THE HENSELIAN CLOSURES OF A PpC FIELD *

by

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Introduction

There are three main types of "pseudo closed fields". They are the "pseudo algebraically closed fields" (PAC), "pseudo real closed fields" (PRC), and "pseudo p–adically closed fields". Recall that a field K is said to be **PAC** (resp., **PRC, PpC**) if every absolutely irreducible variety V has a K-rational point provided it has a simple $\overline{K}$-rational point for each algebraic (resp., real, p–adic) closure $\overline{K}$ of K. It is known that a PAC field K carries no interesting arithmetical structure:

PROPOSITION A: *Let K be a PAC field.*

(a) *K admits no ordering [FJ, Thm. 10.12].*

(b) (Frey – Prestel) *The Henselian closure of each valuation of K is separably closed [FJ, Thm. 10.14].*

It is not difficult to see that the only arithmetical structure of a PRC field emerges from its orderings:

THEOREM B: *Let K be a PRC field and let v be a valuation of K. Then the Henselian closure $\overline{K}$ of K with respect to v is either real closed or separably closed.*

Proof: If $\mathrm{char}(K) \neq 0$, then K is PAC and the theorem reduces to Proposition A(b). So, assume that $\mathrm{char}(K) = 0$.

By Prestel's extension theorem [P, Thm. 3.1], $\overline{K}(\sqrt{-1})$ is a PRC field. Since it has no orderings it is PAC. On the other hand $\overline{K}(\sqrt{-1})$ is Henselian. By Proposition A(b), $\overline{K}(\sqrt{-1})$ is algebraically closed. Conclude from a theorem of Artin [L, p. 223] that $\overline{K}$ is either real closed or algebraically closed. ∎

The goal of this note is to establish the analogue of Theorem B for PpC fields and to show that the only arithmetic of such a field essentially comes from its p–adic valuations:

THEOREM C: *A PpC field K admits no orderings. The Henselian closure of K with respect to any valuation is either p–adically closed or algebraically closed.*

Except from manipulations with Henselian fields and in particular a theorem of F.K. Schmidt and Engler the proof of Theorem C is based on the following result:

PROPOSITION D (Algebraic extension theorem for PpC fields [J, Prop. 8.3]): *Let L be an algebraic extension of a* PpC *field K. Then L is* PpC *if and only if for each p–adic closure $\overline{K}$ of K we have $L \subseteq \overline{K}$ or $\overline{K}L$ is algebraically closed.*

An important ingredient in the proof of this Proposition is the following property of each PpC field K: The compositum of every two p–adic closures of K is the algebraic closure $\widetilde{K}$ of K [HJ, Lemma 4.5(b)]. In Section 2 we point out that this statement fails to be true over an arbitrary field.

1. Reduction of p–adic valuations.

For a valuation w of a field K denote the valuation ring, its maximal ideal and the residue field respectively by O_w, P_w and $\overline{K}_w$. Let also U_w and Γ_w be the group of w–units and the value group of w, respectively. Consider now an additional valuation v of K such that $O_v \subseteq O_w$ (v is **finer** than w and w is **coarser** than v). Then $P_w \subseteq P_v$ and $O_{\bar{v}} = O_v/P_w$ is a valuation ring of $\overline{K}_w = O_w/P_w$ with the maximal ideal $P_{\bar{v}} = P_v/P_w$. The corresponding valuation $\bar{v}$ is defined by $\bar{v}(\bar{x}) = v(x)$ for $x \in U_w$ (the bar denotes reduction modulo P_w). In particular the residue fields of $\bar{v}$ and v coincide. Also, $\Gamma_{\bar{v}} \cong \overline{K}_\omega^\times/U_{\bar{v}} \cong U_\omega/U_v$ is a convex subgroup of $\Gamma_v \cong K^\times/U_v$. Thus

$$(1) \qquad \overline{K}_v \cong O_{\bar{v}}/P_{\bar{v}} \cong O_v/P_v \qquad \text{and} \qquad \Gamma_\omega \cong \Gamma_v/\Gamma_{\bar{v}}.$$

LEMMA 1.1: *The valued field (K, v) is Henselian if and only if (K, w) and $(\overline{K}_w, \bar{v})$ are Henselian.*

Proof: See [R, pp. 210 and 211]. ∎

LEMMA 1.2: *Suppose that an algebraic extension M of a field K is Henselian with respect to valuations v and w such that $O_v \subseteq O_w$. Denote the unique extensions of v and w, respectively, to $\widetilde{K}$ by $\tilde{v}$ and $\tilde{w}$. Then*

(a) $O_{\tilde{v}} \subseteq O_{\tilde{w}}$,

(b) *the decomposition group of $\tilde{v}$ over K is contained in the decomposition group of $\tilde{w}$, and*

(c) *the deomposition field of $\tilde{w}$ over K is contained in the decomposition field of $\tilde{v}$.*

Proof of (a): Each element $x \in O_{\tilde{v}}$ satisfies an equation of the form $x^n = \sum_{i=0}^{n-1} a_i x^i$ with $a_i \in O_v$. Then $a_i \in O_w$, x is integral over O_w, and therefore belongs to $O_{\tilde{w}}$.

