to ψ whenever we want to emphasize the dependence of ψ on $\mathfrak{a}$.

Another choice of the basis changes Δ and ψ by a linear isomorphism of $\mathbb{P}^n$. We say that the map $\psi\colon \Gamma \to \Delta$ is **associated** with $\mathfrak{a}$. Let $\mathbf{p}$ be a point of Γ. We say that the divisor $\mathfrak{a}$ is **very ample** at $\mathbf{p}$ if ψ is biregular at $\mathbf{p}$, i.e., $O_{\Delta,\mathbf{q}} = O_{\Gamma,\mathbf{p}}$. The divisor $\mathfrak{a}$ is **very ample** on Γ if it is very ample at each $\mathbf{p} \in \Gamma$. That is, ψ is an isomorphism.

Lemma 2.1. *Suppose that K is algebraically closed. Let $\mathfrak{a}$ be a Γ-smooth divisor and let $\mathbf{p} \in \Gamma(K)$. Then $\mathfrak{a}$ is very ample at $\mathbf{p}$ in each of the following two cases:*

(A) $\mathbf{p}$ *is simple and* $\deg(\mathfrak{a}) \geq 2g + 1 + \deg(\mathfrak{c})$.

(B) $\mathbf{p} \in \mathbf{S}$ *and* $\deg(\mathfrak{a}) \geq 2g - 1 + 2\deg(\mathfrak{c})$.

Proof. Let $\psi:\Gamma \to \Delta$ be the rational map associated with $\mathfrak{a}$.

Case A: In this case $O_{\Gamma,\mathbf{p}}$ is a valuation ring of F/K. We proceed along classical lines.

Claim A1: *ψ is defined at $\mathbf{p}$ and $O_{\Gamma,\mathbf{p}}$ is the integral closure of $O_{\Delta,\psi(\mathbf{p})}$.* Indeed, let P' be a prime divisor of F/K which is not $\mathbf{p}$. Since K is algebraically closed, $\deg(\mathbf{p}) = \deg(P') = 1$. Hence, by assumption, $\deg(\mathfrak{a} - \mathfrak{c} - \mathbf{p} - P') \geq 2g - 1$. By Riemann-Roch, $\mathcal{L}(\mathfrak{a} - \mathfrak{c} - \mathbf{p} - P')$ is properly contained in both $\mathcal{L}(\mathfrak{a} - \mathfrak{c} - \mathbf{p})$ and $\mathcal{L}(\mathfrak{a} - \mathfrak{c} - P')$. By definition, $\mathcal{L}(\mathfrak{a} - \mathfrak{c} - \mathbf{p}) \cap \mathcal{L}(\mathfrak{a} - \mathfrak{c} - P') = \mathcal{L}(\mathfrak{a} - \mathfrak{c} - \mathbf{p} - P')$. As both $\mathcal{L}(\mathfrak{a} - \mathfrak{c} - \mathbf{p})$ and $\mathcal{L}(\mathfrak{a} - \mathfrak{c} - P')$ are contained in $\mathcal{L}(\mathfrak{a} - \mathfrak{c})$, (3) implies that they are contained in $\mathcal{L}_O(\mathfrak{a})$. Moreover, both $\dim(\mathcal{L}(\mathfrak{a} - \mathfrak{c} - \mathbf{p}))$ and $\dim(\mathcal{L}(\mathfrak{a} - \mathfrak{c} - P'))$ are greater than $\dim(\mathcal{L}(\mathfrak{a} - \mathfrak{c} - \mathbf{p} - P'))$ by 1.

So, we may choose a K-basis $g_0, \ldots, g_n$ for $\mathcal{L}_O(\mathfrak{a})$ such that $g_l \in \mathcal{L}(\mathfrak{a} - \mathfrak{c} - P') \smallsetminus \mathcal{L}(\mathfrak{a} - \mathfrak{c} - \mathbf{p} - P')$ and $g_{l+1} \in \mathcal{L}(\mathfrak{a} - \mathfrak{c} - \mathbf{p}) \smallsetminus \mathcal{L}(\mathfrak{a} - \mathfrak{c} - \mathbf{p} - P')$ for some l between 0 and $n-1$. In particular $v_{\mathbf{p}}(g_i) \geq -v_{\mathbf{p}}(\mathfrak{a})$ for $i = 0, \ldots, n$, $v_{\mathbf{p}}(g_l) = -v_{\mathbf{p}}(\mathfrak{a})$, and $v_{\mathbf{p}}(g_{l+1}) \geq -v_{\mathbf{p}}(\mathfrak{a}) + 1$. Similarly, $v_{P'}(g_i) \geq -v_{P'}(\mathfrak{a})$, for $i = 0, \ldots, n$, $v_{P'}(g_l) \geq -v_{P'}(\mathfrak{a} - \mathfrak{c}) + 1$, and $v_{P'}(g_{l+1}) = -v_{P'}(\mathfrak{a} - \mathfrak{c})$.

It follows that $\frac{g_i}{g_l} \in O_{\Gamma,\mathbf{p}}$, $i = 0, \ldots, n$. Also, $\frac{g_l}{g_l} = 1 \notin M_{\Gamma,\mathbf{p}}$. Hence, ψ can be defined at $\mathbf{p}$ as $\psi(\mathbf{p}) = \frac{\mathbf{g}}{g_l}(\mathbf{p})$. Let $\mathbf{q} = \psi(\mathbf{p})$. Since $v_{P'}\left(\frac{g_{l+1}}{g_l}\right) \leq -1$, we have $\frac{g_{l+1}}{g_l} \notin O_{P'}$. On the other hand, $\frac{g_0}{g_l}, \ldots, \frac{g_n}{g_l}$ generate the coordinate ring of the affine open subset of Δ where the lth coordinate is nonzero. Since $\mathbf{q}$ belongs to this set, $\frac{g_{l+1}}{g_l} \in O_{\Delta,\mathbf{q}}$. Hence, $O_{\Delta,\mathbf{q}} \not\subseteq O_{P'}$.

Thus, $O_{\Gamma,\mathbf{p}}$ is the only valuation ring of F that contains $O_{\Delta,\mathbf{q}}$. Conclude that $O_{\Gamma,\mathbf{p}}$ is the integral closure of $O_{\Delta,\mathbf{q}}$.

Claim A2: *ψ is biregular at $\mathbf{p}$.* By Claim A1, $O_{\Gamma,\mathbf{p}}$ is finitely generated as an $O_{\Delta,\mathbf{q}}$-module [GeJ, Lemma 9.3]. Since K is algebraically closed, we have $K = O_{\Delta,\mathbf{q}}/M_{\Delta,\mathbf{q}} \subseteq O_{\Gamma,\mathbf{p}}/M_{\Gamma,\mathbf{p}} = K$. Hence, $O_{\Gamma,\mathbf{p}} = O_{\Delta,\mathbf{q}} + M_{\Gamma,\mathbf{p}}$. If we find a prime element t for $M_{\Gamma,\mathbf{p}}$ in $O_{\Delta,\mathbf{q}}$, then t will belong to $M_{\Delta,\mathbf{q}}$. Hence, $M_{\Gamma,\mathbf{p}} = tO_{\Gamma,\mathbf{p}} = M_{\Delta,\mathbf{q}}O_{\Gamma,\mathbf{p}}$ and therefore $O_{\Gamma,\mathbf{p}} = O_{\Delta,\mathbf{q}} + M_{\Delta,\mathbf{q}}O_{\Gamma,\mathbf{p}}$. By Nakayama's lemma [Mat1, p. 11], this will imply that $O_{\Gamma,\mathbf{p}} = O_{\Delta,\mathbf{q}}$ and we will be done.

To find t, observe that $\deg(\mathfrak{a} - \mathfrak{c}) > \deg(\mathfrak{a} - \mathfrak{c} - \mathbf{p}) > \deg(\mathfrak{a} - \mathfrak{c} - 2\mathbf{p}) \geq 2g - 1$. Hence, by Riemann-Roch, $\dim(\mathcal{L}(\mathfrak{a} - \mathfrak{c})) = \dim(\mathcal{L}(\mathfrak{a} - \mathfrak{c} - \mathbf{p})) + 1$ and $\dim(\mathcal{L}(\mathfrak{a} - \mathfrak{c} - \mathbf{p})) = \dim(\mathcal{L}(\mathfrak{a} - \mathfrak{c} - 2\mathbf{p})) + 1$. So, by (3), $\mathcal{L}_O(\mathfrak{a}) \supseteq \mathcal{L}(\mathfrak{a} - \mathfrak{c}) \supset \mathcal{L}(\mathfrak{a} - \mathfrak{c} - \mathbf{p}) \supset \mathcal{L}(\mathfrak{a} - \mathfrak{c} - 2\mathbf{p})$.

Choose a K-basis $h_0, \ldots, h_n$ for $\mathcal{L}_O(\mathfrak{a})$ such that $h_r \in \mathcal{L}(\mathfrak{a} - \mathfrak{c} - \mathbf{p}) \smallsetminus \mathcal{L}(\mathfrak{a} - \mathfrak{c} - 2\mathbf{p})$ and $h_{r+1} \in \mathcal{L}(\mathfrak{a} - \mathfrak{c}) \smallsetminus \mathcal{L}(\mathfrak{a} - \mathfrak{c} - \mathbf{p})$ for some r between 0 and $n-1$. Then $v_{\mathbf{p}}(h_i) \geq -v_{\mathbf{p}}(\mathfrak{a})$ for each i, $v_{\mathbf{p}}(h_r) = -v_{\mathbf{p}}(\mathfrak{a}) + 1$, and $v_{\mathbf{p}}(h_{r+1}) = -v_{\mathbf{p}}(\mathfrak{a})$.

The $(n+1)$-tuple $(\frac{h_0}{h_{r+1}}, \ldots, \frac{h_n}{h_{r+1}})$ generates the coordinate ring of an affine open neighborhood of $\psi(\mathbf{p})$. In particular, $t = \frac{h_r}{h_{r+1}}$ belongs to $O_{\Delta,\mathbf{q}}$. By our choice, $v_{\mathbf{p}}(t) = 1$, as desired.

Case B: As $\mathbf{p}$ is singular, we have to use special considerations.

Part B1: *Lifting a basis of O/C^2 to $\mathcal{L}_O(\mathfrak{a})$.* The ideal C^2 of O satisfies

$$\dim_K O/C^2 \leq \dim_K \bar{O}/C^2 = \deg(2\mathfrak{c}) < \infty.$$

Choose $y_0,\ldots,y_l \in O$ such that $y_0,\ldots,y_k$ modulo C^2 form a K-basis for C/C^2 and $y_0,\ldots,y_l$ modulo C^2 form a K-basis for O/C^2.

Next we change the y_i's such that they will belong to $\mathcal{L}_O(\mathfrak{a})$. To do this let $\mathbb{A}$ be the ring of adèles of F/K. Let $\Lambda(\mathfrak{a}-2\mathfrak{c})$ be the vector space over K consisting of all $\alpha \in \mathbb{A}$ such that $v_P(\alpha_P) + v_P(\mathfrak{a}-2\mathfrak{c}) \geq 0$ for all $P \in \mathrm{PrimDiv}(F/K)$. Since $\deg(\mathfrak{a}-2\mathfrak{c}) \geq 2g-1$, the Riemann-Roch theorem implies that $\mathbb{A} = \Lambda(\mathfrak{a}-2\mathfrak{c}) + F$ [FJ2, p. 20]. For each i let α_i be the adèle defined by $\alpha_{i,P} = y_i$ if $P \in \bar{\mathbf{S}}$ and $\alpha_{i,P} = 0$ else. Then there exist $\beta_i \in \Lambda(\mathfrak{a}-2\mathfrak{c})$ and $z_i \in F$ such that $\alpha_i = \beta_i + z_i$. If $P \in \bar{\mathbf{S}}$, then $v_P(\mathfrak{a}) = 0$ and $v_P(y_i - z_i) = v_P(\beta_i) \geq v_P(2\mathfrak{c}) = 2k_P$. Since $y_i \in O$, (1) implies that $z_i \in O$. In particular, $v_P(z_i) + v_P(\mathfrak{a}) \geq 0$. Also,

$$z_i \equiv y_i \mod C^2. \tag{5}$$

If $P \notin \bar{\mathbf{S}}$, then $z_i = -\beta_{i,P}$ and $v_P(\mathfrak{c}) = 0$. Hence, $v_P(z_i)+v_P(\mathfrak{a}) = v_P(\beta_i)+v_P(\mathfrak{a}-2\mathfrak{c}) \geq 0$. Conclude that $\mathrm{div}(z_i) + \mathfrak{a} \geq 0$ and that therefore $z_i \in \mathcal{L}_O(\mathfrak{a})$.

Claim B2: *In the notation of §2.2, $k_P > 0$ for each $P \in \bar{\mathbf{S}}$.* Indeed, by assumption, each $\mathbf{p} \in \mathbf{S}$ is singular. Since K is algebraically closed, $O_{\Gamma,\mathbf{p}}$ is not integrally closed. As the local ring of O at $O \cap M_{\Gamma,P}$ is contained in $O_{\Gamma,P}$, it is also not integrally closed. So, by (2), $k_P > 0$.

Part B3: *Another application of Nakayama's lemma.* Observe that C, as an ideal of the Noetherian ring O, is a finitely generated O-module. Moreover, by Claim B2, C is contained in $\bigcap_{\mathbf{q}\in\mathbf{S}}(O \cap M_{\Gamma,\mathbf{q}})$, which is the Jacobson radical of O. By (5) and by the choice of $y_0,\ldots,y_k$, we have $C = \sum_{i=0}^k Kz_i + C^2 = \sum_{i=0}^k Oz_i + C^2$. Hence, by Nakayama's lemma,

$$C = \sum_{i=0}^k Oz_i. \tag{6}$$

The choice of $y_0,\ldots,y_l$ and (5) implies that

$$O = \sum_{i=0}^l Kz_i + C^2. \tag{7}$$

Let $R = K[z_0,\ldots,z_l]$. Then, by (7), $O = R + C^2$ and therefore $O = R + C$. Suppose by induction that $O = R + C^k$ for some $k \geq 2$. Then, by (6), $C = \sum_{i=0}^k (R + C^k)z_i \subseteq R + C^{k+1}$. Hence, $O = R + C \subseteq R + R + C^{k+1} = R + C^{k+1} \subseteq O$ and therefore $O = R + C^{k+1}$. Thus,

$$O = R + C^r \tag{8}$$

for each $r \geq 1$.

Consider the multiplicative subset $U = \bigcap_{\mathbf{p}'\in\mathbf{S}}(R \smallsetminus M_{\Gamma,\mathbf{p}'})$ of R. The maximal ideals of the quotient ring $U^{-1}R$ belong to the set $\{U^{-1}R \cap M_{\Gamma,\mathbf{p}'} \mid \mathbf{p}' \in \mathbf{S}\}$ [Bou, p. 93, Prop. 17]. In particular there are only finitely many of them. Hence, $U^{-1}R$ is a semilocal ring (by §2.2). By [Ro1, Thm. 2], there exists $r \geq 2$ such that $U^{-1}R$

contains all $y \in F$ such that $v_P(y) \geq r$ for all $P \in \bar{\mathbf{S}}$. Since $k_P > 0$ for each $P \in \bar{\mathbf{S}}$ (Claim B2), $C^r \subseteq U^{-1}R$. Since $U \subseteq O^\times$, (8) implies that $O = U^{-1}R$. It follows that $R_{R \cap M_{\Gamma,\mathbf{p}}} = O_{\Gamma,\mathbf{p}}$.

Part B4: *Conclusion of the proof.* By (5) and by the choice of $y_0, \dots, y_l$, the elements $z_0, \dots, z_l$ of $\mathcal{L}_O(\mathfrak{a})$ are linearly independent over K modulo C^2. Extend $z_0, \dots, z_l$ to a K-basis $z_0, \dots, z_n$ of $\mathcal{L}_O(\mathfrak{a})$. Then assume without loss that $z_0, \dots, z_n$ define ψ.

Next observe by Claim B2 that $f(\mathbf{p}) = 0$ for each $f \in C$. Hence, by (7), there exists j between $k+1$ and l such that $z_j(\mathbf{p}) \neq 0$. Also, for each i, $z_i \in O \subseteq O_{\Gamma,\mathbf{p}}$. Hence, ψ is defined at $\mathbf{p}$ and $\mathbf{q} = \psi(\mathbf{p}) = (z_0(\mathbf{p}) : \cdots : z_n(\mathbf{p}))$. Let $R' = K[z_0, \dots, z_n]$. By Part B3, $O_{\Delta,\mathbf{q}} = R'_{R' \cap M_{\Gamma,\mathbf{p}}} = O_{\Gamma,\mathbf{p}}$. Conclude that ψ is biregular at $\mathbf{p}$. □

The next result will enable us to descend from $\tilde{K}$ to K.

Lemma 2.2. *Let K' be a field extension of K which is linearly disjoint from F, let $F' = FK'$, and let $\Gamma' = \Gamma \times_K K'$. Consider a finite subset $\mathbf{T}$ of Γ and let $\mathbf{T}'$ be the set of all points of Γ' that lie over $\mathbf{T}$. Let $R = \bigcap_{\mathbf{p} \in \mathbf{T}} O_{\Gamma,\mathbf{p}}$ and let $R' = \bigcap_{\mathbf{p}' \in \mathbf{T}'} O_{\Gamma',\mathbf{p}'}$. Then $R' = RK'$ and $R = F \cap R'$. Moreover, for each $\mathbf{p}' \in \mathbf{T}'$, $O_{\Gamma',\mathbf{p}'}$ is the local ring of R' at $R' \cap M_{\Gamma',\mathbf{p}'}$.*

Proof. If $\mathbf{p} \in \mathbf{T}$, $\mathbf{p}' \in \mathbf{T}'$, and $\mathbf{p}'$ lies over $\mathbf{p}$, then $O_{\Gamma,\mathbf{p}} \subseteq O_{\Gamma',\mathbf{p}'}$. Hence, with $A' = RK'$, we have $A' \subseteq R'$. Since Γ is projective, $\mathbf{T}$ is contained in an affine open subset Γ_0 of Γ. Let A_0 be the coordinate ring of Γ_0. Then $A_0 \subseteq \bigcap_{\mathbf{p} \in \mathbf{T}} O_{\Gamma,\mathbf{p}} = R$. Moreover, $A_0' = A_0K'$ is the coordinate ring of $\Gamma_0' = \Gamma_0 \times_K K'$, which is an affine open neighborhood in Γ' of each $\mathbf{p}' \in \mathbf{T}'$. In particular, $O_{\Gamma',\mathbf{p}'}$ is the local ring of A_0' at $A_0' \cap M_{\Gamma',\mathbf{p}'}$. Since $A_0' \subseteq A'$, this implies that $O_{\Gamma',\mathbf{p}'} = A'_{A' \cap M_{\Gamma',\mathbf{p}'}}$.

