PROSOLVABLE SUBGROUPS OF FREE PRODUCTS OF PROFINITE GROUPS

by

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Communications of Algebra **22** (1994), 1467–1494

Introduction

This note is a contribution to the foundation of the theory of pseudo p-adically closed fields as developed in [HJ2] and [J] as well as to the general theory of profinite groups, especially to the theory of free and projective groups with respect to appropriate families of subgroups as developed in [H].

Recall that a field K is said to be **PpC** (**pseudo p-adically closed**) if every absolutely irreducible variety V defined over K which has a simple $\bar{K}$-rational point for each p-adic closure of K has a K-rational point. Theorem 15.1 of [HJ2] asserts that the absolute Galois group $G = G(K)$ of K is **p-adically projective**. This means that the set of all subgroups H of G which are isomorphic to $\Gamma = G(\mathbb{Q}_p)$ is closed in the family of all subgroups of G and that every finite Γ-embedding problem $(\varphi\colon G \to A,\ \alpha\colon B \to A)$ is solvable. Here $(\varphi\colon G \to A,\ \alpha\colon B \to A)$ is a **finite Γ-embedding problem** if φ is a homomorphism, α is an epimorphism of finite groups, and for each subgroup H of G which is isomorphic to Γ there exists a homomorphism $\gamma\colon H \to B$ such that $\alpha \circ \gamma = \varphi$ on H. Conversely, if G is a p-adically projective group, then there exists a PpC field K such that $G(K) \cong G$ [HJ2, Thm. 15.4]. If in addition $\mathrm{rank}(G) \le \aleph_0$, then K can be chosen to be algebraic over $\mathbb{Q}$ [J, Cor. 9.4].

The proofs of both realization theorems of p-adically projective groups as absolute Galois groups of PpC fields depend on certain properties of the group Γ. These properties are formulated in [HJ2] as Assumption 3.1. Some ingredients of this assumption has been found to depend on the rest of this assumption. So Assumption 3.1 of [HJ2] has taken a simplified form in [J] as Assumption 1.5 which we reformulate as Assumption A below. Both versions of the assumption involve the auxiliary groups $D_{e,m} = \Gamma_1 * \cdots * \Gamma_e * \hat{F}_m$ (free product in the category of profinite groups), where Γ_i is an isomorphic copy of Γ and $\hat{F}_m$ is the free profinite group on m generators.

ASSUMPTION A: *The profinite group Γ satisfies the following conditions.*

(a) Γ *is finitely generated and nontrivial,*

(b) *for each e and m, if a subgroup H of $D_{e,m}$ is isomorphic to Γ, then H is conjugate to Γ_i for some i between 1 and e.*

(c) *the center of Γ is trivial, and*

(d) Γ *has a finite quotient* $\overline{\Gamma}$ *such that for each* e *and* m *and for each closed subgroup* H *of* $D_{e,m}$, *if* H *is a quotient of* Γ *and if* $\overline{\Gamma}$ *is a quotient of* H, *then* $H \cong \Gamma$.

The main goal of this note is to simplify Assumption A for a prosolvable group Γ by stating only inner properties of Γ without any reference to auxiliary larger groups:

THEOREM B: *The following conditions on a prosolvable group* Γ *imply Assumption A:*

(a) Γ *is finitely generated.*

(b) *The center of* Γ *is trivial.*

(c) *There exist distinct primes* p, q *such that* Γ_p *(resp.,* Γ_q*) is a torsion free nonfree pro-p (resp, pro-q) group (Here* Γ_p *is a p-Sylow subgroup of* Γ*.)*

(d) Γ *has a finite quotient* $\overline{\Gamma}$ *such that if a subgroup* H *of* Γ *is a quotient of* Γ *and* $\overline{\Gamma}$ *is a quotient of* H, *then* $H = \Gamma$.

We then apply local class field theory to prove that the absolute Galois group of a finite extension of $\mathbb{Q}_p$ satisfies conditions (a)-(d) of Theorem B and therefore also Assumption A. Note, that [HJ2] proves Assumption A for $G(\mathbb{Q}_p)$ by also indirectly using a result from global class field theory, namely, the injectivity of the restriction map of the Brauer group of an algebraic field into the product of the algebraic groups of its Henselizations.

We replace this ingredient of the proof by the injectivity of the restriction map of the second cohomology group of $\mathcal{X}$-projective group G into the direct product of the second cohomology groups of the subgroups in $\mathcal{X}$. Here $\mathcal{X}$ is a, so called, separated family of subgroups of G, and for G to be $\mathcal{X}$-projective means that each finite embedding problem for G which has a local solution for each $H \in \mathcal{X}$ has also a global solution (see Section 1 for a precise definition). In particular a p-adically projective group is projective with respect to the family of all closed subgroups which are isomorphic to $G(\mathbb{Q}_p)$. Also, a free product $G = G_1 * \cdots * G_e$ is projective with respect to the family of all conjugate to G_i, $i = 1, \ldots, e$. Moreover, by Haran's subgroup theorem [H, Thm. 5.1] each closed subgroup H of G is projective with respect to the family of the groups of the form $G_i^x \cap H$, where x ranges over G and $i = 1, \ldots, e$.

The main problem in the proof of Theorem B is to find a criterion under which a prosolvable subgroup H of a free product G as above is conjugate to a subgroup of a free factor G_i. Once we have such a criterion, we can apply it to prove Condition (b) of Assumption A. Condition (d) of Assumption A reduces then to (d) of Theorem B.

By Haran's subgroup theorem, H is an $\mathcal{X}$-projective group for an appropriate family of closed subgroups. It therefore makes sense to try to prove our criterion in the framework of $\mathcal{X}$-projective groups. There are two types of $\mathcal{X}$-projective groups G for which our criterion works:

(1a) G can be $\mathcal{X}$-embedded in a $\mathcal{Y}$-free group (see Section 2 for a definition) (in particular this holds if G is separable), or

(1b) $\mathcal{X}$ is closed in the Boolean space $\mathrm{Subg}(G)$ of closed subgroups of G.

THEOREM C: *Let G be an $\mathcal{X}$-projective group which satisfies condition (1). Let H be a closed prosolvable subgroup of G. Suppose that p, q are distinct primes and $C, C' \in \mathcal{X}$ such that $(C \cap H)_p$ contains an element of infinite order and $(C' \cap H)_q \neq 1$. Then $C = C'$ and $H \leq C$.*

ACKNOWLEDGEMENT: The most important case of Theorem C is when G is the free product of finitely many finitely generated prosolvable groups. This case is due to Florian Pop (oral communication). The author is indebted to Florian Pop for allowing him to incorporate the above central case into Theorem C. The author would also like to draw the attention of the reader to Florian's Pop treatment of relatively projective groups in [P].

1. Minimally generated nonsolvable groups

Let $x_1, \ldots, x_n$ be elements of a group G. We say that $x_1, \ldots, x_n$ **minimally generate** G if $G = \langle x_1^{a_1}, \ldots, x_n^{a_n} \rangle$ for all $a_1, \ldots, a_n \in G$. The goal of this section is to find for each pair p, q of prime numbers an nonsolvable finite group G which is minimally generated by an element of order p and an element of order q. This plays a crucial role in the proof of our main results.

The case where $\{p, q\} \neq \{2, 3\}$ is based on the general theory of permutation groups.

LEMMA 1.1* : *Let $p < q$ be primes such that $\{p, q\} \neq \{2, 3\}$. Let α, β be a p-cycle and q-cycle, respectively, in the symmetric group S_q. Then the subgroup $H = \langle \alpha, \beta \rangle$ of S_q is nonsolvable.*

Proof: As H contains β, it is a primitive group on the set $\Omega = \{1, 2, \ldots, q\}$. Suppose that $\alpha = (a_1\ a_2\ \cdots\ a_p)$. Let $\Gamma = \{a_1, a_2, \ldots, a_p\}$ and $\Delta = \Omega \smallsetminus \Gamma$. Then α belongs to the subgroup $H_\Delta = \{\sigma \in H \| \sigma x = x \text{ for each } x \in \Delta\}$ and H_Δ is primitive on Γ. By a theorem of Jordan from 1871 [W, Thm. 13.2], H is $(q-p+1)$-fold primitive and therefore $(q-p+1)$-fold transitive [W, p. 23]. In particular, since $q - p \geq 2$, H is triply transitive. Given distinct elements $a, b \in \Omega$ use the assumption $\{p, q\} \neq \{2, 3\}$ to choose two more elements $c, d \in \Omega$. Then there exists $\tau \in H$ such that $\tau(a, b, c) = (a, b, d)$. Hence $\tau \in H_{\{a,b\}}$ but $\tau \neq 1$. By a theorem of Galois [W, Thm. 11.6], H is nonsolvable.

Note that if $p = 2$, then $H = S_q$ [W, Thm. 13.3]. If $p < q - 2$, then, by another theorem of Jordan, from 1873 [W, Thm. 13.9], $H = A_q$. ∎

The case $\{p, q\} = \{2, 3\}$ unfortunately involves numerous computations with permutations of A_6. It is due to Dan Haran.

LEMMA 1.2: *Let $\alpha \in S_6$ be a 3-cycle and $\beta \in S_6$ a product of two disjoint 3-cycles, none of which is disjoint to α. Then $\langle \alpha, \beta \rangle = A_6$.*

Proof: Conjugate by an element of S_6, if necessary, to assume that $\beta = (152)(346)$ and either $\alpha = (123)$ or $\alpha = (132) = (123)^{-1}$. Assume without loss that the first option occurs. Observe that $\alpha\beta\alpha\beta\alpha\beta\alpha = (12)(3456)$. Hence, by [CM, p. 67],

$$A_6 = \big\langle (123), (12)(3456) \big\rangle \leq \langle \alpha, \beta \rangle \leq A_6,$$

whence the assertion. ∎

The group S_6 is generated by the elements $s_i = (i6)$, $i = 1, 2, 3, 4, 5$. It is also generated by $v_1 = s_1 = (1n)$ and $v_j = s_1 s_j = (1jn)$, $j = 2, 3, 4, 5$. In terms of the latter system of generators S_6 has the presentation

$$v_1^2 = v_j^3 = (v_i v_j)^2 = 1 \qquad 1 \leq i < j \leq 5 \tag{1}$$

* The proof of this lemma was communicated to the author by Luis Ribes and independently by Florian Pop

[CM, p. 64–65].

Define an automorphism ε of S_6 by the following action on the generators

$$(16)^\varepsilon = (16)(52)(34)$$

$$(26)^\varepsilon = (26)(13)(45)$$

$$(36)^\varepsilon = (36)(24)(51)$$

$$(46)^\varepsilon = (46)(35)(12)$$

$$(56)^\varepsilon = (56)(41)(23)$$

or, equivalently, by (notice that $(16)(j6) = (1j6)$ for $j = 2, 3, 4, 5$)

$$v_1^\varepsilon = (16)^\varepsilon = (16)(52)(34)$$

$$v_2^\varepsilon = (126)^\varepsilon = (124)(356)$$

$$v_3^\varepsilon = (136)^\varepsilon = (132)(546)$$

$$v_4^\varepsilon = (146)^\varepsilon = (145)(236)$$

$$v_5^\varepsilon = (156)^\varepsilon = (153)(426)$$

Then ε is well defined: By (1), it suffices to verify that the following elements are of order 2:

$$\big((124)(356)\big)\big((132)(546)\big) = (26)(34)$$

$$\big((124)(356)\big)\big((145)(236)\big) = (13)(25)$$

$$\big((124)(356)\big)\big((153)(426)\big) = (16)(45)$$

$$\big((132)(546)\big)\big((145)(236)\big) = (16)(24)$$

$$\big((132)(546)\big)\big((153)(426)\big) = (25)(36)$$

$$\big((145)(236)\big)\big((153)(426)\big) = (12)(34)$$

Furthermore, use the rule $(kl) = (l6)(k6)(l6)$ to check that ε is of order 2.