By [Ro1, p. 181, Thm. 11], A' is a semilocal ring of F'/K'. Moreover, the prime divisors of F'/K' with nonzero centers in A' are extensions of the prime divisors of F/K with nonzero centers at R. Hence, by the second paragraph of §2.2, as $\mathbf{p}'$ ranges over $\mathbf{T}'$, $A' \cap M_{\Gamma',\mathbf{p}'}$ ranges over all maximal ideals of A'. It follows from the preceding paragraph that

$$A' = \bigcap_{\mathbf{p}' \in \mathbf{T}'} A'_{A' \cap M_{\Gamma',\mathbf{p}'}} = \bigcap_{\mathbf{p}' \in \mathbf{T}'} O_{\Gamma',\mathbf{p}'} = R'.$$

Finally, $R = F \cap R'$ follows from [Ro1, Thm. 11]. □

Lemma 2.3. *Let $\mathfrak{a}$ be a Γ-smooth divisor of F/K. Let K' be an extension of K which is linearly disjoint from F, let $F' = FK'$, and let $\Gamma' = \Gamma \times_K K'$. Identify $\mathfrak{a}$ as a Γ'-smooth divisor $\mathfrak{a}'$ of FK'/K'. Denote the semilocal ring of singularities of Γ' in F' by O'. Then*

(a) *$O' = OK'$ and $F \cap O' = O$;*

(b) *each K-basis of $\mathcal{L}_O(\mathfrak{a})$ is also a K'-basis of $\mathcal{L}_{O'}(\mathfrak{a}')$;*

(c) *$\psi_{\mathfrak{a}'}(\Gamma')$ is obtained from $\psi_{\mathfrak{a}}(\Gamma)$ by extension of scalars from K to K';*

(d) *$\mathfrak{a}$ is very ample if and only if $\mathfrak{a}'$ is very ample; and*

(e) $\mathrm{genus}(O'/K') = \mathrm{genus}(O/K)$.

Proof of (a). Let $\mathbf{S}'$ be the set of all points of Γ' that lie over points of $\mathbf{S}$. By definition $O' = \bigcap_{\mathbf{p}'\in\mathbf{S}'} O_{\Gamma',\mathbf{p}'}$. By Lemma 2.2, $O' = OK'$ and $O = F \cap O'$.

Proof of (b). Suppose that $f_0,\ldots,f_n$ form a K-basis of $\mathcal{L}_O(\mathfrak{a})$. By the linear disjointness, $f_0,\ldots,f_n$ are linearly independent over K'. So, it suffices to prove that $\mathcal{L}_{O'}(\mathfrak{a}') = \mathcal{L}_O(\mathfrak{a})K'$. Indeed, it suffices to prove that each $g' \in \mathcal{L}_{O'}(\mathfrak{a}')$ belongs to $\mathcal{L}_O(\mathfrak{a})K'$. To that end choose a basis $\{w_i \mid i \in I\}$ for K'/K. By the linear disjointness, the w_i are linearly independent over F. As $g' \in O'$ and $O' = OK'$, we can write $g' = \sum_{i\in I} g_i w_i$, with $g_i \in O$ and almost all of them are 0. It suffices to prove that all g_i are in $\mathcal{L}_O(\mathfrak{a})$.

Consider therefore a Γ-smooth prime divisor $\mathbf{p}$ of F/K. Our convention identifies $\mathbf{p}$ with a smooth point of Γ such that $O_{\Gamma,\mathbf{p}} = O_{\mathbf{p}}$. Hence, by Lemma 2.2, $O_{\mathbf{p}}K'$ is the intersection of all valuation rings $O_{\mathbf{p}'}$ of F', where $\mathbf{p}'$ ranges over all prime divisors of F'/K' which lie over $\mathbf{p}$. Let $\pi_{\mathbf{p}}$ be an element of F such that $v_{\mathbf{p}}(\pi_{\mathbf{p}}) = 1$. Since $\mathbf{p}$ is simple, $v_{\mathbf{p}'}(\pi_{\mathbf{p}}) = 1$ for each $\mathbf{p}'$ as above and $n_{\mathbf{p}} = v_{\mathbf{p}}(\mathfrak{a}) = v_{\mathbf{p}'}(\mathfrak{a}')$. By assumption, $v_{\mathbf{p}'}(g') + n_{\mathbf{p}} \ge 0$. Hence, $v_{\mathbf{p}'}(\pi_{\mathbf{p}}^{n_{\mathbf{p}}} g') \ge 0$ for all $\mathbf{p}'$ over $\mathbf{p}$. So, $\sum_{i\in I} \pi_{\mathbf{p}}^{n_{\mathbf{p}}} g_i w_i = \pi_{\mathbf{p}}^{n_{\mathbf{p}}} g' \in O_{\mathbf{p}}K'$. It follows that $\pi_{\mathbf{p}}^{n_{\mathbf{p}}} g_i \in O_{\mathbf{p}}$ and therefore $v_{\mathbf{p}}(g_i) + v_{\mathbf{p}}(\mathfrak{a}) \ge 0$ for all $i \in I$. Since $\mathfrak{a}$ is Γ-smooth, this implies that the g_i are indeed in $\mathcal{L}_O(\mathfrak{a})$.

Proof of (c). Apply (b).

Proof of (d). Let $\mathbf{p}$ be a point of Γ and let $\mathbf{p}'$ be a point of Γ' over $\mathbf{p}$. By (c), $\psi_{\mathfrak{a}}$ is defined at $\mathbf{p}$ if and only if $\psi_{\mathfrak{a}'}$ is defined at $\mathbf{p}'$. So, assume that this is the case. Let $\mathbf{q} = \psi_{\mathfrak{a}}(\mathbf{p})$ and let $\mathbf{q}' = \psi_{\mathfrak{a}'}(\mathbf{p}')$. By assumption, $O_{\Delta,\mathbf{q}} \subseteq O_{\Gamma,\mathbf{p}}$ and $O_{\Delta',\mathbf{q}'} \subseteq O_{\Gamma',\mathbf{p}'}$. Also, $O_{\Gamma,\mathbf{p}} \subseteq O_{\Gamma',\mathbf{p}'}$ and $O_{\Delta,\mathbf{q}} \subseteq O_{\Delta',\mathbf{q}'}$.

Suppose first that ψ' is very ample at $\mathbf{p}'$. Then $O_{\Delta',\mathbf{q}'} = O_{\Gamma',\mathbf{p}'}$. By [Lan, p. 92], $F \cap O_{\Delta',\mathbf{q}'} = O_{\Delta,\mathbf{q}}$. Hence, $O_{\Delta,\mathbf{q}} \subseteq O_{\Gamma,\mathbf{p}} \subseteq F \cap O_{\Gamma',\mathbf{p}'} = F \cap O_{\Delta',\mathbf{q}'} = O_{\Delta,\mathbf{q}}$ and therefore $O_{\Delta,\mathbf{q}} = O_{\Gamma,\mathbf{p}}$. Conclude that $\psi_{\mathfrak{a}}$ is very ample at $\mathbf{p}$.

Conversely, suppose that $\psi_{\mathfrak{a}}$ is very ample at $\mathbf{p}$, i.e., $O_{\Gamma,\mathbf{p}} = O_{\Delta,\mathbf{q}}$. Then, $O_{\Gamma,\mathbf{p}}K' = O_{\Delta,\mathbf{q}}K'$. By Lemma 2.2, $O_{\Gamma',\mathbf{p}'}$ (resp., $O_{\Delta',\mathbf{q}'}$) is the local ring of $O_{\Gamma,\mathbf{p}}K'$ (resp., $O_{\Delta,\mathbf{q}}K'$) at $O_{\Gamma,\mathbf{p}}K' \cap M_{\Gamma',\mathbf{p}'}$ (resp., $O_{\Delta,\mathbf{q}}K' \cap M_{\Delta',\mathbf{p}'}$). Since the intersections are the same, so are the corresponding local rings. Conclude that ψ' is very ample at $\mathbf{p}'$.

Proof of (e). (See also [Ro1, Thm. 11].) Let $\mathfrak{c}$ (resp., $\mathfrak{c}'$) be the conductor of O (resp., O'). Take a positive Γ-smooth divisor $\mathfrak{a}$ of F/K such that

$$\deg(\mathfrak{a}) > \max\big(2\mathrm{genus}(F/K) - 2 + \deg(\mathfrak{c}),\ 2\mathrm{genus}(F'/K') - 2 + \deg(\mathfrak{c}')\big).$$

By (4b), $\dim_K \mathcal{L}_O(\mathfrak{a}) = \deg(\mathfrak{a}) + 1 - \mathrm{genus}(O/K)$ and $\dim_{K'} \mathcal{L}_{O'}(\mathfrak{a}') = \deg(\mathfrak{a}') + 1 - \mathrm{genus}(O'/K')$. By (b), $\dim_K \mathcal{L}_O(\mathfrak{a}) = \dim_{K'} \mathcal{L}_{O'}(\mathfrak{a}')$. Also, $\deg(\mathfrak{a}) = \deg(\mathfrak{a}')$. Hence, $\mathrm{genus}(O'/K') = \mathrm{genus}(O/K)$. □

Proposition 2.4. *Let Γ be a projective curve over a field K. Then there is a positive integer n_0 such that each Γ-smooth divisor $\mathfrak{a}$ with $\deg(\mathfrak{a}) \ge n_0$ is very ample with respect to Γ.*

Proof. Let $\tilde{\Gamma} = \Gamma \times_K \tilde{K}$, let $\tilde{\mathfrak{c}}$ be the conductor of $\tilde{\Gamma}$, and let $\tilde{g} = \mathrm{genus}(\tilde{\Gamma})$. We claim that $n_0 = \max\{2\tilde{g}+1+\deg(\tilde{\mathfrak{c}}),\ 2\tilde{g}-1+2\deg(\tilde{\mathfrak{c}})\}$ satisfies the conclusion of the proposition.

Indeed, let $\mathfrak{a}$ be a Γ-smooth divisor with $\deg(\mathfrak{a}) \geq n_0$. Let $\tilde{\mathfrak{a}}$ be the extension of $\mathfrak{a}$ to a $\tilde{\Gamma}$-smooth divisor. Observe that $\deg(\tilde{\mathfrak{a}}) = \deg(\mathfrak{a})$. Denote the ring of singularities of $\tilde{\Gamma}$ by $\tilde{O}$. Choose a K-basis $f_0, \ldots, f_n$ for $\mathcal{L}_O(\mathfrak{a})$ and let $\psi_{\mathfrak{a}}$ be the associated rational map of Γ. Since F is a regular extension of K, it is linearly disjoint from $\tilde{K}$ over K. So, by Lemma 2.3, $f_0, \ldots, f_n$ is also a $\tilde{K}$-basis of $\mathcal{L}_{\tilde{O}}(\tilde{\mathfrak{a}})$. Hence, the associated rational map $\psi_{\tilde{\mathfrak{a}}}$ of $\tilde{\Gamma}$ is the extension of $\psi_{\mathfrak{a}}$. By Lemma 2.1, $\tilde{\mathfrak{a}}$ is very ample on $\tilde{\Gamma}$. Hence, by Lemma 2.3, $\mathfrak{a}$ is very ample on Γ. $\square$

Example 2.5. We show that the lower bound given in Lemma 2.1(b) is optimal.

Let K be an algebraically closed field and let t be a transcendental element over K. For an integer $n \geq 2$ consider the rings $S = K[t]$ and $R = K[t^n, \ldots, t^{2n-1}]$. Thus R consists of all sums $a_0 + \sum_{i=n}^{\infty} a_i t^i$, where $a_i \in K$ and almost all of them are 0. Let $\Gamma_0 = \mathrm{Spec}(R)$.

Observe that $t^n S \subseteq R$. Hence, S is the integral closure of R in $K(t)$. Let $\mathbf{p}_0 = tS$ and $\mathbf{q}_0 = t^n S$. Then $\mathbf{q}_0 = \mathbf{p}_0 \cap R \in \Gamma_0$ and $S_{\mathbf{p}_0}$ is the integral closure of $R_{\mathbf{q}_0}$. Since t is in $S_{\mathbf{p}_0}$ but not in $R_{\mathbf{q}_0}$, the latter ring is not integrally closed.

If $\mathbf{q} \in \Gamma_0$ and $\mathbf{q} \neq \mathbf{q}_0$, choose $\mathbf{p} \in \mathrm{Spec}(S)$ above $\mathbf{q}$. Since $t^n \in R \smallsetminus \mathbf{q}$, we have $t = \frac{t^{n+1}}{t^n} \in R_{\mathbf{q}}$. Hence, $R_{\mathbf{q}} = S_{\mathbf{p}}$ is integrally closed. In other words, $\mathbf{q}$ is simple.

Let $O = R_{\mathbf{q}_0}$ and let $\bar{O} = S_{\mathbf{p}_0}$. Then O is a local ring and $\bar{O}$ is its integral closure. Moreover, $t^n \bar{O}$ is the conductor of O in $\bar{O}$. Also, $\bar{O}/t^n\bar{O} = S/t^n S$ has $1, t, \ldots, t^{n-1}$ as a K-basis.

By [Ro1, p. 174, Thm. 5], there exists a projective curve Γ whose function field is $K(t)$ and whose ring of singularities is O. In particular, Γ is birationally equivalent to $\mathbb{A}^1$ and has genus 0. (Note that Γ is a projective completion of the affine curve Γ_0 which has a natural embedding in $\mathbb{A}^n$.) If we denote the zero divisor of t by P_0, we find by the preceding paragraph that $\mathfrak{c} = nP_0$ is the conductor divisor of Γ and that $\deg(\mathfrak{c}) = n$. Lemma 2.1 therefore asserts that if $\mathfrak{a}$ is a divisor of $K(t)/K$ which is relatively prime to P_0 and $\deg(\mathfrak{a}) \geq 2n-1$, then $\mathfrak{a}$ is very ample on Γ. We show below that this is not the case any more if $\deg(\mathfrak{a}) = 2n-2$.

To this end denote the pole divisor of t by ∞. Then $\mathfrak{a} = (2n-2)\infty$ is a Γ-smooth divisor of degree $2n-2$. If $f \in \mathcal{L}_O(\mathfrak{a})$, then $f \in O$ and $\mathrm{div}(f) + \mathfrak{a} \geq 0$. In particular, f has no poles except ∞. Hence, $f \in S \cap O = R$. Moreover, $\deg(f) \leq 2n-2$. Conversely, the two latter conditions suffice for an $f \in K(t)$ to belong to $\mathcal{L}_O(\mathfrak{a})$. Hence, $1, t^n, t^{n+1}, \ldots, t^{2n-2}$ is a basis of $\mathcal{L}_O(\mathfrak{a})$. Let ψ be the rational map of Γ into $\mathbb{P}^{n-1}$ defined by this basis and let $\Delta = \psi(\Gamma)$. We claim that ψ is not an isomorphism. More precisely, ψ is not biregular at $\mathbf{q}_0$.

Indeed, let $\mathbf{r}_0 = \psi(\mathbf{q}_0)$. Since $2n-1$ is not a linear combination of $n, n+1, \ldots, 2n-2$ with non-negative integral coefficients, $t^{2n-1} \in R \smallsetminus K[t^n, \ldots, t^{2n-2}]$. Hence, t^{2n-1} is in $O_{\Gamma,\mathbf{q}_0} = O$ but not in $O_{\Delta,\mathbf{r}_0}$. So, the latter local ring is a proper subring of the former one, which proves our claim.

One observes directly that for $m \geq 2n-1$, the Γ-smooth divisor $m\infty$ is very ample at P_0, as Lemma 2.1 asserts. $\square$

3. Iteration of projections from points

Consider a set of $m+1$ linearly independent linear forms

$$l_i(\mathbf{X}) = \sum_{j=0}^{n} a_{ij} X_j, \qquad i = 0, \ldots, m, \tag{1}$$

with coefficients a_{ij} in a field K. Consider the linear variety L in $\mathbb{P}^n$ of dimension $n-m-1$ defined by the following system of equations:

$$\sum_{j=0}^{n} a_{ij} X_j = 0, \qquad i = 0, \ldots, m. \tag{2}$$

Let $A = (a_{ij})$ be the matrix of coefficients. Let $\pi = \pi_A \colon \mathbb{P}^n \smallsetminus L \to \mathbb{P}^m$ be the morphism

$$\pi(x_0{:}x_1{:}\cdots{:}x_n) = (l_0(\mathbf{x}){:}l_1(\mathbf{x}){:}\cdots{:}l_m(\mathbf{x})). \tag{3}$$

If we abuse notation and write $\mathbf{x}$ also for the $(n+1)$-tuple $(x_0, x_1, \ldots, x_n)$, we may rewrite (3) as $\pi(\mathbf{x}) = A\mathbf{x}^t$, where the exponent t is the transpose operation of matrices. The morphism π is uniquely defined by L up to a linear isomorphism of $\mathbb{P}^m$. Hence, as far as geometric properties are concerned, π depends only on L. So, we abuse notation, denote π also by π_L, and call π_L the **projection from** L. For each closed subvariety V of $\mathbb{P}^n$ over K which is disjoint from L, the restriction $\pi_{L,V}$ of π_L to V is a finite morphism onto a closed subvariety V' of $\mathbb{P}^m$ [Mum, p. 174]. In particular, if V is linear, then so is V' and $\pi_{L,V} \colon V \to V'$ is an isomorphism.

Denote the space of $r \times s$ matrices by $M_{r,s}$. It is naturally isomorphic to $\mathbb{A}^{rs}$. If $r \le s$, let $M^*_{r,s}$ be the set of all matrices in $M_{r,s}$ of rank r, i.e., with independent rows. Thus $M^*_{r,s}$ is a nonempty Zariski-open subset of $M_{r,s}$.

In particular, $A \in M^*_{m+1,n+1}(K)$ acts on K^{n+1} by multiplication from the left and gives a surjective map onto K^{m+1}. Similarly, each $B \in M^*_{l+1,m+1}$ gives a surjective map $K^{m+1} \to K^{l+1}$. Hence, $C = BA$ gives a surjective map $K^{n+1} \to K^{l+1}$. So, $C \in M^*_{l+1,n+1}$. By definition, $\pi_C = \pi_B \circ (\pi_A|_{\mathbb{P}^n \smallsetminus L'})$, where L' is the linear variety of $\mathbb{P}^n$ defined by $C\mathbf{x}^t = 0$.

Note that the map $\mu_{l,m,n} \colon M^*_{l+1,m+1}(K) \times M^*_{m+1,n+1}(K) \to M^*_{l+1,n+1}(K)$ defined by $\mu_{l,m,n}(B, A) = BA$ is surjective. Indeed, denote the unit (resp., zero) matrix in $M_{r,r}$ (resp., in $M_{r,s}$) by I_r (resp., $O_{r,s}$) and let $C \in M^*_{l+1,n+1}$. Let also D be an arbitrary matrix in $M_{m-l,n+1}$. Then

$$\begin{pmatrix} I_{l+1} & O_{l+1,m-l} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = C.$$

Moreover, $\mathrm{rank}(I_{l+1}\ O_{l+1,m-l}) = l+1$ and we may choose D such that $\mathrm{rank}\binom{C}{D} = m+1$.