Also, $(124)(356) = (126)^\varepsilon \in A_6^\varepsilon$ and $(126) = \big((124)(356)\big)^\varepsilon \in A_6^\varepsilon$. Hence, by Lemma 1.2, $A_6 \leq A_6^\varepsilon$. Since both groups have the same order they are equal: $A_6^\varepsilon = A_6$.

Lemma 1.3: *Let G be the semidirect product of A_6 by $\langle\varepsilon\rangle$. Then G is nonsolvable and $\langle\varepsilon, (123)^g\rangle = G$ for each $g \in G$.*

Proof: We may assume that $g \in A_6$, otherwise replace g by $g\varepsilon$. Then $\alpha = (123)^g$ is a 3-cycle. As $\langle\alpha, \alpha^\varepsilon\rangle$ is a subgroup of $\langle\varepsilon, \alpha\rangle$, it suffices to show that $\langle\alpha, \alpha^\varepsilon\rangle = A_6$. By

Lemma 1.2, we have only to verify that α^ε is the product of two disjoint 3-cycles, none of which is disjoint to α. Obviously it suffices to consider for each pair (α, α^{-1}) either α or α^{-1}. So, the following list covers all the possibilities:

$$(123)^\varepsilon = (36)^\varepsilon(16)^\varepsilon(26)^\varepsilon = (136)(254)$$

$$(124)^\varepsilon = (46)^\varepsilon(16)^\varepsilon(26)^\varepsilon(46)^\varepsilon = (162)(345)$$

$$(125)^\varepsilon = (56)^\varepsilon(16)^\varepsilon(26)^\varepsilon(t6)^\varepsilon = (143)(265)$$

$$(126)^\varepsilon = (16)^\varepsilon(26)^\varepsilon = (124)(356)$$

$$(134)^\varepsilon = (46)^\varepsilon(16)^\varepsilon(36)^\varepsilon(46)^\varepsilon = (125)(364)$$

$$(135)^\varepsilon = (56)^\varepsilon(16)^\varepsilon(36)^\varepsilon(56)^\varepsilon = (156)(234)$$

$$(136)^\varepsilon = (16)^\varepsilon(36)^\varepsilon = (132)(546)$$

$$(145)^\varepsilon = (56)^\varepsilon(16)^\varepsilon(46)^\varepsilon(56)^\varepsilon = (164)(253)$$

$$(146)^\varepsilon = (16)^\varepsilon(46)^\varepsilon = (145)(236)$$

$$(156)^\varepsilon = (16)^\varepsilon(56)^\varepsilon = (153)(426)$$

$$(234)^\varepsilon = (46)^\varepsilon(26)^\varepsilon(36)^\varepsilon(46)^\varepsilon = (153)(246)$$

$$(235)^\varepsilon = (56)^\varepsilon(26)^\varepsilon(36)^\varepsilon(46)^\varepsilon = (145)(263)$$

$$(236)^\varepsilon = (26)^\varepsilon(36)^\varepsilon = (164)(235)$$

$$(245)^\varepsilon = (56)^\varepsilon(26)^\varepsilon(36)^\varepsilon(56)^\varepsilon = (123)(465)$$

$$(246)^\varepsilon = (26)^\varepsilon(46)^\varepsilon = (156)(324)$$

$$(256)^\varepsilon = (26)^\varepsilon(56)^\varepsilon = (125)(346)$$

$$(345)^\varepsilon = (56)^\varepsilon(36)^\varepsilon(46)^\varepsilon(56)^\varepsilon = (142)(356)$$

$$(346)^\varepsilon = (36)^\varepsilon(46)^\varepsilon = (134)(265)$$

$$(456)^\varepsilon = (46)^\varepsilon(56)^\varepsilon = (136)(245). \quad \blacksquare$$

Combine Lemma 1.2 with Lemma 1.3:

PROPOSITION 1.4: *Let $p < q$ be primes. Then there exists an nonsolvable finite group S and elements $a, b \in S$ with $\mathrm{ord}(a) = p$, $\mathrm{ord}(b) = q$ such that $S = \langle a^x, b^y \rangle$ for all $x, y \in S$.*

Proof: By Lemmas 1.1 and 1.3 there exists a finite group G and elements $c, d \in G$ with $\mathrm{ord}(c) = p$ and $\mathrm{ord}(d) = q$ such that $\langle c^x, d^y \rangle$ is nonsolvable for all $x, y \in G$. Among all pairs $(x, y) \in G^2$ choose one (x_0, y_0) such that the group S generated by $a = c^{x_0}$ and d^{y_0} is minimal. That is, $\langle c^x, d^y \rangle$ is a proper subgroup of S for no $(x, y) \in G^2$. Then a, b and S satisfy the condition required in the Proposition. ∎

2. $\mathcal{X}$-Projective groups

The concept of projective profinite group has been generalized in two directions to "real-projective group" and "Γ-projective group" in [HJ1] and [HJ2], respectively. Haran [H] generalizes both concepts to what he calls "$\mathcal{X}$-projective groups". Let us repeat his definition.

Let G be a profinite group and $\mathcal{X}$ a family of closed subgroups of G. Then $\mathcal{X}$ is **separated** if for all distinct $\Gamma_1, \Gamma_2 \in \mathcal{X}$

(1a) $\Gamma_1 \cap \Gamma_2 = 1$, and

(1b) there exist disjoint subfamilies $\mathcal{X}_1, \mathcal{X}_2$ such that $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, $\Gamma_i \in \mathcal{X}_i$, and $\bigcup_{\Gamma \in \mathcal{X}_i} \Gamma$ is closed in G, for $i = 1, 2$.

Let $\mathcal{X}$ be a separated family of closed subgroups of a profinite group G. A **finite $\mathcal{X}$-embedding problem for** G is a triple $(\varphi\colon G \to A,\ \pi\colon B \to A,\ \mathcal{B})$ such that

(2a) $\pi\colon B \to A$ is an epimorphism of finite groups,

(2b) $\varphi\colon G \to A$ is a homomorphism, and

(2c) $\mathcal{B}$ is a family of subgroups of B closed under inclusion and under conjugation such that

(2d) for each $\Gamma \in \mathcal{X}$ there is a homomorphism $\gamma_\Gamma\colon \Gamma \to B$ that satisfies $\pi \circ \gamma_\Gamma = \mathrm{res}_\Gamma \varphi$ and $\gamma_\Gamma(\Gamma) \in \mathcal{B}$.

A **solution** to this problem is a homomorphism $\gamma\colon G \to B$ such that $\pi \circ \gamma = \varphi$ and $\gamma(\mathcal{X}) \subseteq \mathcal{B}$.

Suppose now that $\mathcal{X}$ is also closed under conjugation in G. Then G is **$\mathcal{X}$-projective** if every finite $\mathcal{X}$-embedding problem for G has a solution.

Of fundamental importance is Haran's subgroup theorem:

Proposition 2.1 ([H, Prop. 5.1]): *Let G be an $\mathcal{X}$-projective group and let H be a closed subgroup. Then H is projective relative to the family $\{\Gamma \cap H \| \Gamma \in \mathcal{X}\}$.*

In the next sections we show that the following assumption on $(G, \mathcal{X})$ is satisfied in many cases.

If $G_1, \ldots, G_n$ are profinite groups, then we denote their free product in the category of profinite groups by $G_1 * \cdots * G_n$ and also by $\mathop{\ast}_{i=1}^{n} G_i$.

Assumption 2.2: *Let $\Gamma_1, \ldots, \Gamma_n \in \mathcal{X}$ be nonconjugate subgroups of G, A a finite group, $\psi_i \colon \Gamma_i \to A$ a homomorphism, and $A_i = \psi_i(\Gamma_i)$, $i = 1, \ldots, n$. Then there exists a homomorphism $\varphi \colon G \to A$ and elements $a_1, \ldots, a_n \in A$ such that*

(a) $\mathrm{res}_{\Gamma_i} \varphi = [a_i] \circ \psi_i$, $i = 1, \ldots, n$ *($[a_i]$ is the inner automorphism of A induced by a_i), and*

(b) *for each $\Gamma \in \mathcal{X}$ there exists i, $1 \le i \le n$, and $a \in A$ such that $\varphi(\Gamma) \le A_i^a$.*

For each profinite group G and a prime p we choose a p-Sylow subgroup G_p of G. As G_p is unique up to conjugation, the statements we will make about G_p will not depend on the choice of G_p. Denote the maximal pro-p quotient of G by $G(p)$. Denote the cyclic group of order n by C_n.

Proposition 2.3: *Let G be an $\mathcal{X}$-projective group which satisfies Assumption 2.2 and H a closed subgroup. Suppose that there exist nonconjugate subgroups $\Gamma_1, \Gamma_2 \in \mathcal{X}$ and distinct primes p, q such that $(\Gamma_1 \cap H)_p$ and $(\Gamma_2 \cap H)_q$ are nontrivial. Then H is not prosolvable.*

Proof: Choose an element $g \ne 1$ in $(\Gamma_1 \cap H)_p$. Choose an element $h \ne 1$ in $(\Gamma_2 \cap H)_q$. It suffices to prove that $H_0 = \langle g, h \rangle$ is not prosolvable. We do it in three parts.

Part A: *Mapping H_0 onto $C_p \times C_q$.* Choose an open normal subgroup N of G such that $g, h \notin N$. Let $\overline{\Gamma}_i = \Gamma_i N/N$, apply Assumption 2.2 to the canonical maps $\psi_i \colon \Gamma_i \to \overline{\Gamma}_1 \times \overline{\Gamma}_2$, to get a homomorphism $\psi \colon G \to \overline{\Gamma}_1 \times \overline{\Gamma}_2$ such that the restriction of ψ to Γ_i is conjugate to ψ_i, $i = 1, 2$, and for each $\Gamma \in \mathcal{X}$, $\psi(\Gamma)$ is conjugate to a subgroup of $\overline{\Gamma}_1$ or of $\overline{\Gamma}_2$. As $\overline{\Gamma}_i$ is normal in $\overline{\Gamma}_1 \times \overline{\Gamma}_2$, $\psi(\Gamma_i) = \overline{\Gamma}_i$ for $i = 1, 2$, and $\psi(\Gamma)$ is contained in $\overline{\Gamma}_1$ or in $\overline{\Gamma}_2$ for each $\Gamma \in \mathcal{X}$. Denote the restriction of ψ to H_0 by φ_1. In particular

$\bar{g} = \varphi_1(g) \neq 1$ and $\bar{h} = \varphi_1(h) \neq 1$ commute, the order of $\bar{g}$ is a p-power, and the order of $\bar{h}$ is a q-power. So $\varphi_1(H_0) = \langle \bar{g} \rangle \times \langle \bar{h} \rangle$.