In particular, for each $k \ge 1$ let $M^*_k = M^*_{k,k+1}$ be the Zariski-open subset of $M_{k,k+1}$ that consists of all matrices of rank k. Let $\mathbb{M} = M^*_2 \times M^*_3 \times \cdots \times M^*_n$ and define a morphism $\mu \colon \mathbb{M} \to M_{2,n+1}$ by multiplication:

$$\mu(A_2, A_3, \ldots, A_n) = A_2 A_3 \cdots A_n.$$

Since multiplication of matrices is associative, μ maps $\mathbb{M}(K)$ onto $M^*_{2,n+1}(K)$.

If $A = (a_{ij})_{0\le i\le n-1,\, 0\le j\le n} \in M^*_n$, then L becomes a point $\mathbf{o}$ whose homogeneous coordinates can be obtained by solving (2) according to Cramer's rule. The map $\varphi_n\colon M^*_n \to \mathbb{P}^n$, given by $\varphi_n(A) = \mathbf{o}$, is a morphism. Observe that a point $(a_0{:}a_1{:}\cdots{:}a_n) \in \mathbb{P}^n(K)$ with, say, $a_0 \ne 0$ is the unique solution of the system of equations $a_0X_i - a_iX_0 = 0$, $i = 1,\dots,n$. Hence, φ_n maps $M^*_n(K)$ onto $\mathbb{P}^n(K)$. Let

$$\mathbb{P} = \mathbb{P}^2 \times \cdots \times \mathbb{P}^n, \quad \varphi = \varphi_2 \times \cdots \times \varphi_n\colon \mathbb{M} \to \mathbb{P}. \tag{4}$$

Then φ is a morphism that maps $\mathbb{M}(K)$ onto $\mathbb{P}(K)$.

4. Stabilizing elements

By a **function field of one variable over** K we mean a finitely generated regular extension F of K of transcendence degree 1. Thus F has a transcendental element t over K such that $F/K(t)$ is a finite separable extension. Let $\hat F$ be the Galois closure of $F/K(t)$. We say that t **symmetrically stabilizes** F/K if $\mathcal{G}(\hat F\tilde K/\tilde K(t)) \cong S_d$, where $d = [F : K(t)]$. In this case, $\mathcal{G}(\hat F\tilde K/\tilde K(t)) \cong \mathcal{G}(\hat F/K(t))$ [FJ2, Lemma 16.40] and therefore $\hat F/K$ is a regular extension.

Given a curve Γ over K, we denote the curve $\Gamma \times_K \tilde K$ obtained from Γ by extending the field of constants to $\tilde K$ by $\tilde\Gamma$. If the function field of Γ is F, then the function field of $\tilde\Gamma$ is $F\tilde K$. Let $\tilde{\mathbf{p}}$ be a point of $\tilde\Gamma$. Then, for all large n, $\dim_{\tilde K}(M^n_{\tilde\Gamma,\tilde{\mathbf{p}}}/M^{n+1}_{\tilde\Gamma,\tilde{\mathbf{p}}})$ is a fixed positive integer, called the **multiplicity** of $\tilde{\mathbf{p}}$. See [Ful, p. 7, Thm. 2] for the case where $\tilde\Gamma$ is a plane curve. The general case follows from the definition of multiplicity of a prime ideal of an arbitrary Noetherian local ring [Mat2, p. 108]. We say that $\tilde{\mathbf{p}}$ is a **cusp** of $\tilde\Gamma$ if it is singular and if $O_{\tilde\Gamma,\tilde{\mathbf{p}}}$ is contained in a unique valuation ring of $F\tilde K$. We say that Γ is a **cusp curve** if

(1a) the only singular points of $\tilde\Gamma$ are cusps; and

(1b) $\tilde\Gamma$ has at least one cusp.

We say that Γ is a **special cusp curve** if there exists an odd prime number $q \ge \mathrm{genus}(F/K) + 1$ such that

(2a) the only singular points of $\tilde\Gamma$ are cusps of multiplicity at most q; and

(2b) $\tilde\Gamma$ has at least one cusp of multiplicity q.

Suppose now that Γ is a plane curve. A point $\tilde{\mathbf{p}}$ of $\tilde\Gamma$ is a **node** if $\tilde\Gamma$ has two simple tangents at $\tilde{\mathbf{p}}$ [Ful, p. 66]. In particular, $\tilde{\mathbf{p}}$ has multiplicity 2 on $\tilde\Gamma$. We say that Γ is a **node curve** if the singular points of $\tilde\Gamma$ are nodes. We say that Γ is a **cusp-node curve** if it satisfies (1), except that in addition to cusps we allow also nodes as singularities for $\tilde\Gamma$. We say that Γ is a **special cusp-node curve** if it satisfies (2), where in addition to cusps, we allow also nodes.

Suppose again that Γ is a curve in $\mathbb{P}^n$ with a generic point $\mathbf{y} = (y_0{:}y_1{:}\cdots{:}y_n)$ over K with $y_0, y_1, \ldots, y_n \in F$. A **strange point** of $\tilde{\Gamma}$ is a point of $\mathbb{P}^n$ through which infinitely many tangents to $\tilde{\Gamma}$ pass.

Let $\mathbf{p}$ be a simple point of $\tilde{\Gamma}$. Following [GeJ, p. 360], we say that $\mathbf{p}$ is an **inflection point** of $\tilde{\Gamma}$ if

$$\mathrm{rank}(\mathbf{y}(\mathbf{p})\ \mathbf{y}'(\mathbf{p})\ \mathbf{y}^{[2]}(\mathbf{p})) = 2,$$

where we now consider $\mathbf{y}$, $\mathbf{y}'$, and $\mathbf{y}^{[2]}$ as columns of height $n+1$, and $\mathbf{y}'$ and $\mathbf{y}^{[2]}$ are the first and the second derivatives of $\mathbf{y}$, respectively, à la F. K. Schmidt [GeJ, §2]. In the case $n = 2$, this definition agrees with the classical one [GeJ, Lemma 4.1].

Finally, we say that Γ is an **ordinary curve** if $\tilde{\Gamma}$ has only finitely many inflection points, finitely many double tangents, and no strange points.

Lemma 4.1. *Let Γ be a projective ordinary node plane curve over a field K. Let F be the function field of Γ and let y_0, y_1, y_2 be elements of F such that $(y_0{:}y_1{:}y_2)$ is a generic point of Γ over K. Then there is a nonempty Zariski-open subset U of $\mathbb{P}^2$ such that if $\mathbf{o} \in U(K)$ and $(x_0{:}x_1) = \pi_{\mathbf{o}}(y_0{:}y_1{:}y_2)$, then $\frac{x_0}{x_1}$ symmetrically stabilizes F/K. Here $\pi_{\mathbf{o}}$ is the projection from $\mathbf{o}$ defined in §3.*

Proof. Let $\mathcal{L}$ be the set of all double tangents of $\tilde{\Gamma}$, tangents at inflection points, lines through two distinct singular points, and tangents that pass through a singular point of $\tilde{\Gamma}$. By assumption, $\mathcal{L}$ is a finite set. Let A be the union of $\tilde{\Gamma}$ with all lines in $\mathcal{L}$. Then $U = \mathbb{P}^2 \smallsetminus A$ is a nonempty Zariski-open subset of $\mathbb{P}^2$.

Consider now $\mathbf{o} \in U(K)$. Let $n = \deg(\Gamma)$. By Bezout's theorem (see also [FJ1, Lemma 3.3]), finitely many lines in $\mathbb{P}^2$ through $\mathbf{o}$ intersect $\tilde{\Gamma}$ at exactly $n-1$ points and all others intersect $\tilde{\Gamma}$ by exactly n points. (Although [FJ1, Lemma 3.3] is stated only in characteristic 0, its proof applies almost verbatim to our case.) The proof of [FJ1, Lemma 2.1] implies now that if $(x_0{:}x_1) = \pi_{\mathbf{o}}(y_0{:}y_1{:}y_2)$, then $\frac{x_0}{x_1}$ symmetrically stabilizes F/K. □

In the following two lemmas we use the notation of §3.

Lemma 4.2. *Let K be an infinite field and let $n \geq 2$ be an integer. Consider a projective curve Δ in $\mathbb{P}^n$ over K with a function field F. Let $y_0, y_1, \ldots, y_n$ be elements of F such that $\mathbf{y} = (y_0{:}y_1{:}\cdots{:}y_n)$ is a generic point of Δ over K and $F = K(y_0{:}y_1{:}\cdots{:}y_n)$. Suppose that*

(a) *Δ is an ordinary curve;*

(b) *$\tilde{\Delta}$ is contained in no hyperplane of $\tilde{\mathbb{P}}^n$;*

(c) *if $n \geq 3$, then Δ is a smooth curve or a special cusp curve; and*

(d) *if $n = 2$, then Δ is a node curve or a special cusp-node curve.*

Then there exists a nonempty Zariski-open subset U_i of $\mathbb{P}^i$, $i = 2, 3, \ldots, n$, such that with $U = U_2 \times U_3 \times \cdots \times U_n$ and for each $\mathbf{A} \in \varphi^{-1}(U(K))$ and with $\mu(\mathbf{A}) = \binom{\mathbf{b}}{\mathbf{c}}$, the element $t = \sum_{i=0}^{n} b_i y_i / \sum_{i=0}^{n} c_i y_i$ symmetrically stabilizes F/K.

Proof. Suppose first that $n = 2$. Lemma 4.1, in case Δ is a node curve, and [Neu, Prop. 2.12], in case Δ is a cusp-node curve, gives a nonempty Zariski-open subset U_2 of $\mathbb{P}^2$ such that if $\mathbf{o} \in U_2(K)$ and $(x_0{:}x_1) = \pi_{\mathbf{o}}(y_0{:}y_1{:}y_2)$, then $\frac{x_0}{x_1}$ symmetrically stabilizes F/K. Let therefore $\mathbf{o} \in U_2(K)$ and let $A = \binom{b_0\ b_1\ b_2}{c_0\ c_1\ c_2} \in M_2^*(K)$ be a matrix such that $\mathbf{o}$ is the unique solution of the system of two equations: $\sum_{j=0}^2 b_j X_j = 0$, $\sum_{j=0}^2 c_j X_j = 0$. Then $\pi_{\mathbf{o}}(y_0{:}y_1{:}y_2) = \big(\sum_{j=0}^2 b_j y_j{:} \sum_{j=0}^2 c_j y_j\big)$. Hence, $\sum_{j=0}^2 b_j y_j / \sum_{j=0}^2 c_j y_j$ symmetrically stabilizes F/K.

Assume therefore that $n \geq 3$. By (b), $\tilde{\Delta}$ is contained in no hyperplane. Use [Neu, Lemma 2.35] to find a nonempty Zariski-open subset U_n of $\mathbb{P}^n$ such that for each $\mathbf{o} \in U_n(K)$ the projection $\pi_{\mathbf{o}}\colon \mathbb{P}^n \smallsetminus \{\mathbf{o}\} \to \mathbb{P}^{n-1}$ maps Δ onto a curve Γ with the following properties:

(3a) Γ is an ordinary curve.

(3b) If $n \geq 4$ and Δ is smooth, then so is Γ.

(3c) If $n \geq 4$ and Δ is a special cusp curve, then so is Γ.

(3d) If $n = 3$ and Δ is smooth, then Γ is a node curve.

(3e) If $n = 3$ and Δ is a special cusp curve, then Γ is a special cusp-node curve.

Consider $\mathbf{o} \in U_n(K)$. Suppose that $\mathbf{o} = \varphi_n(A_n)$ with $A_n = (a_{ij}) \in M_n^*(K)$. Then Γ is generated over K by $\mathbf{z} = (z_0{:}z_1{:}\cdots{:}z_{n-1})$, where $z_i = \sum_{j=0}^n a_{ij} y_j$, $i = 0, \ldots, n-1$.

Claim: *$\tilde{\Gamma}$ is contained in no hyperplane of $\tilde{\mathbb{P}}^{n-1}$.* Indeed, if $\tilde{\Gamma}$ were contained in a hyperplane, then there would exist $c_0, \ldots, c_{n-1} \in \tilde{K}$ not all zero such that $\sum_{i=0}^{n-1} c_i z_i = 0$. Thus, $\sum_{j=0}^n \big(\sum_{i=0}^{n-1} c_i a_{ij}\big) y_j = 0$. Since $\tilde{\Delta}$ is contained in no hyperplane of $\tilde{\mathbb{P}}^n$, the elements $y_0, y_1, \ldots, y_n$ are linearly independent over $\tilde{K}$. Hence, $\sum_{i=0}^{n-1} c_i a_{ij} = 0$, $j = 0, \ldots, n$. But this means that $\mathrm{rank}(A_n) < n$. This contradiction to the assumption that $A_n \in M_n^*(K)$ proves our claim.

End of proof: An induction hypothesis gives for each i between 0 and $n-1$ a nonempty Zariski-open subset U_i of $\mathbb{P}^i$ such that if $A_i \in M_i(K)$ and $\varphi_i(A_i) \in U_i(K)$, then, with $\binom{\mathbf{b}}{\mathbf{c}} = A_2 A_3 \cdots A_{n-1}$, the element

$$t = \frac{\sum_{i=0}^{n-1} b_i z_i}{\sum_{i=0}^{n-1} c_i z_i} \tag{4}$$

symmetrically stabilizes F/K.

Let $U = U_2 \times U_3 \times \cdots \times U_n$. Consider a point $\mathbf{A} = (A_2, \ldots, A_n)$ of $\mathbb{M}(K)$ such that $\varphi(\mathbf{A}) \in U(K)$. Then $A_n = (a_{ij}) \in M_n^*(K)$ and $\mathbf{o} = \varphi_n(A_n) \in U_n(K)$. Hence, $\pi_{\mathbf{o}}$ satisfies (3) and the Claim. Use the notation of the Claim and substitute $z_i = \sum_{j=0}^n a_{ij} y_j$ in (4) to get a new presentation of t: $t = \sum_{j=0}^n b_j' y_j / \sum_{j=0}^n c_j' y_j$, where $\binom{\mathbf{b}'}{\mathbf{c}'} = \binom{\mathbf{b}}{\mathbf{c}} A_n$. Thus, $\mu(\mathbf{A}) = A_2 \cdots A_{n-1} A_n = \binom{\mathbf{b}'}{\mathbf{c}'}$, and the induction step is complete. $\square$

Remark 4.3. The Veronese curve. Consider the projective line $\mathbb{P}^1$ with a generic point $(x_0{:}x_1)$ over a field K. The Veronese map of degree n is an isomorphism of $\mathbb{P}^1$ onto a smooth projective curve V_n in $\mathbb{P}^n$ with generic point $(x_0^n{:}x_0^{n-1}x_1{:}\cdots{:}x_1^n)$ [Sha. p. 40] which can also be written as $(1{:}t{:}\cdots{:}t^n)$ where $t = \frac{x_1}{x_0}$. Since $1, t_1, \ldots, t^n$ are linearly independent over K, the **Veronese curve** V_n is contained in no hyperplane of $\mathbb{P}^n$.

If $\mathrm{char}(K) \neq 2$, then V_2 is an ordinary curve. But if $\mathrm{char}(K) = 2$, then $(0{:}1{:}0)$ is a strange point for V_2. So, we rather look at V_3. $\square$

Lemma 4.4. *V_3 is an ordinary curve.*

Proof. In the notation of Remark 4.3, $\mathbf{t} = (1{:}t{:}t^2{:}t^3)$ is a generic point of V_3 over K. One checks that the following homogeneous equations define V_3 in $\mathbb{P}^3$:

$$\begin{aligned} Y_1Y_2 - Y_0Y_3 &= 0 \\ Y_1^2 - Y_0Y_2 &= 0 \\ Y_2^2 - Y_1Y_3 &= 0 \end{aligned} \tag{5}$$

The curve V_3 has a unique infinite point $\infty = (0{:}0{:}0{:}1)$. Each finite point has the form $\mathbf{a} = (1{:}a{:}a^2{:}a^3)$. Computing the partial derivatives of the left hand sides of (5) one finds that the following system of linear equations defines the tangent $T_{\mathbf{a}}$ to V_3 at $\mathbf{a}$:

$$\begin{aligned} -a^3Y_0 + a^2Y_1 + aY_2 - Y_3 \quad &= 0 \\ -a^2Y_0 + 2aY_1 - Y_2 \qquad\quad &= 0 \\ -a^3Y_1 + 2a^2Y_2 - aY_3 &= 0 \end{aligned} \tag{6}$$

Of course, the third equation of (6) is redundant. If we take another point $\mathbf{b} = (1{:}b{:}b^2{:}b^3)$ on $V_3(\tilde{K})$ with $b \neq a$, then $T_{\mathfrak{a}}$ and $T_{\mathbf{b}}$ do not intersect. Indeed,

$$\begin{vmatrix} -a^3 & a^2 & a & -1 \\ -a^2 & 2a & -1 & 0 \\ -b^3 & b^2 & b & -1 \\ -b^2 & 2b & -1 & 0 \end{vmatrix} = (a-b)^4 \neq 0.$$

Similarly one computes the equation of T_∞ to be given by the equations $Y_0 = 0$ and $Y_1 = 0$. One observes that T_∞ does not intersect $T_{\mathbf{a}}$. It follows that V_3 has no double tangents nor there exists a strange point for V_3.

Finally we prove that the point $\mathbf{o} = (1{:}0{:}0{:}0)$ of V_3 is not an inflection point.

Indeed, t is a uniformizing parameter at $\mathbf{o}$. Rule (1f) of [GeJ, §2] says that $(t^j)^{[i]} = \binom{j}{i} t^{j-i}$ for each $j \geq i$. Hence,

$$(\mathbf{t}\ \mathbf{t}'\ \mathbf{t}^{[2]}) = \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2 & 2t & 1 \\ t^3 & 3t^2 & 3t \end{bmatrix}$$

In particular,

$$\big(\mathbf{t}(\mathbf{o})\ \mathbf{t}'(\mathbf{o})\ \mathbf{t}^{[2]}(\mathbf{o})\big) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is a matrix of rank 3. Hence, $\mathbf{o}$ is not an inflection point of V_3. Since the set of inflection points is Zariski-closed in V_3, this means that V_3 has only finitely many inflection points. Conclude that V_3 is an ordinary curve. □

Lemma 4.5. *Let F be a function field of one variable over an infinite field K. Let Γ be a projective ordinary model for F/K which is either a smooth curve or a special cusp curve. Let O be the ring of singularities of Γ. Then F/K has a positive divisor $\mathfrak{a}_0$ which is O-smooth such that the following statement holds for each O-smooth divisor $\mathfrak{a} \geq \mathfrak{a}_0$:*

There exists a basis $y_0, y_1, \dots, y_n$ for the linear space $\mathcal{L}_O(\mathfrak{a})$ and there exists a nonempty Zariski-open subset U_i of $\mathbb{P}^i$, $i = 2, 3, \dots, n$ such that with $U = U_2 \times U_3 \times \cdots \times U_n$, for each $\mathbf{A} \in \varphi^{-1}(U(K))$, and with $\mu(\mathbf{A}) = \binom{\mathbf{b}}{\mathbf{c}}$, the element $t = \sum_{i=0}^n b_i y_i / \sum_{i=0}^n c_i y_i$ symmetrically stabilizes F/K.

Proof. If Γ is a line, use Remark 4.3 and Lemma 4.4 to replace it by V_3 (or by V_2 if $\mathrm{char}(K) \neq 2$). Thus, in any case, we may embed Γ in $\mathbb{P}^m$ for some $m \geq 2$ such that Γ is contained in no hyperplane of $\mathbb{P}^m$. Let $\mathbf{x} = (x_0{:}x_1{:}\cdots{:}x_m)$ be a generic point for Γ with $x_0, x_1, \dots, x_m \in F$ such that $F = K(\mathbf{x})$. Then $x_0, x_1, \dots, x_m$ are linearly independent over K.

Use the notation of §2. By the weak approximation theorem there exists $u \in F$ such that $v_P(u) \geq k_P - \min_{0 \leq i \leq m} v_P(x_i)$ for each $P \in \mathrm{PrimDiv}(F/K)$ which lies over a singular point of Γ and where k_P satisfies (1) of §2. Then replace x_i by ux_i, if necessary, to assume that $x_i \in O$, $i = 0, 1, \dots, m$. In particular, $\mathrm{div}_\infty(x_i)$ is relatively prime to the conductor $\mathfrak{c}$ of O, $i = 0, 1, \dots, m$. Let n_0 be the positive integer that appears in Proposition 2.4. Then $\mathfrak{a}_0 = \sum_{i=0}^m n_0 \mathrm{div}_\infty(x_i)$ is a positive O-smooth divisor of F/K with $\deg(\mathfrak{a}_0) \geq n_0$.

Let $\mathfrak{a}$ be a divisor of F/K such that $\mathfrak{a} \geq \mathfrak{a}_0$. Let $y_i = x_i$, $i = 0, 1, \dots, m$. Then $\mathrm{div}(y_i) + \mathfrak{a} \geq -\mathrm{div}_\infty(x_i) + n_0 \mathrm{div}_\infty(x_i) \geq 0$ and therefore $y_i \in \mathcal{L}_O(\mathfrak{a})$. Since $y_0, y_1, \dots, y_m$ are linearly independent over K, they extend to a basis $y_0, y_1, \dots, y_n$ of $\mathcal{L}_O(\mathfrak{a})$. By Proposition 2.4, $\mathfrak{a}$ is very ample on Γ. Thus, the map ψ associated with $\mathfrak{a}$ is an isomorphism of Γ onto the projective curve Δ generated over K in $\mathbb{P}^n$ by $\mathbf{y} = (y_0{:}y_1{:}\cdots{:}y_n)$. In particular, if Γ is smooth, so is Δ; if Γ is a special cusp curve, so is Δ. Since $y_0, y_1, \dots, y_n$ are linearly independent over K and F/K is regular, $\tilde{\Delta}$ is contained in no hyperplane of $\tilde{\mathbb{P}}^n$.

Consider now the linear variety L in $\mathbb{P}^n$ defined by the equations $Y_i = 0$, $i = 0, 1, \dots, m$. The projection $\pi_L{:}\, \mathbb{P}^n \smallsetminus L \to \mathbb{P}^m$ maps each point $\mathbf{q} = (q_0{:}q_1{:}\cdots{:}q_n)$ of $(\mathbb{P}^n \smallsetminus L)(\tilde{K})$ onto the point $(q_0{:}\cdots{:}q_m)$ of $\mathbb{P}^m(\tilde{K})$. If $\mathbf{q} \in \Delta(\tilde{K})$, then $\mathbf{q} = \psi(\mathbf{p})$, with $\mathbf{p} = (p_0{:}\cdots{:}p_m) \in \Gamma(\tilde{K})$. In particular $p_i \neq 0$ for at least one i and $(q_0{:}\cdots{:}q_m) = (p_0{:}\cdots{:}p_m)$. Thus, π_L projects Δ isomorphically onto the ordinary curve Γ. By [Neu, Lemma 2.29], Δ is an ordinary curve. Note that [Neu, Lemma 2.29] is actually stated only in the case where L is a point. In the general case, π_L can be factored into successive projections, first onto the first n coordinates, then onto the first $n-1$ coordinates, and so on. So, we may apply [Neu, Lemma 2.29] $n-m$ times.

With this, Δ satisfies conditions (a), (b), (c), and (d) of Lemma 4.2. So, Lemma 4.2 supplies the Zariski-open subsets U_i of $\mathbb{P}^i$ with the desired properties. $\square$

5. Local existence theorem

Let Γ be a projective curve over a field K with a function field F. In [Ro2], Maxwell Rosenlicht constructs an algebraic group J_Γ over K, known as the **generalized Jacobian variety** of Γ, which coincides with the usual Jacobian variety of Γ if Γ is smooth. As in the latter case, there is an intimate connection between divisors of F/K and the group $J_\Gamma(K)$. We survey this connection as well as the properties of J_Γ that we need in the proof of the local existence theorem. In this survey we use the terminology of §2.

Let O be a semilocal ring of F/K which is contained in the ring of singularities of Γ in F. A priori, the algebraic group that Rosenlicht constructs depends on O (and not only on Γ). So, we denote it by J_O or also by J until we change O. Rosenlicht's construction depends on the following assumption:

(1) $\Gamma(K)$ is infinite.

Let $h = \mathrm{genus}(O/K)$. If $h = 0$, then J is a point [Ro2, p. 519]. We therefore assume throughout that $h > 0$.

Choose a point $\mathbf{o} \in \Gamma_{\mathrm{simp}}(K)$. Then $J = J_O$ has the following properties:

(2a) J is a commutative group variety over K of dimension h [Ro2, Thm. 7].

(2b) There is a morphism $\varphi\colon \Gamma_{\mathrm{simp}} \to J$ over K such that $\varphi(\mathbf{o})$ is the zero point of $J(K)$ [Ro2, Thm. 7].

(2c) For each extension L of K which is linearly disjoint from F let $\mathrm{Div}_O(FL/L)$ (resp., $\mathrm{Div}_{O,0}(FL/L)$) be the group of all OL-smooth divisors (resp., divisors of degree 0) of FL/L. Then $\varphi\colon \Gamma_{\mathrm{simp}}(\tilde{K}) \to J(\tilde{K})$ extends to an epimorphism $\tilde{\varphi}\colon \mathrm{Div}_O(F\tilde{K}/\tilde{K}) \to J(\tilde{K})$ by the rule

$$\tilde{\varphi}\Big(\sum_{i=1}^{m} n_i\mathbf{p}_i\Big) = \sum_{i=1}^{m} n_i\varphi(\mathbf{p}_i),$$

where the sum on the right hand side is addition on the group variety.

(2d) *Symmetric products:* Consider the direct product Γ_{simp}^h of h copies of Γ_{simp}. It is a variety over K of dimension h. The symmetric group S_h acts on Γ_{simp}^h by permuting the coordinates. The factor variety $\Gamma^{(h)} = \Gamma_{\mathrm{simp}}^h/S_h$ is the **symmetric product** of Γ_{simp} with itself h times. It is a variety over K and there is a Galois cover $\rho\colon \Gamma_{\mathrm{simp}}^h \to \Gamma^{(h)}$ over K with Galois group S_h [Ser, p. 48, Prop. 18].

By the generalized Riemann-Roch theorem, φ induces a birational map $\sigma\colon \Gamma^{(h)} \to J$ over K. In particular, there are nonempty Zariski-open subsets B_h, J_0 of $\Gamma^{(h)}, J$, respectively, such that the restriction of σ to B_h is an isomorphism onto J_0. Moreover, the dominant rational map $\psi = \sigma\rho$ of $\Gamma^h_{\rm simp}$ to J satisfies

$$\psi(\mathbf{p}_1, \ldots, \mathbf{p}_h) = \sum_{i=1}^{h} \varphi(\mathbf{p}_i) \tag{2d1}$$

for all $\mathbf{p}_1, \ldots, \mathbf{p}_h \in \Gamma_{\rm simp}$. In particular, the function field of $\Gamma^h_{\rm simp}$ is a Galois extension of the function field of J with Galois group S_h.

(2e) *Abel's theorem.* In the notation of (2c) let L be an algebraic extension of K. Let $\mathrm{Prin}_O(FL/L) = \{\mathrm{div}(f) \mid f \in (OL)^\times\}$ be the group of all principal divisors of FL/L coming from units of OL. It fits into the following short exact sequence

$$1 \longrightarrow L^\times \longrightarrow (OL)^\times \xrightarrow{\mathrm{div}} \mathrm{Prin}_O(FL/L) \to 0. \tag{2e1}$$

Let φ_L be the restriction of $\tilde{\varphi}$ to $\mathrm{Div}_{O,0}(FL/L)$. By [Ro2, p. 517 and p. 519], $\mathrm{Ker}(\varphi_L) = \mathrm{Prin}_O(FL/L)$ and $\mathrm{Im}(\varphi_L) \subseteq J(L)$. By (2d), $\mathrm{Im}(\varphi_L)$ contains a nonempty Zariski-open subset of $J(L)$. But as $\mathrm{Im}(\varphi_L)$ is a subgroup of $J(L)$, it is all of $J(L)$. In other words, the following short sequence is exact:

$$0 \longrightarrow \mathrm{Prin}_O(FL/L) \longrightarrow \mathrm{Div}_{O,0}(FL/L) \xrightarrow{\varphi_L} J(L) \longrightarrow 0. \tag{2e2}$$

(2f) *Changing O:* Intersecting O with finitely many valuation rings of prime divisors of F/K does not change J_0. We may therefore use J_Γ instead of J_O. Indeed, it suffices to consider a prime divisor $\mathbf{q}$ of F/K which does not lie over O, to write $R = O \cap O_{\mathbf{q}}$, and to prove that $J_O = J_R$. This follows from (2e2) (with $L = \tilde{K}$), (2f1), and (2f2):

(2f1) $\mathrm{Prin}_O(F\tilde{K}/\tilde{K}) \cap \mathrm{Div}_{R,0}(F\tilde{K}/\tilde{K}) = \mathrm{Prin}_R(F\tilde{K}/\tilde{K})$ and

(2f2) $\mathrm{Prin}_O(F\tilde{K}/\tilde{K}) + \mathrm{Div}_{R,0}(F\tilde{K}/\tilde{K}) = \mathrm{Div}_{O,0}(F\tilde{K}/\tilde{K})$.

In order to prove (2f1), we apply our convention from §2.1 to $\mathbf{q}$ and identify it with a simple point of Γ such that $O_{\Gamma,\mathbf{q}} = O_{\mathbf{q}}$. Then we prove that $O\tilde{K} \cap O_{\mathbf{q}}\tilde{K} = R\tilde{K}$. To this end let $\{w_i \mid i \in I\}$ be a basis for $\tilde{K}/K$. Each $g \in O\tilde{K} \cap O_{\mathbf{q}}\tilde{K}$ can be written as $g = \sum_{i\in I} g_i w_i$ with $g_i \in O$ and $g = \sum_{i \in I} g'_i w_i$ with $g'_i \in O_{\mathbf{q}}$. So, $g_i = g'_i$ belongs to R and therefore $g \in R\tilde{K}$.

Consider now $f \in (O\tilde{K})^\times$ such that $\mathrm{div}(f) \in \mathrm{Div}_{R,0}(F\tilde{K}/\tilde{K})$. Then, $v_{\tilde{\mathbf{q}}}(f) = 0$ for each prime divisor $\tilde{\mathbf{q}}$ of $F\tilde{K}/\tilde{K}$ that lies over $\mathbf{q}$. By Lemma 2.2, $O_{\mathbf{q}}\tilde{K}$ is the intersection of all valuation rings of $F\tilde{K}$ that lie over $O_{\mathbf{q}}$. So, by the preceding paragraph, $f, f^{-1} \in O\tilde{K} \cap O_{\mathbf{q}}\tilde{K} = R\tilde{K}$. In other words, $f \in (R\tilde{K})^\times$. It follows that $\mathrm{div}(f) \in \mathrm{Prin}_R(F\tilde{K}/\tilde{K})$.

In order to prove (2f2) consider $\mathfrak{b} \in \mathrm{Div}_{O,0}(F\tilde{K}/\tilde{K})$. Use the weak approximation theorem to find $f \in F\tilde{K}$ such that $v_{\tilde{Q}}(f) = v_{\tilde{Q}}(\mathfrak{b})$ for each prime divisor $\tilde{Q}$ of $F\tilde{K}/\tilde{K}$ over $\mathbf{q}$ and $v_{\tilde{P}}(f-1)$ is sufficiently large for all $\tilde{P}$ which lie over $O\tilde{K}$. By (1) of §2, applied to the semilocal ring $O\tilde{K}$, $f \in (O\tilde{K})^\times$. Also, $\mathfrak{b}_1 = -\mathrm{div}(f) + \mathfrak{b} \in \mathrm{Div}_{R,0}(F\tilde{K}/\tilde{K})$, as needed.

(2g) *Comparing two curves:* Let $\pi\colon \Gamma' \to \Gamma$ be a birational morphism of projective curves over K. By (1), $\Gamma'(K)$ is infinite. For each object associated with Γ add a tag on the letter denoting it to denote the corresponding object associated with Γ'. If $\pi(\mathbf{p}') = \mathbf{p}$, then $O_{\Gamma,\mathbf{p}} \subseteq O_{\Gamma',\mathbf{p}'}$. Moreover, if $\mathbf{p}'$ is singular, then so is $\mathbf{p}$. Let O (resp., O') be the ring of singularities of Γ (resp., Γ') and let $R = O' \cap \bigcap_P O_P$ where P ranges over all prime divisors of F/K that lie over singular points of Γ. Then $O \subseteq R \subseteq O'$ and a divisor of F/K is O-smooth if and only if it is R-smooth. In particular, by Lemma 2.2, $\mathrm{Div}_{O\tilde{K},0}(F\tilde{K}/\tilde{K}) = \mathrm{Div}_{R\tilde{K},0}(F\tilde{K}/\tilde{K})$. By (2f), $J' = J_R$. By [Ro2, Thm. 8], there exists a surjective homomorphism $\theta\colon J \to J'$ over K of the corresponding group varieties. Let $H = \mathrm{Ker}(\theta)$ and let $\tilde{\theta}\colon J(\tilde{K}) \to J'(\tilde{K})$ be the homomorphism induced by θ. The short exact sequences (2e1) and (2e2) for $L = \tilde{K}$ yield a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Prin}_O(F\tilde{K}/\tilde{K}) \arrow[r] \arrow[d] & \mathrm{Prin}_{R\tilde{K}}(F\tilde{K}/\tilde{K}) \arrow[r] \arrow[d] & (R\tilde{K})^\times/(O\tilde{K})^\times \arrow[r] & 1 \\
 & \mathrm{Div}_{O,0\tilde{K}}(F\tilde{K}/\tilde{K}) \arrow[r, equal] \arrow[d, "\varphi_K"'] & \mathrm{Div}_{R\tilde{K},0}(F\tilde{K}/\tilde{K}) \arrow[d, "\varphi_{\tilde{K}}"] & & \\
1 \to H(\tilde{K}) \arrow[r] & J(\tilde{K}) \arrow[r, "\tilde{\theta}"] \arrow[d] & J'(\tilde{K}) \arrow[rr] \arrow[d] & & 1 \\
 & 0 & 0 & &
\end{tikzcd}$$

An application of the snake lemma [Eis, p. 640] to the vertical short exact sequences proves that $H(\tilde{K}) \cong (R\tilde{K})^\times/(O\tilde{K})^\times$. This gives the following short exact sequence

(2g1) $$1 \longrightarrow (R\tilde{K})^\times/(O\tilde{K})^\times \longrightarrow J(\tilde{K}) \xrightarrow{\tilde{\theta}} J'(\tilde{K}) \longrightarrow 0.$$

(2h) *The genera satisfy the inequality* $\mathrm{genus}(F/K) \le h' \le h$. Indeed, choose a Γ-smooth divisor $\mathfrak{a}$ of large degree. Let $g = \mathrm{genus}(F/K)$. Then, the Riemann-Roch theorem and the generalized Riemann-Roch theorem (4) of §2. give the following relations:

$$\dim\left(\mathcal{L}_{F/K}(\mathfrak{a})\right) = \deg(\mathfrak{a}) + 1 - g,$$
$$\dim\left(\mathcal{L}_{O'/K}(\mathfrak{a})\right) = \deg(\mathfrak{a}) + 1 - h',$$
$$\dim\left(\mathcal{L}_{O/K}(\mathfrak{a})\right) = \deg(\mathfrak{a}) + 1 - h.$$

Since $O \subseteq O' \subseteq F$, we have $\mathcal{L}_{O/K}(\mathfrak{a}) \subseteq \mathcal{L}_{O'/K}(\mathfrak{a}) \subseteq \mathcal{L}_{F/K}(\mathfrak{a})$. Hence, $g \le h' \le h$.

(2i) *The map $\theta: J \to J'$ of (2g) is separable.* Indeed, by (2h), $h' \leq h$. This gives rise to the following commutative diagram:

$$\begin{tikzcd} \Gamma^h_{\mathrm{simp}} \arrow[r, "\mathrm{pr}"] \arrow[d, "\psi"'] & \Gamma^{h'}_{\mathrm{simp}} & \Gamma'^{h'}_{\mathrm{simp}} \arrow[l, "\pi^{h'}"'] \arrow[d, "\psi'"] \\ J \arrow[rr, "\theta"] & & J' \end{tikzcd}$$

In this diagram pr is a regular map (i.e., the extension of the corresponding function fields is regular), $\pi^{h'}$ is a birational map, and ψ' is a Galois map (by (2d)). Hence, $\psi' \circ (\pi^{h'})^{-1} \circ \mathrm{pr}$ is a separable map. It follows that θ is also a separable map.

Remark 5.1. At the beginning of [Ro2, §3], Rosenlicht considers a smooth projective curve C and a semilocal ring O of the function field F of C. Then he constructs the generalized Jacobian variety of C with respect to the equivalence relation defined by O. However, the construction works even if one allows singularities and only demands that O is contained in the ring of singularities of C. This comes into evidence in [Ro2, Thm. 12], where Rosenlicht explicitly allows singularities on C. The only result which possibly depends on the smoothness of C is [Ro2, Thm. 10] (see the top of p. 523 of [Ro2]).