Let c_p be a generator of C_p. Let c_q be a generator of C_q. Extend the map $(\bar{g}, \bar{h}) \to (c_p, c_q)$ to a homomorphism φ_2: $\langle \bar{g} \rangle \times \langle \bar{h} \rangle \to C_p \times C_q$. Let $\varphi = \varphi_2 \circ \varphi_1$.

$$\begin{tikzcd}
H_0 \arrow[r] \arrow[d, "\varphi_1"'] & G \arrow[d, "\psi"'] \\
\langle \bar{g} \rangle \times \langle \bar{h} \rangle \arrow[r] \arrow[d, "\varphi_2"'] & \bar{\Gamma}_1 \times \bar{\Gamma}_2 \\
C_p \times C_q &
\end{tikzcd}$$

If $\Gamma \in \mathcal{X}$, then $\varphi_1(H_0 \cap \Gamma) \leq (\langle \bar{g} \rangle \times \langle \bar{h} \rangle) \cap \psi(\Gamma)$ and the latter group is contained in $\langle \bar{g} \rangle$ or in $\langle \bar{h} \rangle$. Denote the family of all subgroups of C_p and of C_q by $\overline{\mathcal{B}}$. Then

(1) for each $\Gamma \in \mathcal{X}$ we have $\varphi(H_0 \cap \Gamma) \in \overline{\mathcal{B}}$.

PART B: *An embedding problem for H_0.* By Proposition 2.1, H_0 is projective relative to $\mathcal{Y} = \{\Gamma \cap H_0 \| \Gamma \in \mathcal{X}\}$.

Consider the free product $C_p * C_q$ and construct an epimorphism

$$\pi\colon C_p * C_q \to C_p \times C_q$$

with kernel V by defining the restriction of π to C_p and to C_q as the identity maps. Consider the nonsolvable group S given by Proposition 1.4. Let a, b be its minimal generators of orders p and q, respectively. Construct a homomorphism ρ: $C_p * C_q \to S$ with kernel U by defining $\rho(c_p) = a$ and $\rho(c_q) = b$.

Denote the image of C_p (resp., C_q) in $(C_p * C_q)/(U \cap V)$ by $\bar{C}_p$ (resp., $\bar{C}_q$). Similarly, denote the image of c_p (resp., c_q) in $(C_p * C_q)/(U \cap V)$ by $\bar{c}_p$ (resp., $\bar{c}_q$). Let $\mathcal{B}$ be the family of all subgroups of $(C_p * C_q)/(U \cap V)$ which are conjugate to a subgroup of $\bar{C}_p$ or of $\bar{C}_q$. Let $\bar{\pi}$: $(C_p * C_q)/(U \cap V) \to C_p \times C_q$ be the homomorphism induced by π. Since $\bar{\pi}$ maps $\bar{C}_p$ (resp., $\bar{C}_q$) bijectively onto C_p (resp., C_q) it follows from (1) that for each $\Gamma \in \mathcal{X}$ there exists a commutative diagram of homomorphisms

$$\begin{tikzcd} & & H_0 \cap \Gamma \arrow[dll, "\gamma"'] \arrow[d, "\varphi"] & \\ (C_p * C_q)/(U \cap V) \arrow[rr, "\bar{\pi}"] & & C_p \times C_q \arrow[r] & 1 \end{tikzcd}$$

such that $\gamma(H_0 \cap \Gamma) \in \mathcal{B}$. Conclude that the triple

$$(\varphi \colon H_0 \to C_p \times C_q,\ \bar{\pi} \colon (C_p * C_q)/(U \cap V) \to C_p \times C_q,\ \mathcal{B})$$

is a finite $\mathcal{Y}$-embedding problem for H_0. As H_0 is $\mathcal{Y}$-projective, there exists a homomorphism $\beta \colon H_0 \to (C_p * C_q)/(U \cap V)$ such that $\bar{\pi} \circ \beta = \varphi$ and $\beta(\Gamma \cap H_0) \in \mathcal{B}$ for each $\Gamma \in \mathcal{X}$.

In particular, as $g \in H_0 \cap \Gamma_1$, $\mathrm{ord}(g)$ is a nontrivial p-power, and $p \neq q$, $\langle \beta(g) \rangle$ is conjugate to a subgroup of $\bar{C}_p$. But $\bar{\pi}\beta(g) = \varphi(g) = c_p$ has the same order as $\bar{c}_p$. Hence, $\langle \beta(g) \rangle$ is conjugate to $\bar{C}_p$. Similarly $\langle \beta(h) \rangle$ is conjugate to $\bar{C}_q$.

Part C: *An epimorphism of H_0 onto S.* The map ρ defines a homomorphism $\bar{\rho} \colon (C_p * C_q)/(U \cap V) \to S$ such that $\bar{\rho}(\bar{c}_p) = a$ and $\bar{\rho}(\bar{c}_q) = b$. By Part B, $\langle \bar{\rho} \circ \beta(g) \rangle$ is conjugate to $\langle a \rangle$ and $\langle \bar{\rho} \circ \beta(h) \rangle$ is conjugate to $\langle b \rangle$. Hence, the image of H_0 by $\bar{\rho} \circ \beta$ is generated by conjugates a', b' of a, b, respectively. So, it is the nonsolvable group S. Conclude that H_0 is not prosolvable. ∎

3. $\mathcal{X}$-projective subgroups of free products

We don't know if every $\mathcal{X}$-projective group G satisfies Assumption 2.2. In this section we show however that this is the case if G is isomorphic to a closed subgroup of a free product in the sense of Haran [H].

While free products of finitely many profinite groups is a well defined profinite group there are several definitions for free products of infinitely many profinite group (Neukirch [N1], Gildenhuys and Ribes [GR], Haran [H], and Melnikov [M]). We use Haran's definition (which is equivalent to Melnikov's but more general than the others):

Let $\mathcal{X}$ be a separated family of closed subgroups of a profinite group F. Then F is a **free $\mathcal{X}$-product** if each continuous map ψ of $\bigcup_{\Gamma \in \mathcal{X}} \Gamma$ into a profinite group A whose restriction to each $\Gamma \in \mathcal{X}$ is a homomorphism uniquely extends to a homomorphism $\psi\colon F \to A$.

If $\mathcal{X}$ is a finite set, then each family of homomorphisms $\gamma_\Gamma\colon \Gamma \to A$, with Γ ranges over $\mathcal{X}$, uniquely defines a homomorphism $\gamma\colon F \to A$ whose restriction to each $\Gamma \in \mathcal{X}$ coincides with γ_Γ. Thus, F is the usual free product of the groups in $\mathcal{X}$.

To generalize this statement to the infinite case, we use Haran's other variant of free product.

With each Boolean space E we associate the Boolean space $\exp(E)$ of all closed subsets of E and the Boolean space

$$G(E) = \exp(E) \times \exp(E \times E \times E) \times \exp(E \times E) \times E.$$

An **Etale space** is a pair (E, X), where E is a Boolean space and X is a family of profinite groups contained in E such that

(1a) E is the disjoint union of all $\Gamma \in X$, and

(1b) $X' = \bigcup_{\Gamma \in X} \{\Gamma' \in G(E) \| \Gamma' \leq \Gamma\}$ is closed in $G(E)$. Here each $\Gamma \in X$ is considered as a 4-tuple (Γ, M, I, e), where $M = \{(a, b, c) \in \Gamma \times \Gamma \times \Gamma \| ab = c\}$, $I = \{(a, a') \in \Gamma \times \Gamma \| a' = a^{-1}\}$, and e is the unit of Γ, and thus as a closed subset of $G(E)$.

A **morphism** $\psi\colon (E, X) \to A$ of an etale space (E, X) into a profinite group A is a continuous map $\psi\colon E \to A$ whose restriction to each $\Gamma \in X$ is a homomorphism. In particular $\psi(\Gamma)$ is a subgroup of A. A **free product** over an etale space (E, X) is a profinite group F together with a morphism $\varphi\colon (E, X) \to F$ such that for each morphism $\psi\colon (E, X) \to A$ into a profinite group A there exists a unique homomorphism $\alpha\colon F \to A$ such that $\alpha \circ \varphi = \psi$.

Note that if $\mathcal{X}$ is separated, then $\mathcal{Y} = \mathcal{X} \cup \{1\}$ is also separated, if F is a free $\mathcal{X}$-product, then F is $\mathcal{Y}$-free, and if F is a free product over (E, X), then F is also a free product over the space $(E \mathbin{\dot\cup} \{1\}, X \mathbin{\dot\cup} \{1\})$. So from now on we tacitly assume that each separated family $\mathcal{X}$ contains the trivial subgroup and in each etale space (E, X), X contains the trivial subgroup.

A **morphism** $\varphi\colon (E, X) \to (E', X')$ between etale spaces is a continuous map $\varphi\colon E \to E'$ such that the restriction of φ to each $\Gamma \in X$ is a homomorphism into some group $\Gamma' \in X'$. If $\varphi(E) = E'$ and for each $\Gamma' \in X'$ there exists $\Gamma \in X$ such that $\varphi(\Gamma) = \Gamma'$, then φ is an **epimorphism**.

Similarly let $\mathcal{X}$ (resp., $\mathcal{X}'$) be a separated family of closed subgroups of a profinite group F (resp., F'). Then a morphism of the pair $(F, \mathcal{X})$ into the pair $(F', \mathcal{X}')$ is a homomorphism φ: $F \to F'$ which maps each $\Gamma \in \mathcal{X}$ into some $\Gamma' \in \mathcal{X}'$. It is an **epimorphism** if $\varphi(F) = F'$ and if for each $\Gamma' \in \mathcal{X}'$ there exists $\Gamma \in \mathcal{X}$ such that $\varphi(\Gamma) = \Gamma'$.

LEMMA 3.1: *Let F be an a free $\mathcal{X}$-product, H an open subgroup, $\Gamma_1, \ldots, \Gamma_n \in \mathcal{X}$ distinct subgroups of F, and α_i: $\Gamma_i \to A_i$ an epimorphism onto a finite group A_i such that* $\mathrm{Ker}(\alpha_i) \le H$, $i = 1, \ldots, n$. *Then there exists an epimorphism α: $(F, \mathcal{X}) \to (\bar{F}, \overline{\mathcal{X}})$ such that* $\mathrm{Ker}(\alpha) \le H$, *$\overline{\mathcal{X}}$ is a finite family of finite groups, $\bar{F}$ is $\overline{\mathcal{X}}$-free, $A_i \in \overline{\mathcal{X}}$, and α coincides with α_i on Γ_i for $i = 1, \ldots, n$.*

Proof: Assume without loss that H is normal in F and let η: $F \to F/H$ be the canonical homomorphism. By [H, Prop. 3.7 and Lemma 3.6] there is an etale space (E, X) and a free product φ: $(E, X) \to F$ such that $\mathcal{X} \subseteq X$, $\varphi(X) = \mathcal{X}$, and the restriction of φ to each $\Gamma \in \mathcal{X}$ is the identity map. Consider the map π: $E \to X$ defined by $\pi(x) = \Gamma$ whenever $x \in \Gamma$ and $\Gamma \in X$. Equip X with the topology defined by π. Thus, a subset U of X is open if and only if $\pi^{-1}(U)$ is open in E. This makes X a Boolean space [H, Lemma 1.5].