We are using the letter Γ instead of C. Our only application of Theorem 10 of [Ro2] appears in (2g1) which we have been careful to reprove directly. $\square$

Assumption 5.2.

(3a) *K has a local prime divisor $\mathfrak{p}$ with respect to which it is either Henselian or real closed. Let $\hat{K} = \hat{K}_{\mathfrak{p}}$.*

(3b) *The curve Γ has a simple K-rational point.*

(3c) *Γ is K-normal or $\mathrm{char}(K) > 0$ and Γ is a cusp curve.* $\square$

Conditions (3a) and (3b) imply, by the density theorem 8.2(b), that $\Gamma(K)$ is Zariski-dense in Γ, which means that $\Gamma(K)$ is infinite (Condition (1)). This adds an additional information to (2a)-(2i):

(4) By (2d), Γ^h_{simp} has a Zariski-open subset D such that $\psi|_D: D \to J$ is an étale map [Hrt, p. 271, Lemma 10.5]. (Note that Hartshorne states the lemma only in characteristic 0. However, the proof makes use only of the separability of the map.) Hence, by Proposition 8.2(c), $\psi|_{D(K)}: D(K) \to J(K)$ is a local $\mathfrak{p}$-homeomorphism.

Lemma 5.3. *Make Assumption 5.2 and let $\mathbf{b} \in J(K)$. Then, for each $\mathfrak{p}$-open neighborhood $\mathcal{A}$ of 0 in $J(K)$ there exist infinitely many positive integers n such that $n\mathbf{b} \in \mathcal{A}$.*

Proof. Suppose first that the theorem holds for $\hat{K}$. Then there exist infinitely many positive integers n such that $n\mathbf{b} \in \mathcal{A}(\hat{K})$. For each of these n we also have $n\mathbf{b} \in J(K)$. Hence, $n\mathbf{b} \in \mathcal{A}$. So, we have to prove the theorem only in the case where K is locally compact.

Under this assumption consider first the case that Γ is K-normal. In this case the lemma coincides with [GPR, Lemma 2.2].

Suppose therefore that $\mathrm{char}(K) = p > 0$ and that Γ is a cusp curve. Let $\pi\colon \Gamma' \to \Gamma$ be the normalization of Γ over K. Let $\theta\colon J \to J'$ be the homomorphism of the corresponding generalized Jacobians which is introduced in (2g).

Part A: *There is a power q of p such that $q \cdot \mathrm{Ker}(\tilde{\theta}) = 0$, where* $\mathrm{Ker}(\tilde{\theta}) = \{\mathbf{a} \in J(\tilde{K}) \mid \tilde{\theta}(\mathbf{a}) = 0\}$. Indeed, let O be the ring of singularities of Γ. Then, in the terminology of (2g1), $\mathrm{Ker}(\tilde{\theta}) \cong (R\tilde{K})^\times/(O\tilde{K})^\times$. By Lemma 2.2, $O\tilde{K}$ is the intersection of the local rings $O_{\tilde{\Gamma},\mathbf{p}}$, where $\mathbf{p}$ ranges on Γ_{sing}. Each $\mathbf{p} \in \tilde{\Gamma}_{\mathrm{sing}}$ is a cusp and therefore $O_{\tilde{\Gamma},\mathbf{p}}$ is contained in a valuation ring O_P of some unique prime divisor P of $F\tilde{K}/\tilde{K}$. The ring $R\tilde{K}$ is contained in the intersection of all those O_P. It follows that $(R\tilde{K})^\times/(O\tilde{K})^\times \subseteq \prod_{\mathbf{p}} O_P^\times/O_{\tilde{\Gamma},\mathbf{p}}^\times$, where, again, $\mathbf{p}$ ranges on $\tilde{\Gamma}_{\mathrm{sing}}$. Hence, it suffices to prove that for each $\mathbf{p} \in \tilde{\Gamma}_{\mathrm{sing}}$ there is a power q of p such that $\big(O_P^\times/O_{\tilde{\Gamma},\mathbf{p}}^\times\big)^q = 1$.

Since O_P is the unique valuation ring of $F\tilde{K}/\tilde{K}$ over $O_{\tilde{\Gamma},\mathbf{p}}$, it is the integral closure of $O_{\tilde{\Gamma},\mathbf{p}}$ in $F\tilde{K}$. Hence there exists a positive integer k such that $M_P^k \subseteq M_{\tilde{\Gamma},\mathbf{p}}$ (as follows for example from (1) of §2). Let $q \geq k$ be a power of p. Then $M_P^q \subseteq M_{\tilde{\Gamma},\mathbf{p}}$. Also, $O_P/M_P = \tilde{K} = O_{\tilde{\Gamma},\mathbf{p}}/M_{\tilde{\Gamma},\mathbf{p}}$, i.e., $O_P = O_{\tilde{\Gamma},\mathbf{p}} + M_P$. In particular, each $x \in (O_P)^\times$ has the form $x = a + m$ where $a \in O_{\tilde{\Gamma},\mathbf{p}} \smallsetminus M_{\tilde{\Gamma},\mathbf{p}}$ and $m \in M_P$. So, $x^q = a^q + m^q \in O_{\tilde{\Gamma},\mathbf{p}}^\times$, as desired.

Part B: *A $\mathfrak{p}$-open map.* Since $\theta\colon J \to J'$ is defined over K, it defines a map $\theta_K\colon J(K) \to J'(K)$. We claim that this map is $\mathfrak{p}$-open. Indeed, since, by (2i), $\theta\colon J \to J'$ is a separable map, J and J' have nonzero Zariski-open subsets J_0 and J_0' such that the restriction of θ to J_0 is a smooth map onto J_0' [Hrt, p. 271, Lemma 10.5]. By the open map theorem (Proposition 8.2(c)), $\theta_K\colon J(K) \to J'(K)$ is $\mathfrak{p}$-open in a neighborhood of each point $\mathbf{a} \in J_0(K)$. Since translations by points of $J(K)$ and of $J'(K)$ are $\mathfrak{p}$-isomorphisms, $\theta_K\colon J(K) \to J'(K)$ is $\mathfrak{p}$-open in a neighborhood of each point of $J(K)$ (see also [GPR, p. 51, J5]). So, $\theta_K\colon J(K) \to J'(K)$ is a $\mathfrak{p}$-open map.

Consider now a $\mathfrak{p}$-open neighborhood $\mathcal{A}$ of 0 in $J(K)$. Let $\mathcal{A}_0$ be a $\mathfrak{p}$-open neighborhood of 0 in $\mathcal{A}$ such that $q\mathcal{A}_0 \subseteq \mathcal{A}$. Then $\theta(\mathcal{A}_0)$ is a $\mathfrak{p}$-open neighborhood of 0 in $J'(K)$. By [GPR, Lemma 2.2] there exist infinitely many positive integers l such that $l\theta(\mathbf{b}) \in \theta(\mathcal{A}_0)$. Thus $l\mathbf{b} \in \mathcal{A}_0 + \mathrm{Ker}(\theta)$. Conclude from Part A that $ql\mathbf{b} \in \mathcal{A}$. □

Lemma 5.4. [GPR, Lemma 2.3] *Let G be a topological additive commutative group, let $k \geq 3$, and let $x_1, \ldots, x_k$ be elements of G. For each j let $\mathcal{W}_j$ be a neighborhood of x_j and for each $i \neq j$ let Δ_{ij} be a subset of $\mathcal{W}_i \times \mathcal{W}_j$ whose complement is open and dense in $\mathcal{W}_i \times \mathcal{W}_j$. Then there are $y_j \in \mathcal{W}_j$ such that $\sum_{j=1}^k x_j = \sum_{j=1}^k y_j$ and $(y_i, y_j) \notin \Delta_{ij}$ if $i \neq j$.*

The following result gives an element of $O^\times$ with control on both its pole divisor and its zero divisor.

Lemma 5.5. (Local existence theorem for normal curves and cusp curves) *Under Assumption 5.2 let $\mathcal{U}$ be a nonempty $\mathfrak{p}$-open subset of $\Gamma_{\rm simp}(K)$. Let $\mathfrak{a}$ be a Γ-smooth positive divisor of F/K. Let O be the ring of singularities of Γ. Then there exists a positive integer r_0 such that for each positive integer k there exists $f \in O^\times$ such that $\mathrm{div}(f) = \sum_{i=1}^s \mathbf{p}_i - kr_0\mathfrak{a}$, where $\mathbf{p}_1, \dots, \mathbf{p}_s$ are distinct points in $\mathcal{U} \smallsetminus \mathrm{Supp}(\mathfrak{a})$. In particular, each of them belong to $\Gamma(K)$.*

Proof. We follow the proof of [GPR, Thm. 2.1] which handles the case where Γ is K-normal. Our unified proof handles both the case where Γ is K-normal and the case where Γ is a cusp curve.

By shrinking $\mathcal{U}$, if necessary, we may assume that $\mathcal{U}$ contains no point that belongs to the support of $\mathfrak{a}$. By assumption, $\mathcal{U}^h$ is a nonempty $\mathfrak{p}$-open subset of $\Gamma_{\rm simp}^h(K)$. By (4), $\Gamma_{\rm simp}^h$ has a nonempty Zariski-open subset D such that $\psi|_{D(K)}$ is a local $\mathfrak{p}$-homeomorphism into $J(K)$. By Proposition 8.2(b), $\mathcal{U}^h$ is Zariski-dense in $\Gamma_{\rm simp}^h$. Hence, $\mathcal{U}^h \cap D(K)$ has a nonempty $\mathfrak{p}$-open subset $\mathcal{V}$ which ψ maps $\mathfrak{p}$-homeomorphically onto a $\mathfrak{p}$-open subset $\mathcal{W}$ of $J(K)$. Choose $(\mathbf{q}_1, \dots, \mathbf{q}_h) \in \mathcal{V}$. Then $\mathbf{q}_i \in \Gamma_{\rm simp}(K)$ and $\mathcal{W}_0 = \mathcal{W} - \psi(\mathbf{q}_1, \dots, \mathbf{q}_h)$ is a $\mathfrak{p}$-open neighborhood of 0 in $J(K)$. Also, $\mathfrak{b} = h\mathfrak{a} - \deg(\mathfrak{a}) \sum_{j=1}^h \mathbf{q}_j$ is a divisor of F/K of degree 0. Hence $\mathbf{b} = \tilde\varphi(\mathfrak{b}) \in J(K)$ (by (2e)).

By Lemma 5.3 there exists an integer n such that $n\mathbf{b} \in \mathcal{W}_0$ and $n\deg(\mathfrak{a}) \geq 3$. In other words, there exists $(\mathbf{q}_1', \dots, \mathbf{q}_h') \in \mathcal{V}$ such that $n\mathbf{b} = \psi(\mathbf{q}_1', \dots, \mathbf{q}_h') - \psi(\mathbf{q}_1, \dots, \mathbf{q}_h)$. Hence, $nh\tilde\varphi(\mathfrak{a}) - n\deg(\mathfrak{a})\sum_{j=1}^h \varphi(\mathbf{q}_j) = n\tilde\varphi(\mathfrak{b}) = n\mathbf{b} = \psi(\mathbf{q}_1', \dots, \mathbf{q}_h') - \psi(\mathbf{q}_1, \dots, \mathbf{q}_h) = \sum_{j=1}^h \varphi(\mathbf{q}_j') - \sum_{j=1}^h \varphi(\mathbf{q}_j)$ (use (2d1)). Hence

$$nh\tilde\varphi(\mathfrak{a}) = \sum_{j=1}^h \varphi(\mathbf{q}_j') + \sum_{j=1}^h (n\deg(\mathfrak{a}) - 1)\varphi(\mathbf{q}_j). \tag{5}$$

Let k be a positive integer. Then $m = kn\deg(\mathfrak{a}) \geq 3$. Multiply (5) by k:

$$knh\tilde\varphi(\mathfrak{a}) = \sum_{j=1}^h k\varphi(\mathbf{q}_j') + \sum_{j=1}^h k(n\deg(\mathfrak{a}) - 1)\varphi(\mathbf{q}_j). \tag{6}$$

Thus, with $\mathbf{a} = knh\tilde\varphi(\mathfrak{a})$, there exist $\mathbf{q}_{ij} \in \Gamma_{\rm simp}(K)$, $i = 1, \dots, m$, $j = 1, \dots, h$, such that

$$\mathbf{a} = \sum_{i=1}^m \sum_{j=1}^h \varphi(\mathbf{q}_{ij}) \tag{7}$$

and

$$(\mathbf{q}_{i1}, \dots, \mathbf{q}_{ih}) \in \mathcal{V}, \qquad i = 1, \dots, m. \tag{8}$$

Indeed, we may take $\mathbf{q}_{ij} = \mathbf{q}_j'$ for $i = 1, \dots, k$ and $\mathbf{q}_{ij} = \mathbf{q}_j$ for $i = k+1, \dots, m$.

For each pair (i, r) of distinct integers between 1 and m let

$$\Delta_{ir}' = \{(\mathbf{p}_{i1}, \dots, \mathbf{p}_{ih}, \mathbf{p}_{r1}, \dots, \mathbf{p}_{rh}) \in \Gamma^h \times \Gamma^h \mid \mathbf{p}_{ij} \neq \mathbf{p}_{rs} \text{ for } j, s = 1, \dots, h \text{ and } \mathbf{p}_{ij} \neq \mathbf{p}_{ij'} \text{ and } \mathbf{p}_{rj} \neq \mathbf{p}_{rj'} \text{ if } j \neq j'\}.$$

As Δ'_{ir} is a nonempty Zariski-open subset of $\Gamma^h \times \Gamma^h$, the density theorem (Proposition 8.2(b)) implies that $\Delta'_{ir}(K) \cap (\mathcal{V} \times \mathcal{V})$ is $\mathfrak{p}$-open and $\mathfrak{p}$-dense in $\mathcal{V} \times \mathcal{V}$. Hence, $\psi\big(\Delta'_{ir}(K) \cap (\mathcal{V} \times \mathcal{V})\big)$ is $\mathfrak{p}$-open and $\mathfrak{p}$-dense in $\mathcal{W} \times \mathcal{W}$. By (7) and by (2d1), $\sum_{i=1}^m \psi(\mathbf{q}_{i1}, \ldots, \mathbf{q}_{ih}) = \mathbf{a}$. Lemma 5.4 gives $(\mathbf{q}'_{i1}, \ldots, \mathbf{q}'_{ih}) \in \mathcal{V}$, $i = 1, \ldots, m$, such that $(\mathbf{q}'_{i1}, \ldots, \mathbf{q}'_{ih}, \mathbf{q}'_{r1}, \ldots, \mathbf{q}'_{rh}) \in \Delta'_{ir}(K)$ and such that

$$\sum_{i=1}^m \sum_{j=1}^h \varphi(\mathbf{q}'_{ij}) = \sum_{i=1}^m \psi(\mathbf{q}'_{i1}, \ldots, \mathbf{q}'_{ih}) = \mathbf{a}.$$

Replace $\mathbf{q}_{ij}$ by $\mathbf{q}'_{ij}$, if necessary, to assume that the $\mathbf{q}_{ij}$ are distinct. Rewrite (7) in the form

$$\tilde{\varphi}(-knh\mathfrak{a} + \sum_{i=1}^m \sum_{j=1}^h \mathbf{q}_{ij}) = 0.$$

By Abel's theorem (2e), there exists $f \in O^\times$ such that

$$\mathrm{div}(f) = -knh\mathfrak{a} + \sum_{i=1}^m \sum_{j=1}^h \mathbf{q}_{ij}.$$

By assumption, $\mathfrak{a} > 0$. Hence, $\mathrm{div}_\infty(f) = knh\mathfrak{a}$, the geometric zeros of f are $\mathbf{q}_{ij}$, they are distinct and belong to $\mathcal{U}$. Thus, $r_0 = nh$ satisfies the conclusion of the proposition. □

6. Rumely elements

Let S be a finite set of local primes of a field K. Consider a projective curve Γ over K. Let F be the function field of Γ. Denote the ring of singularities of Γ in F by O. Let $\mathfrak{a}$ be a positive divisor of F/K. Denote the support of $\mathfrak{a}$ by $\mathrm{Supp}(\mathfrak{a})$. For each $\mathfrak{p} \in S$ let $\mathcal{U}_\mathfrak{p}$ be a nonempty $\mathfrak{p}$-open subset of $\Gamma_{\mathrm{simp}}(K_\mathfrak{p})$ and let $\mathcal{U} = \bigcap_{\mathfrak{p}\in S} \bigcap_{\sigma\in G(K)} \mathcal{U}_\mathfrak{p}^\sigma$.

Lemma 6.1. *$\mathcal{U}$ is an S-open subset of* $\Gamma_{\mathrm{simp}}(K_{\mathrm{tot},S})$.

Proof. By the definition of $K_{\mathrm{tot},S}$ (beginning of §1), $\mathcal{U}$ is a subset of $\Gamma_{\mathrm{simp}}(K_{\mathrm{tot},S})$. In order to prove that $\mathcal{U}$ is S-open in $\Gamma_{\mathrm{simp}}(K_{\mathrm{tot},S})$ we may assume that Γ has an affine part Γ_0, say embedded in $\mathbb{A}^n$, such that $\mathcal{U}_\mathfrak{p} \subseteq \Gamma_0(K_\mathfrak{p})$ for each $\mathfrak{p} \in S$. Similarly, we may as well assume that $\mathcal{U}_\mathfrak{p}$ is a basic $\mathfrak{p}$-open set:

$$\mathcal{U}_\mathfrak{p} = \{\mathbf{p} \in \Gamma_0(K_\mathfrak{p}) \mid |\mathbf{p} - \mathbf{a}_\mathfrak{p}|_\mathfrak{p} < \varepsilon_\mathfrak{p}\},$$

where $\mathbf{a}_\mathfrak{p} \in \Gamma_0(K_\mathfrak{p})$ and $\varepsilon_\mathfrak{p} > 0$. Let L be a finite normal extension of K such that $\mathbf{a}_\mathfrak{p}$ is L-rational for each $\mathfrak{p} \in S$. For each $\mathfrak{p} \in S$ choose an extension of $|\ |_\mathfrak{p}$ to an absolute value of $\tilde{K}$. Then choose $\mathbf{a}'_\mathfrak{p} \in K^n$ such that $|\mathbf{a}'_\mathfrak{p} - \mathbf{a}_\mathfrak{p}|_\mathfrak{p} < \varepsilon_\mathfrak{p}$ and let $\mathcal{V}_\mathfrak{p} = \{\mathbf{p} \in \tilde{K}^n \mid |\mathbf{p} - \mathbf{a}'_\mathfrak{p}|_\mathfrak{p} < \varepsilon_\mathfrak{p}\}$. Then $\mathcal{V} = \bigcap_{\mathfrak{p}\in S} \bigcap_{\sigma\in G(K)} \mathcal{V}_\mathfrak{p}^\sigma$ is an S-open subset of

$\tilde{K}^n$ (Remark 1.1) and $\mathcal{U} = \mathcal{V} \cap \Gamma_0(K_{\mathrm{tot},S})$. Conclude that $\mathcal{U}$ is S-open in $\Gamma_{\mathrm{simp}}(K_{\mathrm{tot},S})$. □

A separating transcendental element t of F/K is said to be a **Rumely element** with respect to $S, \mathfrak{a}, \mathcal{U}$ if

(1) $\mathrm{div}(t) = \sum_{k=1}^{m} \mathbf{b}_k - \mathfrak{a}$, where $\mathbf{b}_1, \ldots, \mathbf{b}_m$ are distinct points in $\mathcal{U} \smallsetminus \mathrm{Supp}(\mathfrak{a})$.