For each i between 1 and n of [H, Lemma 1.10(a)] extends α_i to a morphism α_i'': $(E, X) \to A_i$. Let $\bar{\psi} = \eta \circ \varphi$. Since $\mathrm{Ker}(\alpha_i) \le H$, there is a homomorphism β_i: $A_i \to \Gamma_i H/H$ such that $\beta_i \circ \alpha_i = \eta$ on Γ_i. Thus $\bar{\psi}$ and $\beta_i \circ \alpha_i''$ are morphisms from (E, X) to F/H which coincide on Γ_i. By [H, Lemma 1.10], Γ_i has an open-closed neighborhood X_i in X such that with $E_i = \pi^{-1}(X_i)$, we have $\mathrm{res}_{E_i}(\beta_i \circ \alpha_i'') = \mathrm{res}_{E_i}\bar{\psi}$. Thus $\alpha_i' = \mathrm{res}_{E_i}(\alpha_i'')$: $(E_i, X_i) \to A_i$ is a morphism such that $\beta_i \circ \alpha_i' = \eta \circ \varphi$ on E_i. In particular, $\Gamma_i \subseteq E_i$ and $\mathrm{res}_{E_i}\alpha_i' = \alpha_i$.

Making each X_i smaller, if necessary, may further assume that $X_1, \ldots, X_n$ and therefore also $E_1, \ldots, E_n$ are disjoint.

Let $X_0 = X \smallsetminus X_1 \cup \cdots \cup X_n$ and $E_0 = E \smallsetminus E_1 \cup \cdots \cup E_n$. Then (E_0, X_0) is an etale space. By [H, Prop. 1.11] there is an epimorphism α_0' of (E_0, X_0) onto a finite etale space $(\bar{E}_0, \bar{X}_0)$ such that the partition $E_0 = \dot{\bigcup}_{\bar{e} \in \overline{E}_0} (\alpha_0')^{-1}(\bar{e})$ of E_0 is finer than the partition $E_0 = \dot{\bigcup}_{x \in R} (E_0 \cap \varphi^{-1}(xH))$, where R is a system of representatives of the left cosets of F modulo H. In particular φ induces a map $\bar{\psi}_0$: $(\bar{E}_0, \bar{X}_0) \to F/H$ such that $\bar{\psi}_0 \circ \alpha_0' = \eta \circ \varphi$ on E_0. Necessarily, $\bar{\psi}_0$ is a morphism. Let $\bar{F}_0$ be the free product of the finitely many finite groups in $\bar{X}_0$ and let $\bar{\varphi}_0$: $(\bar{E}_0, \bar{X}_0) \to \bar{F}_0$ be the corresponding morphism. Then there is a unique homomorphism β_0: $\bar{F}_0 \to F/H$ such that $\beta_0 \circ \bar{\varphi}_0 = \bar{\psi}_0$.

Now let $\bar{E} = \bar{E}_0 \cup A_1 \cup \cdots \cup A_n$ and $\bar{X} = \bar{X}_0 \cup \{A_1, \ldots, A_n\}$. Then $(\bar{E}, \bar{X})$ is a finite etale space and the maps α_i', $i = 0, 1, \ldots, n$, combine to an epimorphism α': $(E, X) \to (\bar{E}, \bar{X})$.

Construct the free product $\bar{F} = \bar{F}_0 * A_1 * \cdots * A_n$. Then let $\bar{\varphi}$: $(\bar{E}, \bar{X}) \to \bar{F}$ be the unique morphism whose restriction to $\bar{E}_0$ is $\bar{\varphi}_0$ and to A_i is the identity map, $i = 1, \ldots, n$. By the universal property of F there is a homomorphism α: $F \to \bar{F}$ such that $\bar{\varphi} \circ \alpha' = \alpha \circ \varphi$. In particular $\alpha = \alpha_i$ on Γ_i, for $i = 1, \ldots, n$, and α is surjective.

Finally let $\bar{\psi}$: $(\bar{E}, \bar{X}) \to F/H$ be the unique morphism whose restriction to $\bar{E}_0$ is $\bar{\psi}_0$ and to A_i is β_i, for $i = 1, \ldots, n$. Then there exists a unique homomorphism β: $\bar{F} \to F/H$ such that $\beta \circ \bar{\varphi} = \bar{\psi}$. Thus $\beta = \beta_0$ on $\bar{F}_0$ and $\beta = \beta_i$ on A_i, $i = 1, \ldots, n$.

$$\begin{tikzcd}
(E, X) \arrow[r, "\varphi"] \arrow[d, "\alpha'"'] & F \arrow[d, "\alpha"] \\
(\bar{E}, \bar{X}) \arrow[r, "\bar{\varphi}"] \arrow[dr, "\bar{\psi}"'] & \bar{F} \arrow[d, "\beta"] \\
& F/H
\end{tikzcd}$$

We claim that $\beta \circ \alpha = \eta$. Indeed, on E_0 we have: $\beta \circ \alpha \circ \varphi = \beta_0 \circ \bar{\varphi}_0 \circ \alpha_0' = \bar{\psi}_0 \circ \alpha_0' = \eta \circ \varphi$. For $i \geq 1$ we have on E_i: $\beta \circ \alpha \circ \varphi = \beta \circ \bar{\varphi} \circ \alpha_i' = \beta \circ \alpha_i' = \eta \circ \varphi$. Thus $\beta \circ \alpha \circ \varphi = \eta \circ \varphi$ on all E. Since $\varphi(E)$ generates F, [H, p. 272] this implies $\beta \circ \alpha = \eta$. Conclude that $\mathrm{Ker}(\alpha) \leq H$, as desired. ∎

Lemma 3.2: *Let F be an $\mathcal{X}$-projective group, G a closed subgroup and A, B groups in $\mathcal{X}$ such that $A \cap G$ is not conjugate in G to $B \cap G$. Then F has an open subgroup F_0 which contains G such that for every open subgroup E of F_0 which contains G the groups $A \cap E$ and $B \cap E$ are not conjugate in E.*

Proof: As $A \cap G$ and $B \cap G$ are nonconjugate at least one of them is not trivial. Assume therefore without loss that so is the other one.

For each closed subgroup E_0 of F which contains G consider the continuous map $f\colon E \to \mathrm{Subg}(E)$ defined by $f(x) = (A \cap E)^x$. Then $S(E) = f^{-1}(B \cap E) = \{x \in E \| (A \cap E)^x = B \cap E\}$ is a closed subset of E.

If $E \le H \le F$ and $x \in S(E)$, then $A^x \cap E = B \cap E$, hence $A^x \cap B$ contains $B \cap G$ and therefore it is nontrivial. Since $\mathcal{X}$ is separated $A^x = B$. Thus $A^x \cap H = B \cap H$ and therefore $x \in S(H)$. Conclude that $S(E) \subseteq S(H)$.

If the lemma were false, then there would exist a family $\{E_i \| i \in I\}$ of open subgroups of F which contain G such that every open subgroup of F which contains G contains E_i for some $i \in I$ and $S(E_i) \neq \emptyset$ for each $i \in I$.

By compactness $S(G) = \bigcap_{i \in I} S(E_i) \neq \emptyset$. This contradiction proves that the lemma is true. ∎

Proposition 3.3: *Let F be a free $\mathcal{X}$-product, G be a closed subgroup, and $\mathcal{Y} = \{\Gamma^x \cap G \| \Gamma \in \mathcal{X},\ x \in F\}$. Then G is $\mathcal{Y}$-projective and satisfies Assumption 2.2.*

Proof: By [H, Prop. 4.3], F is $\mathcal{X}^F$-projective. Hence, by Proposition 2.1, G is $\mathcal{Y}$-projective.

We prove that $(G, \mathcal{Y})$ satisfies Assumption 2.2:

(2) Let $\Delta_1, \ldots, \Delta_n \in \mathcal{Y}$ be nonconjugate subgroups of G, A a finite group, $\psi_i\colon \Delta_i \to A$ a homomorphism, and $A_i = \psi_i(\Delta_i)$, $i = 1, \ldots, n$. Then there exists a homomorphism $\varphi\colon G \to A$ and elements $a_1, \ldots, a_n \in A$ such that

  (2a) $\mathrm{res}_{\Delta_i} \varphi = [a_i] \circ \psi_i$, $i = 1, \ldots, n$ ($[a_i]$ is the inner automorphism of A induced by a_i), and

  (2b) for each $\Delta \in \mathcal{Y}$ there exists i, $1 \le i \le n$, and $a \in A$ such that $\varphi(\Delta) \le A_i^a$.

Indeed, $\Delta_i = \Gamma_{j(i)}^{x_{j(i)}} \cap G$, with $\Gamma_{j(i)} \in \mathcal{X}$ and $x_{j(i)} \in F$. The proof of (2) naturally brakes up now into two parts.

Part A: *G is open in F.* Take an open normal subgroup H of F which is contained in G such that $\Gamma_{j(i)}^{x_{j(i)}} \cap H \le \mathrm{Ker}(\psi_i)$, $i = 1, \dots, n$ and $\Gamma_{j(1)}^{x_{j(1)}} \cap G, \dots, \Gamma_{j(n)}^{x_{j(n)}} \cap G$ are non-conjugate in G modulo H. Apply Lemma 2.1 to the canonical maps $\Gamma_{j(i)} \to \Gamma_{j(i)}H/H$, $i = 1, \dots, n$ to get epimorphism α: $(F, \mathcal{X}) \to (\bar{F}, \overline{\mathcal{X}})$, where $\overline{\mathcal{X}}$ is a finite family of finite groups and $\bar{F}$ is an $\overline{\mathcal{X}}$-free product such that $N = \mathrm{Ker}(\alpha)$ is contained in H and $\alpha(\Gamma_{j(i)}) \in \overline{\mathcal{X}}$, $i = 1, \dots, n$. Put a bar on each element (resp., subgroup) of F to denote its image under α. Thus $\overline{\mathcal{X}} = \{\bar{\Gamma}_1, \dots, \bar{\Gamma}_q\}$, with $\Gamma_1, \dots, \Gamma_q \in \mathcal{X}$, and $\bar{F} = \bar{\Gamma}_1 * \cdots * \bar{\Gamma}_q$. Also, $\bar{G}$ is an open subgroup of $\bar{F}$. By Kurosh's subgroup theorem [BNW, p. 105]

$$\bar{G} = \coprod_{j=1}^{q}{}^{*} \coprod_{k \in K_j}{}^{*} (\bar{\Gamma}_j^{\bar{y}_{jk}} \cap \bar{G}) * \hat{F}_m,$$

where $\hat{F}_m$ is a free profinite group on m elements, and for each j, K_j is a finite set and y_{jk} are elements of F which give a double class decomposition of $\bar{F}$:

$$\bar{F} = \dot{\bigcup_{k \in K_j}} \bar{\Gamma}_j \bar{y}_{jk} \bar{G}.$$

As $N \le G$, this gives also a double class decomposition of F:

$$F = \dot{\bigcup_{k \in K_j}} \Gamma_j y_{jk} G.$$

For each i, $1 \le i \le n$, there exist $c_i \in \Gamma_{j(i)}$, $k(i) \in K_{j(i)}$, and $g_i \in G$ such that $x_{j(i)} = c_i y_{j(i),k(i)} g_i$. Then $\Gamma_{j(i)}^{x_{j(i)}} \cap G = (\Gamma_{j(i)}^{y_{j(i),k(i)}} \cap G)^{g_i}$ and $\bar{\Gamma}_{j(i)}^{\bar{x}_{j(i)}} \cap \bar{G} = (\bar{\Gamma}_{j(i)}^{\bar{y}_{j(i),k(i)}} \cap \bar{G})^{\bar{g}_i}$. Since $\Gamma_{j(i)}^{x_{j(i)}} \cap H \le \mathrm{Ker}(\psi_i)$, there is an epimorphism $\bar{\psi}_i$: $\bar{\Gamma}_{j(i)}^{\bar{x}_{j(i)}} \cap \bar{G} \to A_i$ such that $\bar{\psi}_i \circ \alpha = \psi_i$ on $\Gamma_{j(i)}^{x_{j(i)}} \cap G$. Then, $\bar{\varphi}_i = \bar{\psi}_i \circ [\bar{g}_i]$: $\bar{\Gamma}_{j(i)}^{\bar{y}_{j(i),k(i)}} \cap \bar{G} \to A_i$ is an epimorphism which satisfies $\bar{\varphi}_i \circ \alpha = \psi_i \circ [g_i]$ on $\Gamma_{j(i)}^{y_{j(i),k(i)}} \cap G$.

$$\begin{tikzcd}
\Gamma_{j(i)}^{y_{j(i),k(i)}} \cap G \arrow[r, "{[g_i]}"] \arrow[d, "\alpha"'] & \Gamma_{j(i)}^{x_{j(i)}} \cap G \arrow[r, "\psi_i"] \arrow[d, "\alpha"'] & A_i \arrow[d, equal] \\
\bar{\Gamma}_{j(i)}^{\bar{y}_{j(i),k(i)}} \cap \bar{G} \arrow[r, "{[\bar{g}_i]}"] & \bar{\Gamma}_{j(i)}^{\bar{x}_{j(i)}} \cap \bar{G} \arrow[r, "\bar{\psi}_i"] & A_i
\end{tikzcd}$$

As $\bar\Gamma_{j(1)}^{\bar x_{j(1)}} \cap \bar G, \ldots, \bar\Gamma_{j(n)}^{\bar x_{j(n)}} \cap \bar G$ are nonconjugate in $\bar G$, $(j(1), k(1)), \ldots, (j(n), k(n))$ are distinct. Hence, $\bar\varphi_1, \ldots, \bar\varphi_n$ simultaneously extend to a homomorphism $\bar\varphi\colon \bar G \to A$ which is trivial on $\hat F_m$ and on each free factor $\bar\Gamma_j^{\bar y_{jk}} \cap \bar G$ such that $(j,k) \notin \{(j(1), k(1)), \ldots, (j(n), k(n))\}$.

We prove that $\varphi = \bar\varphi \circ \alpha\colon G \to A$ and $a_i = \varphi(g_i)$, $i = 1, \ldots, n$ satisfy (2a) and (2b).

Indeed, $\varphi = \psi_i \circ [g_i]$ on $\Gamma_{j(i)}^{y_{j(i),k(i)}} \cap G$. Hence, for $z \in \Gamma_{j(i)}^{x_{j(i)}} \cap G$ we have $\varphi(z) = \psi_i(z^{g_i}) = \psi_i(z)^{a_i}$. This proves (2a).

For an arbitrary $\Gamma \in \mathcal{X}$ and $x \in F$ there exists j between 1 and q such that $\bar\Gamma \le \bar\Gamma_j$, and there exists $c \in \Gamma_j$, $k \in K_j$, and $g \in G$ such that $x = c y_{jk} g$. If $(j,i) = (j(i), k(i))$ for some i between 1 and n, then $\varphi(\Gamma^x \cap G) = \bar\varphi(\bar\Gamma_j^{\bar y_{j,k}} \cap \bar G)^{\varphi(g)} \le A_i^{\varphi(g)}$. Otherwise $\varphi(\Gamma^x \cap G) = 1$, and (2b) is proved.

Part B: *The general case.* Each ψ_i extends to a homomorphism of an open subgroup of $\Gamma_{j(i)}^{x_{k(i)}}$ into A_i. Hence, F has an open subgroup E that contains G such that ψ_i extends to a homomorphism $\varphi_i\colon \Gamma_{j(i)}^{x_{k(i)}} \cap E \to A_i$, $i = 1, \ldots, n$. By making E smaller if necessary we may assume that $\Gamma_{j(i)}^{x_{j(i)}} \cap E$, $i = 1, \ldots, n$ are nonconjugate in E (Lemma 3.2). Take a homomorphism φ which satisfies (2) with respect to φ_i and E. Its restriction to G satisfies (2) with respect to ψ_i and G. ∎

Let G be an $\mathcal{X}$-projective group. We say that G can be $\mathcal{X}$-embedded into a $\mathcal{Y}$-free group F if G can be embedded in F such that $\mathcal{X} = \mathcal{Y}^F \cap G = \{\Gamma^x \cap G \| \Gamma \in \mathcal{Y},\ x \in F\}$.

Corollary 3.4: *Let G be an $\mathcal{X}$-projective group which can be $\mathcal{X}$-embedded in a $\mathcal{Y}$-free group F. Then, for every closed subgroup U of G the pair $(U, \mathcal{X} \cap U)$ satisfies Assumption 2.2.*

Proof: Just note that $\mathcal{X} \cap U = \mathcal{Y}^F \cap U$ and apply Proposition 3.3. ∎

We don't know if every $\mathcal{X}$-projective group can be $\mathcal{X}$-embedded in a $\mathcal{Y}$-free group. By [H, Thm. 8.5], we know at least that this is the case if G is separable.

Corollary 3.5: *Let G be a separable $\mathcal{X}$-projective group. Then, for every closed subgroup U of G the pair $(U, \mathcal{X} \cap U)$ satisfies Assumption 2.2.* ∎

4. $\mathcal{X}$-projective groups with $\mathcal{X}$ closed

Assumption 2.2 depends on finitely many homomorphisms from an $\mathcal{X}$-projective group G into a finite group. It is therefore conceivable that it is possible to reduce the assumption to separable $\overline{\mathcal{X}}$-projective quotient of G which by Corollary 3.5 does satisfy the assumption. In this section we succeed to carry out this idea in the case where $\mathcal{X}$ is closed in the topology induced from that of $\mathrm{Subg}(G)$.

LEMMA 4.1: *Let G be an $\mathcal{X}$-projective group. Suppose that $\mathcal{X}$ is closed in* $\mathrm{Subg}(G)$. *Then, for each open normal subgroup M of G there exists an open normal subgroup N which is contained in M such that*

$$\Gamma, \Delta \in \mathcal{X} \text{ and } \Gamma M \neq \Delta M \quad \text{imply} \quad \Gamma N \cap \Delta N \leq M.$$

Proof: Choose representatives $\Lambda_1, \ldots, \Lambda_m$ for the groups in $\mathcal{X}$ modulo M. For each i consider the closed and open subset $\mathcal{X}_i = \{\Lambda \in \mathcal{X} \| \Lambda M = \Lambda_i M\}$ of $\mathcal{X}$.

Consider distinct i, j between 1 and m, and groups $\Gamma \in \mathcal{X}_i$, $\Delta \in \mathcal{X}_j$. Then $\Gamma \neq \Delta$ and therefore $\Gamma \cap \Delta = 1$ ((1a) of Section 2). In particular $\Gamma \cap (\Delta - M) = \emptyset$. Since both Γ and $\Delta - M$ are closed sets, G has an open normal subgroup $N(\Gamma, \Delta)$ which is contained in M such that $\Gamma N(\Gamma, \Delta) \cap (\Delta - M)N(\Gamma, \Delta) = \emptyset$. This implies that

$$\Gamma N(\Gamma, \Delta) \cap \Delta N(\Gamma, \Delta) \leq M \tag{1}$$

Consider now the open neighborhood of (Γ, Δ) in $\mathcal{X} \times \mathcal{X}$:

$$\mathcal{X}(\Gamma, \Delta) = \{(\Gamma', \Delta') \in \mathcal{X} \times \mathcal{X} \| (\Gamma', \Delta')N(\Gamma, \Delta) = (\Gamma, \Delta)N(\Gamma, \Delta)\}.$$

It is contained in $\mathcal{X}_i \times \mathcal{X}_j$. As the latter set is compact, there exist $\Gamma_k, \Delta_k \in \mathcal{X}$, $k = 1, \ldots, n$, such that

$$\bigcup_{i \neq j} \mathcal{X}_i \times \mathcal{X}_j = \bigcup_{k=1}^{n} \mathcal{X}(\Gamma_k, \Delta_k).$$

The group $N = \bigcap_{k=1}^{n} N(\Gamma_k, \Delta_k)$ is open, normal and contained in M. If $\Gamma, \Delta \in \mathcal{X}$ and $\Gamma M \neq \Delta M$, then there are $i \neq j$ such that $\Gamma \in \mathcal{X}_i$ and $\Delta \in \mathcal{X}_j$. Hence there is k between 1 and n such that $(\Gamma, \Delta) \in \mathcal{X}(\Gamma_k, \Delta_k)$. Conclude by (1) that

$$\Gamma N \cap \Delta N \leq \Gamma N(\Gamma_k, \Delta_k) \cap \Delta N(\Gamma_k, \Delta_k) = \Gamma_k N(\Gamma_k, \Delta_k) \cap \Delta_k N(\Gamma_k, \Delta_k) \leq M,$$

as desired. ∎

Let $\mathcal{X}$ be a separated family of closed subgroups of a profinite group G. We say that the $\mathcal{X}$-embedding problem ($\hat{\varphi}$: $G \to \hat{A}$, $\hat{\pi}$: $\hat{B} \to \hat{A}$, $\hat{\mathcal{B}}$) **dominates** the $\mathcal{X}$-embedding problem (φ: $G \to A$, π: $B \to A$, $\mathcal{B}$) if there exist homomorphisms α: $\hat{A} \to A$ and β: $\hat{B} \to B$ such that $\pi \circ \beta = \alpha \circ \hat{\pi}$, $\varphi = \alpha \circ \hat{\varphi}$, and $\beta(\hat{\mathcal{B}}) \subseteq \mathcal{B}$.

If this is the case, then every solution $\hat{\gamma}$ of the former embedding problem leads to a solution $\beta \circ \hat{\gamma}$ of the latter.