The existence of Rumely elements implies that $\mathcal{U}$ is nonempty. This is assured by the following criterion.

Proposition 6.2. *Let K, S, Γ, F, O, and $\mathcal{U}$ be as above. Suppose that Γ is K-normal or $\mathrm{char}(K) > 0$ and Γ is a cusp curve. Let $\mathfrak{a}_0$ be a positive Γ-smooth divisor of F/K. Then there exists a positive integer k_0 such that for each multiple k of k_0, for $\mathfrak{a} = k\mathfrak{a}_0$, and for each basis $u_0, \ldots, u_n$ of $\mathcal{L}_O(\mathfrak{a})$ there exists a nonempty S-open set $\mathcal{A} \subseteq K^{n+1}$ such that for each $\mathbf{a} \in \mathcal{A}$, $\sum_{i=0}^{n} a_i u_i$ is a Rumely element with respect to $S, \mathfrak{a}, \mathcal{U}$.*

Proof. The proof naturally breaks up into two parts.

Part A: *The local case.* Consider $\mathfrak{p} \in S$. Let $\Gamma_\mathfrak{p} = \Gamma \times_K K_\mathfrak{p}$ be the projective curve obtained from Γ by extending K to $K_\mathfrak{p}$. Since F, as a regular extension of K, is linearly disjoint from $K_\mathfrak{p}$ over K, $FK_\mathfrak{p}$ is the function field of $\Gamma_\mathfrak{p}$ and we may view $\mathfrak{a}_0$ also as a divisor of $FK_\mathfrak{p}/K_\mathfrak{p}$. By [Mum, p. 158], $\Gamma_\mathfrak{p}(K_\mathfrak{p}) = \Gamma(K_\mathfrak{p})$. By Lemma 2.3, $OK_\mathfrak{p}$ is the ring of singularities of $\Gamma_\mathfrak{p}$ in $FK_\mathfrak{p}$.

By Proposition 5.5 there exists a positive integer $k_\mathfrak{p}$ such that for each multiple k of $k_\mathfrak{p}$ there exists a function $f_\mathfrak{p} \in OK_\mathfrak{p}$ with the following property:

(2) $\mathrm{div}(f_\mathfrak{p}) = \sum_{i=1}^{m} \mathbf{b}_{\mathfrak{p},i} - k\mathfrak{a}_0$, where $\mathbf{b}_{\mathfrak{p},1}, \ldots, \mathbf{b}_{\mathfrak{p},m}$ are distinct points in $\mathcal{U}_\mathfrak{p} \smallsetminus \mathrm{Supp}(\mathfrak{a}_0)$.

Suppose now that $u_0, \ldots, u_n$ is a basis of $\mathcal{L}_{OK_\mathfrak{p}}(k\mathfrak{a}_0)$. Write $f_\mathfrak{p} = \sum_{j=0}^{n} a_{\mathfrak{p},j} u_j$ with $a_{\mathfrak{p},j} \in K_\mathfrak{p}$. Let $\mathbf{a}_\mathfrak{p} = (a_{\mathfrak{p},0}, \ldots, a_{\mathfrak{p},n})$. By Proposition 8.3

(3) there exists a $\mathfrak{p}$-open neighborhood $\mathcal{A}_\mathfrak{p}$ of $\mathbf{a}_\mathfrak{p}$ in $K_\mathfrak{p}^{n+1}$ such that if $\mathbf{a}' \in \mathcal{A}_\mathfrak{p}$ and $f'_\mathfrak{p} = \sum_{j=0}^{n} a'_j u_j$, then (2) remains valid if we replace $f_\mathfrak{p}$ by $f'_\mathfrak{p}$.

Part B: *The global case.* Let k_0 be the least common multiple of all $k_\mathfrak{p}$ from Part A with $\mathfrak{p} \in S$. Consider a multiple k of k_0 and let $\mathfrak{a} = k\mathfrak{a}_0$. Let $u_0, \ldots, u_n$ be a basis of $\mathcal{L}_O(\mathfrak{a})$. By Lemma 2.3, $u_0, \ldots, u_n$ is also a basis of $\mathcal{L}_{OK_\mathfrak{p}}(\mathfrak{a})$. For each $\mathfrak{p} \in S$ let $f_\mathfrak{p} \in OK_\mathfrak{p}$ be a function as in (2). Condition (3) gives a $\mathfrak{p}$-open neighborhood $\mathcal{A}_\mathfrak{p}$ of $\mathbf{a}_\mathfrak{p}$ in $K_\mathfrak{p}^{n+1}$ such that if $\mathbf{a} = (a_0, \ldots, a_n) \in \mathcal{A}_\mathfrak{p}$ and $f = \sum_{j=0}^{n} a_j u_j$, then

(4) $\mathrm{div}(f) = \sum_{k=1}^{m} \mathbf{b}_k - \mathfrak{a}$ with distinct $\mathbf{b}_1, \ldots, \mathbf{b}_m \in \mathcal{U}_\mathfrak{p} \smallsetminus \mathrm{Supp}(\mathfrak{a})$.

To conclude the proof use the $\mathfrak{p}$-density of K in $K_\mathfrak{p}$ and the weak approximation theorem [CaF, p. 48] to choose a nonempty S-open subset $\mathcal{A}$ of K^{n+1} which is contained in $\mathcal{A}_\mathfrak{p}$ for each $\mathfrak{p} \in S$. Consider $\mathbf{a} \in \mathcal{A}$ and let $f = \sum_{j=0}^{n} a_j u_j$. Then (4) holds for each $\mathfrak{p} \in S$. In particular, $\mathbf{b}_1, \ldots, \mathbf{b}_m \in \mathcal{U}_\mathfrak{p}$ for each $\mathfrak{p} \in S$.

Since F/K is regular, we may extend each $\sigma \in G(K)$ to an automorphism of $F\tilde{K}/F$. Since $f \in F$, σ permutes $\mathbf{b}_1, \dots, \mathbf{b}_m$. Hence, $\mathbf{b}_1, \dots, \mathbf{b}_m \in \mathcal{U}_{\mathfrak{p}}^{\sigma}$. Conclude that $\mathbf{b}_1, \dots, \mathbf{b}_m \in \mathcal{U}$. □

The combination of Proposition 6.2 and Lemma 4.5 yields a stabilizing element for F/K with control on the location of its zeros and poles.

Theorem 6.3. *Let K, S, Γ, F, and O be as above. Suppose that Γ is smooth or $\mathrm{char}(K) > 0$ and Γ is an ordinary special cusp curve. For $i = 1, 2$ and for each $\mathfrak{p} \in S$ let $\mathcal{U}_{i,\mathfrak{p}}$ be a nonempty $\mathfrak{p}$-open subset of $\Gamma_{\mathrm{simp}}(K_{\mathfrak{p}})$. Write $\mathcal{U}_i = \bigcap_{\mathfrak{p}\in S} \bigcap_{\sigma\in G(K)} \mathcal{U}_{i,\mathfrak{p}}^{\sigma}$, $i = 1, 2$. Let d_0 be a positive integer. Then there exists $d \geq d_0$ and F/K has a stabilizing element t such that*

(a) $\mathrm{div}_0(t) = \sum_{i=1}^{d} \mathbf{z}_i$, *where $\mathbf{z}_1, \dots, \mathbf{z}_d$ are distinct points in $\mathcal{U}_1$; and*

(b) $\mathrm{div}_\infty(t) = \sum_{i=1}^{d} \mathbf{p}_i$, *where $\mathbf{p}_1, \dots, \mathbf{p}_d$ are distinct points in $\mathcal{U}_2$.*

(c) *Moreover, let $\hat{F}$ be the Galois closure of $F/K(t)$. Then $\mathcal{G}(\hat{F}/K(t)) \cong S_d$.*

Proof. Let $\mathfrak{a}_0$ be the positive divisor of F/K that Lemma 4.5 gives. For $i = 1, 2$, Proposition 6.2 gives a positive integer k_i such that for each multiple k of k_i, for $\mathfrak{a} = k\mathfrak{a}_0$, and for each basis $y_0, y_1, \dots, y_n$ of $\mathcal{L}_O(\mathfrak{a})$

(5) there exists a nonempty S-open set $\mathcal{A}_i \subseteq K^{n+1}$ such that for each $\mathbf{b} \in \mathcal{A}_i$, $\sum_{j=0}^{n} b_j y_j$ is a Rumely element with respect to $S, \mathfrak{a}, \mathcal{U}_i$.

Let k be a common multiple of k_1 and k_2 and let $\mathfrak{a} = k\mathfrak{a}_0$. Section 3 defines two morphisms of varieties over K:

$$\mathbb{P} \xleftarrow{\varphi} \mathbb{M} \xrightarrow{\mu} M_{2,n+1}^{*}.$$

In particular, $M_{2,n+1}^{*}$ consists of all $2 \times (n+1)$ matrices of rank 2. By Lemma 4.5, $\mathcal{L}_O(\mathfrak{a})$ has a basis $y_0, y_1, \dots, y_n$ and there exists a nonempty Zariski-open subset U of $\mathbb{P}$ such that

(6) for each $\mathbf{A} \in \varphi^{-1}(U(K))$ and with $\mu(\mathbf{A}) = \binom{\mathbf{b}_1}{\mathbf{b}_2}$ the element

$$t = \frac{\sum_{j=0}^{n} b_{1j} y_j}{\sum_{j=0}^{n} b_{2j} y_j}$$

symmetrically stabilizes F/K.

Make $\mathcal{U}_{i,\mathfrak{p}}$ smaller, if necessary, to assume that $\mathcal{U}_1$ and $\mathcal{U}_2$ are disjoint. For $i = 1, 2$ let $\mathcal{A}_i$ be the set that (5) gives. Then $\mathcal{A}_1 \times \mathcal{A}_2$ is a nonempty S-open subset of $M_{2,n+1}^{*}(K)$. Since $\mu(\mathbb{M}(K)) = M_{2,n+1}^{*}(K)$ (by §3), the S-open subset $\mu^{-1}(\mathcal{A}_1 \times \mathcal{A}_2)$ of $\mathbb{M}(K)$ is not empty.

Since $\mathbb{M}$ is a Zariski-open subset of an affine space, $\mu^{-1}(\mathcal{A}_1 \times \mathcal{A}_2)$ is Zariski-dense in $\mathbb{M}$ (by the weak approximation theorem). In particular, there exists

$$\mathbf{A} \in \mu^{-1}(\mathcal{A}_1 \times \mathcal{A}_2) \cap \varphi^{-1}(U(K)).$$

Let $\mu(\mathbf{A}) = \binom{\mathbf{b}_1}{\mathbf{b}_2}$. Then $\mathbf{b}_i \in \mathcal{A}_i$, $i = 1, 2$. By (5), $t_i = \sum_{j=0}^{n} b_{ij} y_j$ is a Rumely element with respect to $S, \mathfrak{a}, \mathcal{U}_i$. Thus, $\mathrm{div}_\infty(t_i) = \mathfrak{a}$ and $\mathrm{div}_0(t_i) = \sum_{j=1}^{d} \mathbf{p}_{ij}$, where $\mathbf{p}_{i1}, \ldots, \mathbf{p}_{id}$ are distinct points of $\mathcal{U}_i$, $i = 1, 2$. Let $t = t_1/t_2$. Then $\mathrm{div}(t) = \mathrm{div}_0(t_1) - \mathrm{div}_\infty(t_1) - \mathrm{div}_0(t_2) + \mathrm{div}_\infty(t_2) = \mathrm{div}_0(t_1) - \mathrm{div}_0(t_2)$. Since $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$, the positive divisors $\mathrm{div}_0(t_1)$ and $\mathrm{div}_0(t_2)$ are relatively prime. Hence $\mathrm{div}_0(t) = \mathrm{div}_0(t_1) = \sum_{j=1}^{d} \mathbf{p}_{1j}$. Similarly, $\mathrm{div}_\infty(t) = \mathrm{div}_0(t_2) = \sum_{j=1}^{d} \mathbf{p}_{2j}$. So both (a) and (b) hold.

Finally, since $\varphi(\mathbf{A}) \in U(K)$, (6) implies that the element t symmetrically stabilizes F/K. For the Galois closure $\hat{F}$ of $F/K(t)$, this means that $\mathcal{G}(\hat{F}\tilde{K}/\tilde{K}(t)) \cong S_d$, where $d = [F : K(t)] = \deg(\mathrm{div}_0(t)) = \deg(\mathrm{div}_0(t_1)) = \deg(\mathrm{div}_\infty(t_1)) = \deg(\mathfrak{a}) = k \deg(\mathfrak{a}_0)$. Hence, $\mathcal{G}(\hat{F}/K(t)) \cong S_d$. Taking k large, d becomes large. This proves (c) and concludes the proof of the theorem. □

7. The main result

Let A be a subset of a field M. We say that M is **PAC over** A if for each dominant separable rational map $\varphi: V \to \mathbb{A}^r$ of varieties of dimension r over M there exists $\mathbf{x} \in V(M)$ such that $\varphi(\mathbf{x}) \in A^r$.

Proposition 7.1. (Razon [Ra1, Thm. 2.3]) *Let S be a finite set of local primes of a field K such that none of them is complex. Let M be a field which is PAC over K. Then M is PAC over each nonempty S-open subset of K.*

Lemma 7.2. ([Ra1, Lemma 3.6]) *Let S be a finite set of local primes of an infinite field K. Let $\tau: \Gamma \to \mathbb{P}^1$ be a rational map of degree d of projective curves over K. Suppose that there exists $\mathbf{p} \in \mathbb{P}^1(K)$ such that $\tau^{-1}(\mathbf{p})(\tilde{K})$ consists of d distinct points which are in $\Gamma_{\mathrm{simp}}(K_{\mathrm{tot},S})$. Then $\mathbb{P}^1(K)$ has a nonempty S-open set $\mathcal{A}$ such that for each $\mathbf{p}' \in \mathcal{A}$, the fibre $\tau^{-1}(\mathbf{p}')(\tilde{K})$ consists of d distinct points which are in $\Gamma_{\mathrm{simp}}(K_{\mathrm{tot},S})$.*

Remark 7.3. Sections 1–2 and 3.1–3.6 of [Ra1] are independent of our work. The use of Theorem 2.3 and Lemma 3.6 of [Ra1] here is therefore valid.

Likewise, Sections 1–6 and 8 of our work are independent of [Ra1]. Hence, the use of Proposition 7.2(b) and Proposition 7.2 of our work in [Ra1, Lemma 3.7 and Prop. 3.8] is justified. □

Theorem 7.4. (= Theorem B) *Let S be a finite set of local primes of a field K. Let M be a field which is PAC over K. Denote the maximal Galois extension of K which is contained in $M \cap K_{\mathrm{tot},S}$ by N. Then N is PSC.*

P r o o f. Omitting the complex primes from S does not change $K_{\mathrm{tot},S}$. So, assume S contains no complex primes. Let C be a curve over K. If $\mathrm{char}(K) = 0$, take Γ to be the projective K-normal curve which is birationally equivalent over K to C.

Then Γ is smooth. If $\mathrm{char}(K) > 0$, then C is birationally equivalent over K to an ordinary special cusp curve Γ [Neu, Prop. 2.22]. Suppose that for each $\mathfrak{p} \in S$ there exists $\mathbf{a}_\mathfrak{p} \in \Gamma_{\mathrm{simp}}(K_\mathfrak{p})$. By Lemma 1.6, it suffices to prove that $\Gamma(N)$ is Zariski-dense in Γ. So, consider a nonempty Zariski-open subset Γ_0 of Γ. We have to prove that $\Gamma_0(N) \neq \emptyset$.

By Proposition 8.2(a), $\mathbf{a}_\mathfrak{p}$ has a $\mathfrak{p}$-open neighborhood $\mathcal{U}_\mathfrak{p}$ in $\Gamma_{\mathrm{simp}}(K_\mathfrak{p})$. Then $\mathcal{U} = \bigcap_{\mathfrak{p}\in S} \bigcap_{\sigma\in G(K)} \mathcal{U}_\mathfrak{p}^\sigma$ is an S-open subset of $\Gamma_{\mathrm{simp}}(K_{\mathrm{tot},S})$ (Lemma 6.1).

Let F be the function field of Γ over K. Theorem 6.3 with $\mathcal{U}_{1,\mathfrak{p}} = \mathcal{U}_{2,\mathfrak{p}} = \mathcal{U}_\mathfrak{p}$ gives a stabilizing element t for F/K such that all geometric zeros of t belong to $\mathcal{U}$ and each of them has multiplicity 1. In other words, let $d = [F : K(t)] = [F\tilde{K} : \tilde{K}(t)]$. Let $\hat{F}$ be the Galois closure of $F/K(t)$. Then the following holds:

(1a) $\hat{F}$ is a regular extension of K.

(1b) If we consider t as an element of $F\tilde{K}$, then $\mathrm{div}_0(t) = \sum_{i=1}^d \mathbf{p}_{0i}$, where $\mathbf{p}_{01}, \ldots, \mathbf{p}_{0d}$ are distinct and each of them is in $\mathcal{U}$, hence in $\Gamma_{\mathrm{simp}}(K_{\mathrm{tot},S})$.

As an element of F, t may be identified with a separable rational map $\tau\colon \Gamma \to \mathbb{P}^1$ over K, where

$$\deg(\tau) = [F : K(t)] = [F\tilde{K} : \tilde{K}(t)] = \deg(\mathrm{div}_0(t)) = d.$$

Moreover, denote the zero of t, when considered as a rational function of $\mathbb{P}^1$, by $\mathbf{p}_0$. Then $\tau^{-1}(\mathbf{p}_0)(\tilde{K}) = \{\mathbf{p}_{01}, \ldots, \mathbf{p}_{0d}\} \subseteq \Gamma_{\mathrm{simp}}(K_{\mathrm{tot},S})$.