Lemma 4.2: *Let $\mathcal{X}$ be a separated family of closed subgroups of a profinite group G. Consider two finite $\mathcal{X}$-embedding problems for G:*

$$(\varphi_i\colon G \to A_i,\ \pi_i\colon B_i \to A_i,\ \mathcal{B}_i), \qquad i = 1,2.$$

Suppose that α: $A_2 \to A_1$ is a homomorphism such that $\alpha \circ \varphi_2 = \varphi_1$. Then there exists an $\mathcal{X}$-embedding problem (φ_2: $G \to A_2$, π: $B \to A_2, \mathcal{B}$) which dominates the two given ones.

Proof: Let $B = B_1 \times_{A_1} B_2$ be the cartesian product of B_1 and B_2 over A_1 with respect to π_1 and $\alpha \circ \pi_2$. Denote the projection of B on B_i by p_i. The family $\mathcal{B} = \{C \in \mathrm{Subg}(B) \| p_i(C) \in \mathcal{B}_i,\ i = 1,2\}$ contains with each C all the subgroups of C. Also, $\mathcal{B}$ is closed under conjugation. We prove that (φ_2: $G \to A_2$, $p_2 \circ \pi_2$: $B \to A_2$, $\mathcal{B}$) is an $\mathcal{X}$-embedding problem, which obviously dominates the given ones.

Indeed, for $\Gamma \in \mathcal{X}$ there exist homomorphisms γ_i: $\Gamma \to B_i$ such that $\pi_i \circ \gamma_i = \varphi_i$ on Γ and $\gamma_i(\Gamma) \in \mathcal{B}_i$ for $i = 1,2$. In particular $\pi_1 \circ \gamma_1 = \varphi_1 = \alpha \circ \varphi_2 = \alpha \circ \pi_2 \circ \gamma_2$ on Γ. Hence, there exists a homomorphism γ: $\Gamma \to B$ such that $p_i \circ \gamma = \gamma_i,\ \ i = 1,2$. It satisfies $\pi_2 \circ p_2 \circ \gamma = \pi_2 \circ \gamma_2 = \varphi_2$ on Γ and $p_i(\gamma(\Gamma)) = \gamma_i(\Gamma) \in \mathcal{B}_i$, for $i = 1,2$. Hence $\gamma(\Gamma) \in \mathcal{B}$, as desired. ∎

Proposition 4.3: *Let G be an $\mathcal{X}$-projective group such that $\mathcal{X}$ is closed in* Subg(G)*. Suppose that a closed subgroup K of G is the intersection of countably many open subgroups of G. Then G has a closed normal subgroup N which is contained in K such that G/N is separable and $\mathcal{X}/N = \{\Gamma N/N \| \Gamma \in \mathcal{X}\}$-projective.*

Proof: Let $K_1, K_2, K_3, \ldots$, be a sequence of open subgroups of G whose intersection is K. We construct by induction a descending double sequence $G = M_0 = N_0 \ge M_1 \ge N_1 \ge M_2 \ge N_2 \ge \cdots$ of open normal subgroups such that

(2a) $M_i \le K_i$, $i = 1, 2, 3, \ldots$,

(2b) if $\Gamma, \Delta \in \mathcal{X}$ and $\Gamma M_i \ne \Delta M_i$, then $\Gamma N_i \cap \Delta N_i \le M_i$, $i = 1, 2, 3, \ldots$ and

(2c) for each i we order the finite $\mathcal{X}$-embedding problems over G/N_i in a sequence

$$(3) \qquad (G \to G/N_i,\ \pi_{ij}\colon B_{ij} \to G/N_i,\ \mathcal{B}_{ij})$$

such that for each $i, j \le n$ the problem (3) has a solution which decomposes through G/M_{n+1}.

Indeed, suppose that we have already constructed M_i, N_i, B_{ij}, π_{ij}, and $\mathcal{B}_{ij}$ for $i \le n$ and for each j such that they satisfy requirement (2). Use Lemma 4.2 to construct a finite $\mathcal{X}$-embedding problem $(G \to G/N_n, \pi\colon B \to G/N_n, \mathcal{B})$ which dominates (3) for all $i, j \le n$. Since G is $\mathcal{X}$-projective this problem has a solution γ. This solution decomposes through the open normal subgroup $M_{n+1} = \mathrm{Ker}(\gamma)$ which is contained in N_n. Moreover, γ leads to a solution of (3) for each $i, j \le n$. By Lemma 4.1, there exists an open normal subgroup N_{n+1} which is contained in $K_{n+1} \cap M_{n+1}$ and satisfies (2b) for $i = n + 1$.

Let $N = \bigcap_{n=1}^{\infty} N_n = \bigcap_{n=1}^{\infty} M_n$. We use a bar to denote reduction modulo N and prove that $\bar{G}$ is $\mathcal{X}/N$-projective.

Note that $\mathcal{X}/N$ is closed in $\mathrm{Subg}(\bar{G})$. Hence, in order to prove that $\mathcal{X}/N$ is separated it suffices to prove that $\overline{\Gamma} \cap \overline{\Delta} = 1$ for each $\Gamma, \Delta \in \mathcal{X}$ such that $\overline{\Gamma} \ne \overline{\Delta}$ [J, Remark 5.1]. Indeed, if $\bar{x} \in \overline{\Gamma} \cap \overline{\Delta}$ and x is a lifting of $\bar{x}$ to G, then $\Gamma N \ne \Delta N$ and $x \in \Gamma N \cap \Delta N$. Hence there exist n such that $\Gamma M_n \ne \Delta M_n$. Hence, for each $i \ge n$, $\Gamma M_i \ne \Delta N_i$ and $x \in \Gamma N_i \cap \Delta N_i$. Conclude from (2b) that $x \in M_i$. Hence $x \in N$ and $\bar{x} = 1$, as desired.

Finally we prove that $\bar{G}$ is $\mathcal{X}/N$-projective. Let

$$(4) \qquad (\varphi\colon \bar{G} \to A,\ \pi\colon B \to A,\ \mathcal{B})$$

be a finite $\mathcal{X}/N$-embedding problem. Denote the canonical projection of G onto $\bar{G}$ by

ν. Then

$$(\varphi \circ \nu\colon G \to A,\ \pi\colon B \to A, \mathcal{B}) \tag{5}$$

is a finite $\mathcal{X}$-embedding problem. The kernel of φ contains N_i/N for some i. Use Lemma 4.2 to replace (5) by an $\mathcal{X}$-embedding problem over G/N_i which dominates (4). Thus, without loss, assume that $A = G/N_i$. Then $B = B_{ij}$, $\pi = \pi_{ij}$ and $\mathcal{B} = \mathcal{B}_{ij}$ for some j. Let $n = \max\{i, j\}$. By (2c), (5) has a solution γ which decomposes through G/M_{n+1} and therefore also through G/N. Conclude that (4) has a solution. ∎

PROPOSITION 4.4: *Every $\mathcal{X}$-projective group G such that $\mathcal{X}$ is closed in* Subg(G) *satisfies Assumption 2.2.*

Proof: Let $\Gamma_1, \ldots, \Gamma_n \in \mathcal{X}$ be nonconjugate subgroups of G, A a finite group, $\psi_i\colon \Gamma_i \to A$ a homomorphism, and $A_i = \psi_i(\Gamma_i)$, $i = 1, \ldots, n$. Take an open normal subgroup K such that $\Gamma_i \cap K \le \mathrm{Ker}(\psi_i)$ for $i = 1, \ldots, n$ and $\Gamma_1, \ldots, \Gamma_n$ are nonconjugate modulo K. By Proposition 4.3, G has a closed normal subgroup N which is contained in K such that $\bar{G} = G/N$ is separable and $\mathcal{X}/N$-projective. Let $\nu\colon G \to \bar{G}$ be the canonical map. For each i there exists an epimorphism $\bar{\psi}_i$ from $\overline{\Gamma}_i$ onto A_i such that $\bar{\psi}_i \circ \nu = \psi_i$ on Γ_i. By Corollary 3.5, $(\bar{G}, \mathcal{X}/N)$ satisfies Assumption 2.2. Thus, there exists a homomorphism $\bar{\varphi}\colon \bar{G} \to A$ and elements $a_1, \ldots, a_n \in A$ such that $\mathrm{res}_{\overline{\Gamma}_i} \bar{\varphi} = [a_i] \circ \bar{\psi}_i$, $i = 1, \ldots, n$, and for each $\Gamma \in \mathcal{X}$ there exists i such that $\bar{\varphi}(\nu(\Gamma))$ is conjugate to a subgroup of A_i. The homomorphism $\varphi = \nu \circ \bar{\varphi}$ from G to A satisfies the requirements of Assumption 2.2.
∎

LEMMA 4.5: *Let H be an open subgroup of a profinite group G. If $\mathcal{X}$ is a closed subfamily of* Subg(G), *then $\mathcal{X} \cap H$ is a closed subfamily of* Subg(H).

Proof: Since both $\mathcal{X}$ and Subg(H) are profinite spaces, it suffices to prove that the map $\Gamma \mapsto \Gamma \cap H$ from $\mathcal{X}$ into Subg(H) is continuous. Consider therefore $\Gamma \in \mathcal{X}$ and an open normal subgroup N of H. Take an open normal subgroup M of G which is contained in N. It suffices to prove that if $\Gamma' \in \mathcal{X}$ and $\Gamma' M = \Gamma M$, then $(\Gamma' \cap H)N = (\Gamma \cap H)N$.

Indeed, the assumption implies that $\Gamma' N = \Gamma N$ (Note that $\Gamma N = \{cn \| c \in \Gamma,\ n \in N\}$ need not be a subgroup of G). Hence $(\Gamma \cap H)N = (\Gamma N) \cap H = (\Gamma' N) \cap H = (\Gamma' \cap H)N$, as desired. ∎

Remark 4.6: We suspect that Lemma 4.5 does not hold for an arbitrary closed subgroup H of G. ∎

Combine Lemma 4.4 with Proposition 2.1 and Lemma 4.5:

COROLLARY 4.7: *Let G be an $\mathcal{X}$-projective group such that $\mathcal{X}$ is closed in* $\mathrm{Subg}(G)$*. Then each open subgroup H of G satisfies Assumption 2.2 with respect to $\mathcal{X} \cap H$.*

5. Prosolvable subgroups of $\mathcal{X}$-projective groups

The theorems that we prove from now on about an $\mathcal{X}$-projective group G are true if Assumption 2.2 holds for each open subgroup H of G. However, we prefer to formulate them for the two types of $\mathcal{X}$-projective groups which satisfy this assumption according to Sections 3 and 4.

LEMMA 5.1: *Let G be an $\mathcal{X}$-projective group which can be $\mathcal{X}$-embedded in a free $\mathcal{Y}$-product or such that $\mathcal{X}$ is closed in* $\mathrm{Subg}(G)$*. Let H be a closed prosolvable subgroup of G. Suppose that p, q are distinct primes, $\Gamma \in \mathcal{X}$ and t is an element of G such that $(\Gamma \cap H)_p$ is infinite, and $(\Gamma^t \cap H)_q \neq 1$. Then $t \in \Gamma$.*

Proof: We assume that $t \notin \Gamma$ and draw a contradiction by showing that H is not prosolvable.

Indeed, $t\Gamma^t$ is the intersection of all sets tU, where U ranges over all open subgroups of G that contain Γ^t. As $t\Gamma^t \cap \Gamma = \emptyset$, there exists, by compactness, an open subgroup U of G that contains Γ^t such that

$$tU \cap \Gamma = \emptyset. \tag{1}$$

It follows that

(2) $\Gamma \cap U$ and $\Gamma^t = \Gamma^t \cap U$ are nonconjugate in U.

Indeed if $\Gamma^{tu} = \Gamma \cap U$ for some $u \in U$, then $\Gamma^{tu} \cap \Gamma \neq 1$. As $\mathcal{X}$ is separated, $\Gamma^{tu} = \Gamma$ and therefore $tu \in \Gamma$ [H, Lemma 4.6], a contradiction to (1). So, (2) is true.

Next observe that $(\Gamma \cap H \cap U)_p$ contains $(\Gamma \cap H)_p \cap U$ which is open in $(\Gamma \cap H)_p$. Hence $(\Gamma \cap H \cap U)_p$ is infinite and in particular nontrivial.

By Proposition 2.1, U is $\mathcal{X} \cap U$-projective. By Corollary 3.4 and Corollary 4.7, $(U, \mathcal{X} \cap U)$ satisfies Assumption 2.2. Apply Proposition 2.3 to U, $\mathcal{X} \cap U$, $H \cap U$, $\Gamma \cap U$, Γ^t instead of to G, $\mathcal{X}$, H, Γ_1, Γ_2, respectively, to conclude that H is not prosolvable. This is the required contradiction. ∎

THEOREM 5.2: *Let G be an $\mathcal{X}$-projective group which can be $\mathcal{X}$-embedded into a free $\mathcal{Y}$-product or such that $\mathcal{X}$ is closed in* $\mathrm{Subg}(G)$. *Let H be a closed prosolvable subgroup of G. Suppose that p, q are distinct primes and $\Gamma_1, \Gamma_2 \in \mathcal{X}$ such that $(\Gamma_1 \cap H)_p$ is infinite and $(\Gamma_2 \cap H)_q \neq 1$. Then $\Gamma_1 = \Gamma_2$ and $H \leq \Gamma_1$.*

Proof: By Proposition 2.3, Γ_1 and Γ_2 are conjugate. Thus $\Gamma_1 = \Gamma$ and $\Gamma_2 = \Gamma^t$ with $t \in G$. By Lemma 5.1, $t \in \Gamma$. Thus, $(\Gamma \cap H)_p$ is infinite and $(\Gamma \cap H)_q \neq 1$. Hence, for each $h \in H$ the group $(\Gamma^h \cap H)_q$ is nontrivial. Use lemma 5.1 again to conclude that $h \in \Gamma$. Thus $H \leq \Gamma$. ∎

THEOREM 5.3: *Let G be an $\mathcal{X}$-projective group which can be $\mathcal{X}$-embedded into a free $\mathcal{Y}$-product, or for which $\mathcal{X}$ is closed in* $\mathrm{Subg}(G)$. *Let H be a closed prosolvable subgroup of G for which there are infinitely many primes p which divide the order of some group in $\mathcal{X} \cap H$. Then H is contained in some group Γ which belongs to $\mathcal{X}$.*

Proof: Use Proposition 2.1 to assume without loss that $H = G$. By assumption there is an infinite set $\{p_i \| i \in I\}$ of primes such that for each $i \in I$ there exists $\Gamma_i \in \mathcal{X}$ such that $(\Gamma_i)p_i \neq 1$. It follows from Proposition 2.3 that all the Γ_i are conjugate, say to a group Γ. In particular, each p_i divides the order of Γ. We prove that $G = \Gamma$.

Indeed, assume that there exists $g \in G \smallsetminus \Gamma$. As in the proof of Lemma 5.1, G has an open subgroup U which contains Γ^g such that $gU \cap \Gamma = \emptyset$. Again, this means that $\Gamma \cap U$ and Γ^g are nonconjugate in U.

As $\Gamma \cap U$ is open in Γ, its index is finite. Hence almost all p_i divide the order of $\Gamma \cap U$. Obviously, each p_i divides the order of Γ^g. In particular, there exists distance primes p, q such that $(\Gamma \cap U)_p$ and $(\Gamma^g)_q$ are nontrivial. But this contradicts Proposition 2.3. Conclude that $G = \Gamma$. ∎

6. Cohomological criterion

In the setup of Theorem 5.2 it is in general not easy to determine whether there exist p, q and Γ_1, Γ_2 such that $(\Gamma_1 \cap H)_p$ is infinite and $(\Gamma_2 \cap H)_q$ is nontrivial. The problem lies in the difficulty to analyze the intersections of H with the groups $\Gamma \in \mathcal{X}$. In this section we use cohomology to find conditions on H which will insure the existence of p and q as above. As in previous sections, whenever we speak about an $\mathcal{X}$-group, we assume that $\mathcal{X}$ is separated.

LEMMA 6.1: *Let G be a $\mathcal{X}$-projective group and A a finite G-module. Then the restriction map*

$$\text{res}\colon H^2(G, A) \to \prod_{\Gamma \in \mathcal{X}} H^2(\Gamma, A) \tag{1}$$

composed by restrictions on each factor is injective.

Proof (After Neukirch's proof of [N1, Satz 4.1]): Let $x \in H^2(G, A)$. Choose a factor system $f\colon G \times G \to A$ which represents x. Associate with x the short exact sequence $0 \to A \to \hat{G} \xrightarrow{\pi_G} G \to 1$ where $\hat{G} = \{(a, g) \| a \in A,\ g \in G\}$, the (noncommutative) addition on $\hat{G}$ is given by the formula

$$(a_1, g_1) + (a_2, g_2) = (a_1 + g_1 a_2 + f(g_1, g_2),\ g_1 g_2) \tag{2}$$

and π_G is the projection on G [R, p. 103]. Let Γ be a subgroup of G. The extension of Γ that corresponds to $\text{res}_\Gamma x$ gives the following commutative diagram of exact rows and inclusions as vertical maps:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & \hat{\Gamma} \arrow[r, "\pi_\Gamma"] \arrow[d, "i"'] & \Gamma \arrow[r] \arrow[d] & 1 \\ 0 \arrow[r] & A \arrow[r] & \hat{G} \arrow[r, "\pi_G"] & G \arrow[r] & 1 \end{tikzcd} \tag{3}$$

Now choose an open normal subgroup N of $\hat{G}$ such that $N \cap A = 1$. Increase (3) to a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & \hat{\Gamma} \arrow[r, "\pi_\Gamma"] \arrow[d, "i"'] & \Gamma \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & \hat{G} \arrow[r, "\pi_G"] \arrow[d, "\rho"'] & G \arrow[r] \arrow[d, "\varphi"'] & 1 \\
0 \arrow[r] & A \arrow[r] & \hat{G}/N \arrow[r, "\hat{\pi}_G"] & \bar{G} \arrow[r] & 1
\end{tikzcd}
\tag{4}
$$

such that the right lower rectangle is cartesian [FJ, Section 20].

If $x \in \mathrm{Ker}(\mathrm{res})$, then $\mathrm{res}_\Gamma x = 0$ for each $\Gamma \in \mathcal{X}$. Hence π_Γ has a section θ_Γ and therefore $\hat{\pi}_G \circ \rho \circ \theta_\Gamma = \varphi$ on Γ. Since G is $\mathcal{X}$-projective, there exists a homomorphism $\gamma\colon G \to \hat{G}/N$ such that $\hat{\pi}_G \circ \gamma = \varphi$. As the lower right rectangle of (4) is cartesian, π_G has a section θ. This means that $x = 0$.

Conclude that res is injective. ∎

THEOREM 6.2: *Let G be an $\mathcal{X}$-projective group which can be embedded in a $\mathcal{Y}$-free profinite group, or such that $\mathcal{X}$ is closed in* $\mathrm{Subg}(G)$. *Suppose that H is a prosolvable closed subgroup of G*

(a) *and p, q are distinct primes such that H_p is torsionfree but not free pro–p, and H_q is not free pro–q, or*

(b) *there exist infinitely many primes p such that H_p is not free pro–p.*

Then there exists $\Gamma \in \mathcal{X}$ such that $H \leq \Gamma$.

Proof: Let p be a prime such that H_p is not a free pro–p group. Then $\mathrm{cd}(H_p) \geq 2$ [R, pp. 235-236] and therefore $H^2(H_p, \mathbb{Z}/p\mathbb{Z}) \neq 0$ [R, p. 220, Cor. 4.3]. By Proposition 2.1, H_p is $(\mathcal{X} \cap H_p)$-projective. Hence, by Lemma 6.1, restriction

$$\mathrm{res}\colon H^2(H_p, \mathbb{Z}/p\mathbb{Z}) \to \prod_{\Gamma \in \mathcal{X}} H^2(\Gamma \cap H_p, \mathbb{Z}/p\mathbb{Z})$$

is injective. Thus, there exists $\Gamma \in \mathcal{X}$ such that $H^2(\Gamma \cap H_p, \mathbb{Z}/p\mathbb{Z}) \neq 0$. In particular $\Gamma \cap H_p \neq 1$.

If this is the case for infinitely many p, then, by Theorem 5.3, H is contained in some Γ which belongs to $\mathcal{X}$.

So, suppose that (a) hold. As H_p is torsionfree, and with the above notation, $\Gamma \cap H_p$ contains an element of infinite order. Hence, $(\Gamma \cap H)_p$, which contains $\Gamma \cap H_p$, is infinite. Similarly, there exists $\Gamma' \in \mathcal{X}$ such that $(\Gamma' \cap H)_q$ is nontrivial. Conclude from Theorem 5.2 that $H \leq \Gamma$. ∎

COROLLARY 6.3: *Let F be a free $\mathcal{X}$-product. Suppose that H is a prosolvable subgroup of F*

(a) *and p, q are distinct primes such that H_p is torsionfree but not free pro–p, and H_q is not free pro–q, or*

(b) *there exist infinitely many primes p such that H_p is not free pro–p.*

Then H is conjugate to a subgroup of some $\Gamma \in \mathcal{X}$.

Proof: F is $\mathcal{X}^F$-projective [H, Prop. 5.3]. ∎

Example 6.4: Projective prosolvable groups. Let H be a projective prosolvable group (e.g, a free pro–p group). Then H is isomorphic to a subgroup of a free profinite group F [FJ, Cor. 20.14]. If X is a basis of F [FJ, p. 190] and $\mathcal{X}$ is the family of all closed procyclic groups generated by the elements of X, then F a free $\mathcal{X}$-product. However, H is conjugate to no subgroup of a $\Gamma \in \mathcal{X}$ unless H is procyclic. Of course, for each p, the p-Sylow subgroup of H is pro–p free [FJ, Prop. 20.37]. So, the hypothesis of Corollary 6.3 is not satisfied. ∎

7. Large quotients

The study of p-adically projective groups and pseudo p-adically projective fields in [HJ2] and in [J] depends on special properties that the group $\Gamma = G(\mathbb{Q}_p)$ has. They involve however information on subgroups of the groups $D_{e,m} = \Gamma_1 * \cdots * \Gamma_e * \hat{F}_m$, where each Γ_i is an isomorphic copy of Γ and $\hat{F}_m$ is the free profinite group on m generators. The exact condition is formulated in [J] as assumption A of the introduction.