Let $\hat{\Gamma}$ be a model of $\hat{F}/K$. Since $\hat{F}/K$ is regular, $\hat{\Gamma}$ is a curve (i.e., absolutely irreducible). Also, there exist separable rational maps $\pi\colon \hat{\Gamma} \to \Gamma$ and $\hat{\tau}\colon \hat{\Gamma} \to \mathbb{P}^1$ over K such that $\tau \circ \pi = \hat{\tau}$. In addition, there exists a nonempty Zariski-open subset A of $\mathbb{P}^1$ such that if $\hat{\mathbf{q}} \in \hat{\Gamma}(\tilde{K})$ and $\hat{\tau}(\hat{\mathbf{q}}) \in A(K)$, then $K(\hat{\mathbf{q}})/K$ is a Galois extension, π is defined at $\hat{\mathbf{q}}$, and $\pi(\hat{\mathbf{q}}) \in \Gamma_0(\tilde{K})$.

Let $\mathcal{A}$ be as in Lemma 7.2. By Proposition 7.1 there exists $\hat{\mathbf{q}} \in \hat{\Gamma}_{\mathrm{simp}}(M)$ such that $\mathbf{p} = \hat{\tau}(\hat{\mathbf{q}}) \in \mathcal{A} \cap A(K)$. By Lemma 7.2, $\tau^{-1}(\mathbf{p})(\tilde{K}) \subseteq \Gamma(K_{\mathrm{tot},S})$. Let $\mathbf{q} = \pi(\hat{\mathbf{q}})$. Then $\mathbf{q}$ is in $\tau^{-1}(\mathbf{p})(\tilde{K})$ and therefore also in $\Gamma_0(K_{\mathrm{tot},S})$. Moreover, $K(\mathbf{q}) \subseteq K(\hat{\mathbf{q}}) \cap K_{\mathrm{tot},S} \subseteq M \cap K_{\mathrm{tot},S}$. Since $K(\hat{\mathbf{q}})$ is Galois over K, so is $K(\hat{\mathbf{q}}) \cap K_{\mathrm{tot},S}$. Hence, $K(\mathbf{q}) \subseteq N$. Conclude that $\mathbf{q} \in \Gamma_0(N)$. □

Let S be a set of local primes of a field K. Let N be a field between K and $K_{\mathrm{tot},S}$. We say that N has the **S-density property** if it satisfies the following condition: Let V be a variety over K. For each $\mathfrak{p} \in S$ let $\mathcal{U}_\mathfrak{p}$ be a nonempty $\mathfrak{p}$-open subset of $V_{\mathrm{simp}}(K_\mathfrak{p})$. Then

$$V(N) \cap \bigcap_{\mathfrak{p}\in S} \bigcap_{\sigma\in G(K)} \mathcal{U}_\mathfrak{p}^\sigma \neq \emptyset. \tag{2}$$

Corollary 7.5. (S-density property) *Let K, S, M, and N be as in Theorem 7.4. Suppose S contains no complex primes. Then N has the S-density property.*

Proof. Since N is PSC, the corollary follows from Theorem 7.4 and from Razon's density theorem [Ra1, Thm. 3.9]. Alternatively, one can use the arguments of the proof of Theorem 7.4 to prove the corollary directly. □

Corollary 7.6. *Let K be a field and let S be a finite set of local primes of K. Then $K_{\mathrm{tot},S}$ is PSC. Moreover, if S contains no complex primes, then $K_{\mathrm{tot},S}$ has the S-density property.*

Proof. By the Hilbert Nullstellensatz, the field $M = K_s$ is PAC over K. Moreover, in the notation of Theorem 7.4, $N = K_{\mathrm{tot},S}$. So, Corollary 7.6 is a special case of Theorem 7.4 and Corollary 7.5. □

Note that the first statement of Corollary 7.6 is the main result of [Po1] (if $\mathrm{char}(K) = 0$). So, Theorem 7.4 is a generalization of Pop's result.

Corollary 7.7.*(= Theorem) Let K be a countable separably Hilbertian field and let S be a finite set of local primes of K. Let e be a positive integer. Then, for almost all $\boldsymbol{\sigma} \in G(K)^e$, the field $N = K_s[\boldsymbol{\sigma}] \cap K_{\mathrm{tot},S}$ is PSC. Moreover, N has the S-density property.*

Proof. Recall that for each $\boldsymbol{\sigma} \in G(K)^e$ the field $K_s[\boldsymbol{\sigma}]$ is the maximal Galois extension of K which is contained in $K_s(\boldsymbol{\sigma})$. Hence, N is the maximal Galois extension of K which is contained in $K_s(\boldsymbol{\sigma}) \cap K_{\mathrm{tot},S}$. By [JR1, Prop. 3.1], $K_s(\boldsymbol{\sigma})$ is PAC over K for almost all $\boldsymbol{\sigma} \in G(K)^e$. The corollary is therefore a special case of Theorem 7.4 and Corollary 7.5. □

Recall that a field N is **ample** if every curve C over N with a simple N-rational point has infinitely many N-rational points [Po2, p. 2 or HJ2, Def. 6.3]. If N is PSC and C is a curve over N with a simple N-rational point, then this point is also $N_{\mathfrak{p}}$-rational for each $\mathfrak{p} \in S$. By Proposition 1.5, $C(N)$ is infinite. Thus N is ample. As an ample field, N has several nice properties, which we may apply to the field appearing in Corollary 7.7.

Corollary 7.8. *Let K be a countable separably Hilbertian field and let S be a finite set of local primes of K. Let e be a positive integer. Then, for almost all $\boldsymbol{\sigma} \in G(K)^e$, the field $N = K_s[\boldsymbol{\sigma}] \cap K_{\mathrm{tot},S}$ has the following properties:*

(a) *N is ample; in particular, N is existentially closed in the field of formal power series $N((x))$.*

(b) *Let E be a finitely generated regular extension of N of transcendence degree 1. Then every finite split embedding problem over E is solvable (even in a regular way).*

Proof of (a). Note that the statement about the field of formal power series is actually equivalent to N being ample [Po2, Prop. 1].

Proof of (b). We refer the reader to [HJ3, Thm. B] for an exact formulation of a split embedding problem and for an algebraic proof of the statement. Alternatively, see [Po3] for a proof which uses methods of rigid analytical geometry. □

Another application of the main result is motivated by a peculiar result of Razon. He proves in [Ra2, Thm. 4.8] that for each finite set S of prime numbers and for almost all $\boldsymbol{\sigma} \in G(\mathbb{Q})^e$ the absolute Galois group of each field between $(\tilde{\mathbb{Q}}(\boldsymbol{\sigma}) \cap \mathbb{Q}_{\mathrm{tot},S})_{\mathrm{cycl}}$

and $\tilde{\mathbb{Q}}(\boldsymbol{\sigma})_{\mathrm{cycl}}$ is isomorphic to $\hat{F}_\omega$. (Here L_{cycl} denotes the field obtained from L by adjoining all roots of unity.) The proof of this result relies, among others, on the property of $\tilde{\mathbb{Q}}(\boldsymbol{\sigma})$ being PAC over $\mathbb{Q}$. We do not know whether $\tilde{\mathbb{Q}}[\boldsymbol{\sigma}]$ is PAC over $\mathbb{Q}$. Nevertheless, we can substitute the latter property of $\tilde{\mathbb{Q}}(\boldsymbol{\sigma})$ by $\tilde{\mathbb{Q}}[\boldsymbol{\sigma}]/\mathbb{Q}$ being Galois and by a result of Haran:

Corollary 7.9. *Let K be an infinite field which is finitely generated over its prime field. Let S be a finite set of local primes of K. Let e be a positive integer. Then, for almost all $\boldsymbol{\sigma} \in G(K)^e$ and for each field E between $(K_s[\boldsymbol{\sigma}] \cap K_{\mathrm{tot},S})_{\mathrm{cycl}}$ and $K_s[\boldsymbol{\sigma}]_{\mathrm{cycl}}$ we have: $G(E) \cong \hat{F}_\omega$.*

Proof. By [FJ2, Cor. 12.8 and Thm. 12.10], K is Hilbertian. By Corollaries 7.7 and 7.8, for almost all $\boldsymbol{\sigma} \in G(K)^e$ the field $N = K_s[\boldsymbol{\sigma}] \cap K_{\mathrm{tot},S}$ is PSC and therefore ample. Since K_{cycl} is an infinite extension of K, we may assume, in addition, that $K_{\mathrm{cycl}} \not\subseteq K_s[\boldsymbol{\sigma}]$.

Let E be a field between N_{cycl} and $K_s[\boldsymbol{\sigma}]_{\mathrm{cycl}}$. As a separable algebraic extension of an ample field, E is ample [Po2, Prop. 1.2]. In addition, let $\mathcal{E} = \{EK_\mathfrak{p}^\sigma \mid \mathfrak{p} \in S,\ \sigma \in G(K)\}$. By [Ja2, Lemma 8.2], E is P$\mathcal{E}$C.

We prove below that since E contains K_{cycl}, the absolute Galois group $G(EK_\mathfrak{p}^\sigma)$ is projective for each $\mathfrak{p} \in S$ and all $\sigma \in G(K)$. It will follow from Proposition 1.7(b) that $G(E)$ is projective. As $K_\mathfrak{p}$ is, any way, determined only up to K-conjugation, we may as well assume that $\sigma = 1$.

Suppose first that $\hat{K}_\mathfrak{p}$ is a finite extension of $\mathbb{Q}_p$ for some prime number p. Since E contains all roots of unity, l^∞ divides $[E\hat{K}_\mathfrak{p} : E]$ for each prime number l. By [Rib, p. 291, Cor. 7.4], $G(E\hat{K}_\mathfrak{p})$ is projective. Since restriction maps $G(\hat{K}_\mathfrak{p})$ bijectively onto $G(K_\mathfrak{p})$ (by Krasner's Lemma), it maps $G(E\hat{K}_\mathfrak{p})$ bijectively onto $G(EK_\mathfrak{p})$. Hence, $G(EK_\mathfrak{p})$ is projective.

If $\hat{K}_\mathfrak{p}$ is $\mathbb{R}$ or $\mathbb{C}$, then $EK_\mathfrak{p}$ is algebraically closed and therefore $G(EK_\mathfrak{p})$ is projective.

The third possibility is that $\hat{K}_\mathfrak{p}$ is a finite extension of $\mathbb{F}_p((t))$. Then $E\hat{K}_\mathfrak{p}$ is a separable algebraic extension of $\tilde{\mathbb{F}}_p((t))$. The latter field is complete with respect to a discrete valuation and with residue field $\tilde{\mathbb{F}}_p$ of cohomological dimension 0. Hence, by [Rib, p. 277, Thm. 6.1], $G(\tilde{\mathbb{F}}_p((t)))$ is projective. Let $L = K_s \cap \tilde{\mathbb{F}}_p((t))$. Then $G(L)$ is isomorphic to $G(\tilde{\mathbb{F}}_p((t)))$ and is therefore also projective. Finally, as $EK_\mathfrak{p}$ contains L, its absolute Galois group is also projective.

Finally, a result of Haran [Har, Thm. 4.1] says that if L_1 and L_2 are Galois extensions of a Hilbertian field L and F is a field between L and L_1L_2 which is not contained in L_1 or in L_2, then F is Hilbertian. In our case, K is Hilbertian and both K_{cycl} and $K_s[\boldsymbol{\sigma}]$ are Galois extensions of K. By the choice of $\boldsymbol{\sigma}$, E is not contained in $K_s[\boldsymbol{\sigma}]$. Hence, if E is also not contained in K_{cycl}, then, by Haran's result, E is Hilbertian.

By avoiding another set of measure 0, one may choose $\boldsymbol{\sigma}$ such that $K_s[\boldsymbol{\sigma}] \cap K_{\mathrm{tot},S} \not\subseteq K_{\mathrm{cycl}}$ and hence $E \not\subseteq K_{\mathrm{cycl}}$. We do not prove this fact here and instead note that if $E = K_{\mathrm{cycl}}$, then E is Hilbertian (by a theorem of Kuyk [FJ2, Thm. 15.6]).

In any case, it follows now from [HJ2, Thm. 6.5] that $G(E) \cong \hat{F}_\omega$. □

Remark 7.10.

(a) It is interesting to note that, in the setup of Corollary 7.8, $G(E) \cong \hat{F}_\omega$ for each field E between $K_s[\boldsymbol{\sigma}]$ and $K_s[\boldsymbol{\sigma}]_{\rm cycl}$.

Indeed, for almost all $\boldsymbol{\sigma} \in G(K)^e$, $K_s[\boldsymbol{\sigma}]$ is PAC and $G(K_s[\boldsymbol{\sigma}]) \cong \hat{F}_\omega$ [Ja4, Thm. 2.7], $K_{\rm cycl} \not\subseteq K_s[\boldsymbol{\sigma}]$, and $K_s[\boldsymbol{\sigma}] \not\subseteq K_{\rm cycl}$ (because $K_s[\boldsymbol{\sigma}]/K$ has Galois subextensions L/K with $\mathcal{G}(L/K) \cong S_n$, as follows e.g. from [JR1, Prop. 6.2]). If $K_s[\boldsymbol{\sigma}] \subset E \subseteq K_s[\boldsymbol{\sigma}]_{\rm cycl} = K_{\rm cycl} \cdot K_s[\boldsymbol{\sigma}]$, then, by Haran's result, E is Hilbertian. As E is an algebraic extension of a PAC field, E is itself PAC [FJ2, Cor. 10.7]. By [HJ2, Thm. A], $G(E) \cong \hat{F}_\omega$.

(b) If we take $K = \mathbb{Q}$ in Corollary 7.9 and let S be the set whose only element is the ordering of $\mathbb{Q}$, then $\mathbb{Q}_{{\rm tot},S} = \mathbb{Q}_{\rm tr}$ is the maximal totally real extension of $\mathbb{Q}$. For almost all $\boldsymbol{\sigma} \in G(\mathbb{Q})^e$ Corollary 7.7 asserts that $E = \tilde{\mathbb{Q}}[\boldsymbol{\sigma}] \cap \mathbb{Q}_{\rm tr}$ is PRC (Remark 1.4(f)). Hence, $F = (\tilde{\mathbb{Q}}[\boldsymbol{\sigma}] \cap \mathbb{Q}_{\rm tr})(\sqrt{-1})$ is a PRC field [Pre, Thm. 3.1] and therefore PAC. By a result of Weissauer [FJ2, Cor. 12.15] (which preceded that of Haran), $F = E(\sqrt{-1})$ is Hilbertian. So, again by [HJ2, Thm. 6.5], $G(F) \cong \hat{F}_\omega$. This is an analog of a result of Pop which says that $G\big(\mathbb{Q}_{\rm tr}(\sqrt{-1})\big) \cong \hat{F}_\omega$. □

8. Appendix — Generalized local fields

A **generalized local field** is a pair $(K, \mathfrak{p})$ which satisfies one of the following three conditions:

(1a) K is a field and $\mathfrak{p}$ is an equivalence class of Henselian valuations of K. In this case we choose a valuation $v_\mathfrak{p}$ representing $\mathfrak{p}$.

(1b) K is a real closed field and $\mathfrak{p}$ is the unique ordering $<$ of K. The latter defines a **generalized metric absolute value** $|\ |$ of K by $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x \leq 0$.

(1c) K is the algebraic closure of a real closed field R and $\mathfrak{p}$ is the corresponding generalized metric absolute value $|\ |$. Thus, if $z = x + y\sqrt{-1}$ with $x, y \in R$, then $|z| = \sqrt{x^2 + y^2}$.

In particular, each local field is a generalized local field.

In each case $\mathfrak{p}$ defines a topology on K. A basic $\mathfrak{p}$-open neighborhood of an element $a \in K$ is $\{x \in K \mid v_\mathfrak{p}(x - a) > \alpha\}$ for some α in the value group of $\mathfrak{p}$ in the Henselian case, and $\{x \in K \mid |x - a| < \varepsilon\}$ for some positive $\varepsilon \in R$ in the generalized metric case.

Two of the main tools that we use in the proof of the main result of this work are the density theorem and the open map theorem for varieties over local fields. In addition, we use the continuity of zeros of algebraic functions. Proofs of these theorems for Henselian valuations appear in [GPR]. However, they are also true for generalized metric absolute values. Here we reduce the proof of the latter case to the Henselian case. The reduction depends on the following construction.

Remark 8.1. (Ultrapowers) Let $(K, |\ |)$ be a generalized metric absolute valued field. Choose a nonprincipal ultrafilter $\mathcal{D}$ of $\mathbb{N}$ and let $(K^*, |\ |^*) = (K, |\ |)^\mathbb{N}/\mathcal{D}$ be the corresponding ultrapower. (See [FJ2, §6.7] for definition and basic properties of ultrapowers.) Then $(K^*, |\;|^*)$ is an elementary extension of $(K, |\;|)$ [FJ2, Prop. 6.15]. In particular K^* is real (resp., algebraically) closed if K is real (resp., algebraically) closed and $|\;|^*$ is a generalized metric absolute value. Let

$$O = \{x \in K^* \mid \exists n \in \mathbb{N}\colon |x| \le n\} \quad \text{and} \quad M = \big\{x \in K^* \mid \forall n \in \mathbb{N}\colon |x| \le \frac{1}{n}\big\}.$$

Then O is a valuation ring of K^* and M is the maximal ideal of O. In particular, the equivalence class in K^* of $(1, 2, 3, 4, \ldots)$ is not in O. Denote the corresponding valuation by v.

If K^* is real closed, then, by [Ja1, Lemma 16.2], v is Henselian. Alternatively note that if $x \in O \smallsetminus M$, then $x^2 + 1 \not\equiv 0 \mod M$. Thus, O/M is not algebraically closed. Now use the inequality $efg \le [\tilde{K}^* : K^*]$ for extensions of valuations [Rbn, p. 228] to conclude that v has a unique extension to $\tilde{K}^*$. In other words, (K^*, v) is Henselian.

Claim: *The v-topology of K^* coincides with the $|\;|^*$-topology.* It suffices to prove that each v-open neighborhood of 0 in K^* is contained in an $|\;|^*$-open neighborhood of K^* and vica versa.

Indeed, if $a \in K^*$, $a \ne 0$, then $v(x) > v(a)$ implies that $v\big(\frac{x}{a}\big) > 0$, hence $\frac{x}{a} \in M$, hence $|\frac{x}{a}| < 1$ and therefore $|x| < |a|$.

Conversely, choose $b \in M$, $b \ne 0$. If $|x| < |a|$, then $|\frac{x}{a}| < 1$, hence $|\frac{xb}{a}| < |b| < \frac{1}{n}$ for all $n \in \mathbb{N}$. It follows that $\frac{xb}{a} \in M$ and therefore $v(x) > v\big(\frac{a}{b}\big)$. □

As usual, each of the above field topologies induce compatible topologies on the sets of rational points of varieties [Mum, p. 81]. Recall that rational functions $f_1, \ldots, f_r$ of a variety V of dimension r are **local parameters** at a simple point $\mathbf{a}$ of V if they generate the maximal ideal of the local ring $O_{V,\mathbf{a}}$.