It is pointed out in [J] that this assumption implies the seemingly stronger Assumption 3.1 of [HJ2]. The purpose of this section is to apply the previous results to show that Assumption A follows from assumptions on the group Γ without any reference to the auxiliary groups $D_{e,m}$. We denote the maximal pro-p quotient of a profinite group Γ by $\Gamma(p)$.

PROPOSITION 7.1: *Let G be an $\mathcal{X}$-projective group which can be embedded in a $\mathcal{Y}$-free group, or such that $\mathcal{X}$ is closed in* $\mathrm{Subg}(G)$. *Suppose that Γ is a profinite group that satisfies the following conditions:*

(a) Γ *is prosolvable and finitely generated.*

(b) *There exist distinct primes p, q such that $\Gamma(p)$ is not a free pro-p group and $\Gamma(q)$ is not a free pro-q group.*

Then Γ has a finite quotient $\overline{\Gamma}$ such that if a subgroup H of G is a quotient of Γ and $\overline{\Gamma}$ is a quotient of H, and if H_p is torsionfree, then H is a subgroup of some $\Delta \in \mathcal{X}$. In particular, this conclusion holds if $H \cong \Gamma$ and Γ_p is torsionfree.

Proof: By [HJ2, Lemma 11.2], Γ has open normal subgroups N_p and N_q such that Γ/N_p is a p-group, Γ/N_q is a q-group, and for each closed normal subgroup N of Γ which is contained in N_p (resp., in N_q) Γ/N is not a free pro-p (resp., pro-q) group. Let N_0 be the intersection of all open subgroups of Γ of index at most $\max\{(\Gamma : N_p), (\Gamma : N_q)\}$. As Γ is finitely generated N_0 is open and normal. Put $\overline{\Gamma} = \Gamma/N_0$.

Suppose that H satisfies the conditions of the proposition. Then H, as a quotient of a prosolvable group, is prosolvable. If H_p were free pro-p, then $\mathrm{cd}_p H \le 1$, and therefore $H(p)$ were free pro-p [R, p. 255]. Let M_p be the intersection of all open normal subgroups N_p' of Γ such that $\Gamma/N_p' \cong \Gamma/N_p$. Then $N_0 \le M_p$ and Γ/M_p is a quotient of $H(p)$. Let N be the kernel of the composed homomorphism $\Gamma \to H \to H(p)$. Let M be the kernel of the composed map $\Gamma \to H \to H(p) \to \Gamma/M_p$. Then $N \le M$, $\Gamma/M \cong \Gamma/M_p$. Hence M is the intersection of open normal subgroups N_p' of Γ such that $\Gamma/N_p' \cong \Gamma/N_p$. It follows that $M_p \le M$. As both subgroups have the same index in Γ, they coincide. Hence $N \le N_p$ and $\Gamma/N \cong H(p)$ is free pro-p. This contradiction to the choice of N_p proves that H_p is not free pro-p. Similarly H_q is not a free pro-q group.

Conclude from Theorem 6.2 that H is a subgroup of some $\Delta \in \mathcal{X}$. ∎

THEOREM 7.2: *The following conditions on a profinite group Γ imply Assumption A:*

(a) Γ *is prosolvable and finitely generated.*

(b) *The center of* Γ *is trivial.*

(c) *There exists a prime* p *such that* Γ_p *is a torsionfree nonfree pro-*p *group.*

(d) *There exists a prime* $q \neq p$ *such that* Γ_q *is not a free pro-*q *group.*

(e) Γ *has a finite quotient* $\overline{\Gamma}$ *such that if a subgroup* H *of* Γ *is a quotient of* Γ *and* $\overline{\Gamma}$ *is a quotient of* H*, then* $H \cong \Gamma$.

Proof: By taking a larger quotient, if necessary, we may assume that the kernel of $\Gamma \to \overline{\Gamma}$ is contained in the intersection N_0 mentioned in the first paragraph of the proof of Proposition 7.1. In particular $\overline{\Gamma}$ satisfies the conclusion of Proposition 7.1.

Let H be a subgroup of $D_{e,m}$ such that H is a quotient of Γ and $\overline{\Gamma}$ is a quotient of H. We have to prove that H is conjugate to one of the groups Γ_i.

By a theorem of Ribes and Herfort [HR, Thm. 1], each element of $D_{e,m}$ of a finite order is conjugate to an element of one of the factors $\Gamma_1, \ldots, \Gamma_e, \hat{F}_m$. Since $\hat{F}_m$ is torsionfree and since by (c), Γ_p is torsionfree, so is H_p. By Proposition 7.1, H is conjugate to a subgroup H' of $\hat{F}_m$ or to a subgroup H' of some Γ_i. In the former case H_p is free pro-p [FJ, Prop. 47]. Hence, $H(p)$ is also free pro-p group [R, p. 255]. On the other hand, by the choice of $\bar{\Gamma}$, the kernel of $\Gamma \to H$ is contained in N_0 and hence in N_p (in the notation of the proof of Proposition 7.1). Hence, so is the kernel of the composed map $\Gamma \to H \to H(p)$. So, $H(p)$ is not a free pro-p-group. This contradiction proves that $H' \leq \Gamma_i$ for some i between 1 and e. Conclude from (e) that $H' = \Gamma_i$, that is H is conjugate to Γ_i. ∎

Remark 7.3: Note that condition (b) is not involved in the proof. So, Theorem 7.2 remains valid if we omit both condition (c) of Assumption A and condition (b) of the Theorem. ∎

Methods from both local and global class field theory are applied in [HJ2] to prove that $\Gamma = G(\mathbb{Q}_p)$ satisfies Assumption A. We replace the methods from global class field theory by Theorem 7.2 (which is proved by pure group theoretic methods) to prove the same result for open subgroups of $G(\mathbb{Q}_p)$.

THEOREM 7.4: *Let K be a finite extension of $\mathbb{Q}_p$. Then $\Gamma = G(K)$ satisfies conditions (a)–(e) of Theorem 7.2 and therefore also Assumption A.*

Proof: It is an implicit result of Iwasawa that $G(K)$ is finitely generated [S, p. III-30]. Jannsen [Ja, Satz 3.6] gives an explicit proof of this statement. (See also [JR, Introduction] for a simpler proof.)

To prove that the center of $G(K)$ is trivial we repeat Ikeda's arguments [I, p. 7] for the triviality of the center of $G(\mathbb{Q}_p)$. Suppose that σ is an element of $Z(G(K))$. Let L be a finite Galois extension of K and consider the maximal abelian extension $L_{\rm ab}$ of L. Then $L_{\rm ab}$ is a Galois extension of K. By local class field theory [N2, p. 69] the reciprocity map θ: $L^\times \to \mathcal{G}(L_{\rm ab}/L)$ is injective. Denote the restriction of σ to $L_{\rm ab}$ by $\bar\sigma$. Then for each $x \in L^\times$ we have $\theta(x) = \theta(x)^{\bar\sigma} = \theta(x^\sigma)$ [N2, p. 26]. Hence, $x = x^\sigma$. As L and x are arbitrary, $\sigma = 1$ and $Z(G(K))$ is trivial.

As ${\rm cd}_p(G(K)) = 2$ for each prime p [R, p. 291-292], conditions (c) and (d) are true for $G(K)$.

Finally, the proof of condition (e) of Theorem 6.3 is a simplified version of the proof of [HJ2, Prop. 11.5]:

Let K_0 be the compositum of all finite extensions of K of degree at most $p-1$. In particular K_0 contains a primitive p-th root, ζ_p, of 1. For a profinite group G we denote the rank of $G(p)$ by ${\rm rank}_p G$. By [HJ2, Lemma 11.1], K_0 has a finite extension K_1 such that for each Galois extension K_1' of K_0 that contains K_1

$$(1) \qquad {\rm rank}_p \mathcal{G}(K_1'/K_0) = {\rm rank}_p G(K_0).$$

As $\zeta_p \in K_0$, the maximal p-quotient of $G(K_0)$ is not free pro-p [S, p. II-30]. By [HJ2, Lemma 11.2], K_0 has a proper finite p-extension K_p such that for each Galois extension K_p' of K_0 that contains K_p the group $\mathcal{G}(K_p'/K_0)$ is not free pro-p. Denote the compositum of all extensions of K of degree at most $m = \max\{[K_1 : K], [K_p : K]\}$ by E. Then E is a finite Galois extension of K. We prove that $\overline{\Gamma} = \mathcal{G}(E/K)$ satisfies the requirements of condition (e).

Indeed, suppose that H is a subgroup of $G(K)$ and there exist epimorphisms $G(K) \xrightarrow{\varphi} H \xrightarrow{\psi} \mathcal{G}(E/K)$. Denote the fixed field of H in $\tilde K$ by L. Denote the fixed field of $\mathrm{Ker}(\varphi)$ by N. Then $\mathcal{G}(N/K) \cong G(L)$. Also, for the fixed field E' of $\mathrm{Ker}(\psi \circ \varphi)$, we have $\mathcal{G}(E'/K) \cong \mathcal{G}(E/K)$. Hence E' is a compositum of extensions of K of degree at most m. Hence $E' \subseteq E$. Since both fields have the same degree over K, $E' = E$. As $\mathrm{Ker}(\varphi) \leq \mathrm{Ker}(\psi \circ \varphi)$, we have $E \subseteq N$. We proceed to prove that $L = K$ and that therefore $H = G(K)$.

By construction, p divides $[N : K_0]$. Let $K_0^{(p)}$ be the maximal p-extension of K_0. Then $K_p \subseteq N \cap K_0^{(p)}$. Hence, the maximal pro-p quotient $\mathcal{G}(N \cap K_0^{(p)}/K_0)$ of $\mathcal{G}(N/K_0)$ is not pro-p free. It follow from [R, p. 255] that $\mathrm{cd}_p\mathcal{G}(N/K_0) > 1$. Let L_0 be the fixed field of $\varphi(G(K_0))$ in $\tilde{\mathbb{Q}}_p$. As $\mathcal{G}(N/K_0) \cong G(L_0)$, we have $\mathrm{cd}_pG(L_0) > 1$. Hence p^∞ does not divide $[L_0 : K]$ [R, pp. 291–292]. Also, L_0 is the compositum of all extensions of L of degree at most $p-1$. In particular $\zeta_p \in L_0$. By [N2, Satz 4] and (1)

$$2 + [L_0 : \mathbb{Q}_p] = \mathrm{rank}_pG(L_0) = \mathrm{rank}_p\mathcal{G}(N/K_0) = \mathrm{rank}_pG(K_0) = 2 + [K_0 : \mathbb{Q}_p].$$

Hence, $[K_0 : \mathbb{Q}_p] = [L_0 : \mathbb{Q}_p]$. As $[K_0 : K] = [L_0 : L]$, this implies that $[K : \mathbb{Q}_p] = [L : \mathbb{Q}_p]$. So, conclude from $K \subseteq L$ that $K = L$. ∎

PROBLEM 7.5: *Suppose that a profinite group Γ satisfies conditions (a)—(e) of Theorem 7.2. Is Γ isomorphic to the absolute Galois group of a finite extension of $\mathbb{Q}_p$?*

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9 February, 2007