Proposition 8.2. *Let $(K, \mathfrak{p})$ be a generalized local field. Let V be a variety over K and let $\mathbf{a} \in V_{\mathrm{simp}}(K)$.*

(a) *Local parameters: Let $t_1, \ldots, t_r$ be a system of local parameters of V at $\mathbf{a}$. View $\mathtt{t} = (t_1, \ldots, t_r)$ as a rational map of V into $\mathbb{A}^r$. Then $\mathbf{a}$ has a $\mathfrak{p}$-open neighborhood $\mathcal{V}$ in $V(K)$ which $\mathtt{t}$ maps $\mathfrak{p}$-homeomorphically onto a $\mathfrak{p}$-open neighborhood of 0 in K^r.*

(b) *Density theorem: Each $\mathfrak{p}$-open neighborhood of $\mathbf{a}$ in $V(K)$ is Zariski-dense in V. In particular, $V(K)$ is Zariski-dense in V.*

(c) *Open map theorem: Let $\varphi\colon V \to W$ be a dominant morphism of varieties over K. Suppose that φ is smooth at a simple point $\mathbf{a}$ of $V(K)$. Then $\mathbf{a}$ has a $\mathfrak{p}$-open neighborhood $\mathcal{V}$ in $V(K)$ such that $\varphi|_{\mathcal{V}}$ is a $\mathfrak{p}$-open map onto a $\mathfrak{p}$-open subset $\mathcal{W}$ of $W(K)$. In particular, if $\dim(V) = \dim(W)$, then $\mathcal{V}$ and $\mathcal{W}$ can be chosen such that φ maps $\mathcal{V}$ $\mathfrak{p}$-homeomorphically onto $\mathcal{W}$.*

Proof. As all parts of the theorem are of Zariski-local nature, we may assume that both V and W are affine.

Proof of (a)**:** The case where $(K, \mathfrak{p})$ is Henselian is [GPR, Thm. 9.2]. So, suppose that $(K, \mathfrak{p})$ is a generalized metric absolute valued field.

Take a non-principal ultrapower $(K^*, |\ |^*)$ of $(K, |\ |)$. By Remark 8.1, K^* has a valuation v such that the v-topology of K^* coincides with the $|\ |^*$-topology of K^*. Moreover, K^* is Henselian with respect to v.

The point $\mathbf{a}$ belongs also to $V_{\mathrm{simp}}(K^*)$ and we can view $t_1, \ldots, t_r$ as local parameters of V over K^* (i.e., of $V \times_K K^*$) at $\mathbf{a}$. By the Henselian case, $\mathbf{a}$ has a v-open neighborhood $\mathcal{V}$ in $V(K^*)$ which $\mathbf{t}$ maps v-homeomorphically onto a v-open neighborhood $\mathcal{W}$ of 0 in $(K^*)^r$. Since the $|\ |^*$-topology of K^* coincides with its v-topology, $\mathcal{V}$ and $\mathcal{W}$ are also $|\ |^*$-open sets and $\mathbf{t}$ is an $|\ |^*$-homeomorphism. In other words, the following statement holds:

(2) There exists $\varepsilon > 0$ such that $\mathbf{t}$ maps the set $\mathcal{E} = \{\mathbf{x} \in V(K^*) \mid |\mathbf{x} - \mathbf{a}|^* <^* \varepsilon\}$ injectively into $(K^*)^r$, $0 \in \mathbf{t}(\mathcal{E})$, and for each $\mathbf{a}' \in \mathcal{E}$ and for each $\varepsilon' > 0$ there exists $\delta > 0$ such that for all $\mathbf{y} \in (K^*)^r$ we have

$$\{\mathbf{y} \in (K^*)^r \mid |\mathbf{y} - \mathbf{t}(\mathbf{a}')|^* < \delta\} \subseteq \mathbf{t}\big(\{\mathbf{x} \in \mathcal{E} \mid |\mathbf{x} - \mathbf{a}'|^* < \varepsilon'\}\big).$$

Since $(K^*, |\ |^*)$ is an elementary extension of $(K, |\ |)$ and (2) is an elementary statement on K^* (in the language of ordered fields) with parameters in K, (2) is also true if we replace $(K^*, |\ |^*)$ by $(K, |\ |)$. Conclude that $\mathbf{a}$ has an $|\ |$-open neighborhood in $V(K)$ which $\mathbf{t}$ maps $|\ |$-homeomorphically onto an $|\ |$-open neighborhood of 0 in K^r.

Proof of (b)**:** Let $\mathcal{V}$ be a $\mathfrak{p}$-open neighborhood of $\mathbf{a}$ in $V(K)$. By (a) we can make $\mathcal{V}$ smaller, if necessary, to assume that $\mathbf{t}$ induces a $\mathfrak{p}$-homeomorphism of $\mathcal{V}$ onto a $\mathfrak{p}$-open neighborhood $\mathcal{W}$ of 0 in K^r. Suppose that V is a Zariski-closed subset of $\mathbb{A}^n$ and let $\mathbf{x} = (x_1, \ldots, x_n)$ be a generic point of V over K. Then $\mathbf{t}$ can be also considered as a separating transcendence base of $K(\mathbf{x})/K$.

Let now $g(\mathbf{x})$ be a nonzero element of $K[\mathbf{x}]$. Then $h(\mathbf{t}) = \mathrm{Norm}_{K(\mathbf{x})/K(\mathbf{t})} g(\mathbf{x})$ is a nonzero element of $K(\mathbf{t})$. Since K is infinite and $\mathcal{W}$ contains a $\mathfrak{p}$-ball, $h(\mathbf{t})$ does not vanish on $\mathcal{W}$. Hence, $g(\mathbf{x})$ does not vanish on $\mathcal{V}$. Conclude that $\mathcal{V}$ is Zariski-dense in V.

Proof of (c)**:** The case where $(K, \mathfrak{p})$ is Henselian is [GPR, Thm. 9.4]. Assume again that $(K, \mathfrak{p})$ is a generalized metric absolute valued field and, as in the proof of (a), consider a non-principal ultrapower $(K^*, |\ |^*)$ of $(K, |\ |)$.

We may consider $\varphi\colon V \to W$ as a morphism of the corresponding varieties over K^*. By the Henselian case, $\mathbf{a}$ has a v-open neighborhood $\mathcal{V}$ in $V(K^*)$ such that $\varphi|_{\mathcal{V}}$ is a v-open map into $W(K^*)$. Since φ is defined by polynomials, with coefficients in K, we may proceed as in (a) to conclude that φ induces an $|\ |$-open map of a $|\ |$-open neighborhood of $\mathbf{a}$ in $V(K)$ into $W(K)$. $\square$

Proposition 8.3. (Continuity of zeros of algebraic functions) *Let $\mathfrak{a}$ be a positive, Γ-smooth divisor of F/K. Let $(K, \mathfrak{p})$ be a generalized local field. Let Γ be a projective curve over K. Denote its function field (resp., semilocal ring of singularities) by F (resp., O). Let $\mathfrak{a}$ be a positive Γ-smooth divisor of F/K. Let f be an element of O such that* $\mathrm{div}(f) = \sum_{i=1}^m \mathbf{b}_i - \mathfrak{a}$, *where $\mathbf{b}_1, \ldots, \mathbf{b}_m$ are distinct points in* $\Gamma_{\mathrm{simp}}(K) \smallsetminus \mathrm{Supp}(\mathfrak{a})$.

For each i let $\mathcal{U}_i$ be a $\mathfrak{p}$-open neighborhood of $\mathbf{b}_i$ in $\Gamma_{\mathrm{simp}}(K)$. Let $u_0,\ldots,u_n$ be a K-basis of $\mathcal{L}_O(\mathfrak{a})$. Write $f=\sum_{j=0}^n a_j u_j$ with $a_j\in K$. Then $(a_0,\ldots,a_n)$ has a $\mathfrak{p}$-open neighborhood $\mathcal{A}$ in K^{n+1} such that for each $(a'_0,\ldots,a'_n)\in\mathcal{A}$ the function $f'=\sum_{j=0}^n a'_j u_j$ satisfies $\mathrm{div}(f')=\sum_{i=1}^m \mathbf{b}'_i-\mathfrak{a}$, where $\mathbf{b}'_i\in\mathcal{U}_i\smallsetminus\mathrm{Supp}(\mathfrak{a})$, $i=1,\ldots,m$. Moreover, $\mathbf{b}'_1,\ldots,\mathbf{b}'_m$ are distinct.

Proof. The case where $(K,\mathfrak{p})$ is Henselian follows from [GPR, Cor. 7.2]. So, suppose that $(K,\mathfrak{p})=(K,|\ |)$ is a generalized metric absolute valued field. Let $(K^*,\mathfrak{p}^*)$ and v be as in the second paragraph of the proof of Proposition 8.2(a). Shrink $\mathcal{U}_1,\ldots,\mathcal{U}_m$, if necessary, to assume that they are mutually disjoint and also disjoint from $\mathrm{Supp}(\mathfrak{a})$. Moreover, suppose that $\Gamma\subseteq\mathbb{P}^r$, choose a generic point $\mathbf{x}=(x_0{:}\ldots{:}x_r)$ for Γ over K with $x_0,\ldots,x_r\in F$ and choose homogeneous polynomials of the same degree $g_j,h_j\in K[X_0,\ldots,X_r]$ such that $h_j(\mathbf{x})\neq 0$ and $u_j=\frac{g_j(\mathbf{x})}{h_j(\mathbf{x})}$ for $j=0,\ldots,n$. Since none of the points $\mathbf{b}_1,\ldots,\mathbf{b}_m$ is a pole of u_j, we may choose h_j such that $h_j(\mathbf{b}_i)\neq 0$ for all i and j. Then shrink $\mathcal{U}_1,\ldots,\mathcal{U}_m$ further until $h_j(\mathbf{X})$ does not vanish on $\mathcal{U}_i$ for all i and j.

Note that OK^* is the semilocal ring of singularities of $\Gamma\times_K K^*$ and $u_0,\ldots,u_n$ is also a K^*-basis for $\mathcal{L}_{OK^*}(\mathfrak{a})$ (Lemma 2.3). Each of the sets $\mathcal{U}_i(K^*)$ is $\mathfrak{p}^*$-open and therefore also v-open. Hence, by the Henselian case, $(a_0,\ldots,a_n)$ has a v-open neighborhood $\mathcal{A}^*$ in $(K^*)^{n+1}$ such that for each $(a^*_0,\ldots,a^*_n)\in\mathcal{A}^*$ the element $f^*=\sum_{j=0}^n a^*_j u_j$ of FK^* has for each i exactly one zero $\mathbf{b}^*_i$ in each $\mathcal{U}_i(K^*)$. Thus $\sum_{j=0}^n a^*_j\frac{g_j(\mathbf{b}^*_i)}{h_j(\mathbf{b}^*_i)}=f^*(\mathbf{b}^*_i)=0$.

Since the v-topology coincides with the $\mathfrak{p}^*$-topology, we may assume that $\mathcal{A}^*$ is a basic $\mathfrak{p}^*$-open neighborhood of $(a_0,\ldots,a_n)$ in $(K^*)^{n+1}$. As $(K^*,\mathfrak{p}^*)$ is an elementary extension of $(K,\mathfrak{p})$, the point $(a_0,\ldots,a_n)$ has a $\mathfrak{p}^*$-open neighborhood $\mathcal{A}$ in K^{n+1} such that if $(a'_0,\ldots,a'_n)\in\mathcal{A}$ and $f'=\sum_{j=0}^n a'_j u_j$, then for each i there exists a unique $\mathbf{b}'_i\in\mathcal{U}_i$ such that $f'(\mathbf{b}'_i)=\sum_{j=0}^n a'_j\frac{g_j(\mathbf{b}'_i)}{h_j(\mathbf{b}'_i)}=0$. Thus $\sum_{i=1}^m\mathbf{b}'_i\le\mathrm{div}_0(f')$. Since $u_j\in\mathcal{L}_O(\mathfrak{a})$, $\mathrm{div}_\infty(f')\le\mathfrak{a}$. By assumption, $\deg(\mathfrak{a})=\deg(\mathrm{div}_\infty(f))=\deg(\mathrm{div}_0(f))=m$. It follows that $m\le\deg(\mathrm{div}_0(f'))=\deg(\mathrm{div}_\infty(f'))\le\deg(\mathfrak{a})=m$. Hence, $\mathrm{div}(f')=\sum_{i=1}^m\mathbf{b}'_i-\mathfrak{a}$, as desired. □

References

[Art] E. Artin, *Algebraic Numbers and Algebraic Functions,* Gordon and Breach, New York, 1967.

[Bou] N. Bourbaki, *Commutative Algebra, Chapters 1–7,* Springer, Berlin, 1989.

[CaF] J.W.S. Cassels and A. Fröhlich, *Algebraic Number Theory,* Academic Press, London, 1967.

[Che] C. Chevalley, *Introduction to the theory of algebraic functions of one variable,* Mathematical Surveys VI, AMS, Providence, 1951.

[Deu] M. Deuring, *Algebren,* Ergebnisse der Mathematik und ihrer Grenzgebiete **4**, Springer, Berlin, 1935.

[Eis] D. Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry, Graduate Texts in Mathematics **150**, Springer-Verlag, New York, 1995.

[FJ1] M. Fried and M. Jarden, *Diophantine properties of subfields of $\tilde{\mathbb{Q}}$,* American Journal of Mathematics **100** (1978), 653–666.

[FJ2] M. D. Fried and M. Jarden, *Field Arithmetic,* Ergebnisse der Mathematik (3) **11**, Springer, Heidelberg, 1986.

[Ful] W. Fulton, *Algebraic Curves,* Addison Weseley, Redwood City, 1989.

[GeJ] W.-D. Geyer and M. Jarden, *On stable fields in positive characteristic,* Geometria Dedicata **29** (1989), 335–375.

[GPR] B. Green, F. Pop, and P. Roquette, *On Rumely's local-global principle,* Jahresbericht der Deutschen Mathematiker-Vereinigung **97** (1995), 43–74.

[Har] D. Haran, *Hilbertian fields under separable algebraic extensions,* to appear in Inventiones Mathematicae.

[HJ1] D. Haran and M. Jarden, *The absolute Galois group of a pseudo p-adically closed field,* Journal für die reine und angewandte Mathematik **383** (1988), 147–206.

[HJ2] D. Haran and M. Jarden, *Regular split embedding problems over complete valued fields,* Forum Mathematicum **10** (1998), 329–351.

[HJ3] D. Haran and M. Jarden, *Regular split embedding problems over function fields of one variable over ample fields,* Journal of Algebra **208** (1998), 147-164.

[Hrt] R. Hartshorne, *Algebraic Geometry,* Graduate Texts in Mathematics **52**, Springer, New York, 1977.

[Ja1] M. Jarden, *Intersection of local algebraic extensions of a Hilbertian field (A. Barlotti et al., eds),* NATO ASI Series C **333** 343–405, Kluwer, Dordrecht, 1991.

[Ja2] M. Jarden, *Algebraic realization of p-adically projective groups,* Compositio Mathematica **79** (1991), 21–62.

[Ja3] M. Jarden, *The inverse Galois problem over formal power series fields,* Israel Journal of Mathematics **85** (1994), 263–275.

[Ja4] M. Jarden, *Large normal extensions of Hilbertian fields,* Mathematische Zeitschrift **224** (1997), 555-565.

[Jac] N. Jacobson, *Finite Dimensional Division Algebras over Fields,* Springer-Verlag, Berlin, 1996

[JR1] M. Jarden and A. Razon, *Pseudo algebraically closed fields over rings,* Israel Journal of Mathematics **86** (1994), 25–59.

[JR2] M. Jarden and A. Razon, *Rumely's local global principle for algebraic PSC fields over rings,* Transactions of AMS **350** (1998), 55–85.

[Kow] H. J. Kowalsky, *Zur topologischen Kennzeichnung von Körpern,* Mathematische Nachrichten **9** (1953), 261–320.

[Lan] S. Lang, *Introduction to algebraic geometry,* Interscience Publishers, New York, 1958.

[Mat1] H. Matsumura, *Commutative Algebra,* Benjamin, New York, 1970.

[Mat2] H. Matsumura, *Commutative ring theory,* Cambridge studies in advanced mathematics **8**, Cambridge University Press, Cambridge 1994.

[MoB] L. Moret-Bailly, *Groupes de Picard et problm̀es de Skolem II,* Annales Scientifiques de l'Ecole Normale Superieure **22** (1989), 181–194.

[Mum] D. Mumford, *The Red Book of Varieties and Schemes,* Lecture Notes in Mathematics **1358**, Springer, Berlin, 1988.

[Neu] K. Neumann, *Every finitely generated regular field extension has a stable transcendence base,* Israel Journal of Mathematics **104** (1998), 221–260.

[Po1] F. Pop, *Fields of totally Σ-adic numbers,* manuscript, Heidelberg, 1992.

[Po2] F. Pop, *Embedding problems over large fields,* Annals of Mathematics **144** (1996), 1–34.

[Po3] F. Pop, *The geometric case of a conjecture of Shafarevich, — $G_{\tilde{\kappa}(t)}$ is profinite free —,* preprint, Heidelberg, 1993.

[Pre] A. Prestel, *Pseudo real closed fields,* Set Theory and Model Theory, Lecture Notes **872**, (1981), 127–156, Springer.

[Ra1] A. Razon, *On the density property of PSC fields,* manuscript, Essen, 1998.

[Ra2] A. Razon, *Abundance of Hilbertian domains,* manuscripta mathematica **94** (1997), 531–542.

[Rbn] P. Ribenboim, *Théorie des valuations,* Les Presses de l'Université de Montréal, Montréal, 1964.

[Rib] L. Ribes, *Introduction to Profinite Groups and Galois Cohomology,* Queen's papers in Pure and Applied Mathematics **24**, Queen's University, Kingston, 1970.

[Ro1] M. Rosenlicht, *Equivalence relations on algebraic curves,* Annals of Mathematics **56** (1952), 169–191.

[Ro2] M. Rosenlicht, *Generalized Jacobian varieties,* Annals of Mathematics **59** (1954), 505–530.

[Ser] J.-P. Serre, *Algebraic Groups and Class Fields,* Graduate Texts in Mathematics **117**, Springer, New-York, 1988.

[Sha] I. R. Shafarevich, *Basic Algebraic Geometry,* Grundlehren der mathematischen Wissenschaften **213**, Springer Berlin, 1977.

[Wei] A. Weil, *Foundation of Algebraic Geometry,* American Mathematical Society, Providence, 1962.

[ZaS] O. Zariski and P. Samuel, *Commutative Algebra I,* Springer, New York, 1975.

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