Proof of (b): Suppose that an automorphism $\sigma \in G(K)$ belongs to the decomposition group of $\tilde{v}$, that is $\sigma O_{\tilde{v}} = O_{\tilde{v}}$. Then $O_{\tilde{v}} \subseteq \sigma O_{\tilde{w}}$. It is known that the set of all valuation rings of $\widetilde{K}$ that contain $O_{\tilde{v}}$ is linearly ordered [R, p. 58]. In particular $\sigma O_{\tilde{w}} \subseteq O_{\tilde{w}}$ or $O_{\tilde{w}} \subseteq \sigma O_{\tilde{w}}$. Replace σ by σ^{-1} if necessary to assume that $\sigma O_{\tilde{w}} \subseteq O_{\tilde{w}}$. Then, for each positive integer n we have $\sigma^n O_{\tilde{w}} \subseteq \sigma^{n-1} O_{\tilde{w}} \subseteq \cdots \subseteq \sigma O_{\tilde{w}}$. Now, for each $x \in O_{\tilde{w}}$ there exists a positive integer n such that $\sigma^n x = x$. Hence $x \in \sigma O_{\tilde{w}}$. Conclude that $\sigma O_{\tilde{w}} = O_{\tilde{w}}$ and σ belongs to the decomposition group of $\tilde{w}$.

Proof of (c): Assertion (c) is a reinterpretation of (b). ∎

Two valuations v and v' of a field K are **comparable** if one of them is finer than the other.

The following result was proved by F.K. Schmidt [S] for valuations of rank 1 and then generalized by Engler [E] for higher rank valuations.

PROPOSITION 1.3 (F.K. Schmidt – Engler): *If a field K which is not separably closed is Henselian with respect to incomparable valuations v and v', then these valuations are finer than a common valuation w which has a separably closed residue field. In particular, K can not be Henselian with respect to two distinct valuations of rank 1.*

Thus, if L/K is a Galois extension, L is Henselian with respect to a valuation v of rank 1, but L is not separably closed, then K is Henselian with respect to the restriction of v to K.

Recall that a valuation v of a field K is called p–**adic** if $v(p)$ is the smallest positive element of $v(K^\times)$ and $\overline{K}_v \cong \mathbb{F}_p$. In particular $\mathrm{char}(K) = 0$. A p–adic valued field (K, v) is p–**adically closed** if it has no proper algebraic extension to a p–adic field. An ordered abelian group Γ is a $\mathbb{Z}$**-group** if it contains $\mathbb{Z}$ as a convex subgroup and for each $\gamma \in \Gamma$ and each positive integer n there exists $\delta \in \Gamma$ such that $\gamma \cong n\delta \bmod \mathbb{Z}$.

LEMMA 1.4 (Prestel – Roquette [PR, p. 34]): *A p–adic field (K, v) is p–adically closed if and only if it is Henselian and $v(K^\times)$ is a $\mathbb{Z}$-group.*

We denote the algebraic closure of a field K by $\widetilde{K}$ and its absolute Galois group by $G(K)$.

LEMMA 1.5: *Let (K, v) be a p–adically closed field and let w be a strictly coarser valuation of K than v. Then w is unramified in $\widetilde{K}$. Moreover, $G(K)$ is the decomposition group of the unique extension of w to $\widetilde{K}$ (which we also denote by w) and the map $G(K) \to G(\overline{K}_w)$ that w induces is an isomorphism. In particular, for each algebraic extension M of K we have an isomorphism $G(M) \cong G(\overline{M}_w)$.*

Proof: By assumption $1 = v(p)$ belongs to the convex subgroup $\Gamma_{\bar{v}}$ of Γ_v. By Lemma 1.4, Γ_v is a $\mathbb{Z}$-group. Hence, for each $\bar{\gamma} \in \Gamma_w$ there exists $\bar{\delta} \in \Gamma_w \cong \Gamma_v/\Gamma_{\bar{v}}$ such that $\bar{\gamma} = n\bar{\delta}$. In other words Γ_w is a divisible group.

As (K, w) is a Henselian field (Lemma 1.1) with residue field $\overline{K}_w$ of characteristic 0 the formula $[L : K] = e(L/K)f(L/K)$ holds for each finite extension L of K [R, p. 236]. By the preceding paragraph $e(L/K) = 1$. Hence $[L : K] = [\overline{L}_w : \overline{K}_w]$ and w is unramified in L.

If in addition L is Galois over K, then $\mathcal{G}(L/K)$ is the decomposition group of w (since (K, w) is Henselian). By the preceding paragraph $\mathcal{G}(L/K)$ is isomorphic to $\mathcal{G}(\overline{L}_w/\overline{K}_w)$.

On the other hand, each finite extension of $\overline{K}_w$ is the residue field $\overline{L}$ of a finite extension of L with respect to the unique extension of w to K. Indeed, as $\mathrm{char}(\overline{K}_w) = 0$, $\overline{L}$ has a primitve element $\bar{z}$ over $\overline{K}_w$. Let $\bar{f} = \mathrm{irr}(\bar{z}, \overline{K}_w)$ and lift $\bar{f}$ to a monic polynomial $f \in K[Z]$ of the same degree. Then take L as $K(z)$, where z is any root of f.

Thus let L range over all finite Galois extensions of K to conclude that w induces an ismorphism of $G(K)$ onto $G(\overline{K}_w)$. ∎

LEMMA 1.6: *Let (K, v_p) be a p–adically closed field. Let M be an algebraic extension of K which is not algebraically closed. If M is Henselian with respect to a valuation v, then v is coarser than the unique extension of v_p to M (which we also denote by v_p).*

Proof: The residue field of M with respect to v_p is algebraic over $\mathbb{F}_p$ and therefore has no nontrivial valuations. Hence v is not strictly finer than v_p.

Assume that v is not coarser than v_p. Then v and v_p are incomparable. By Lemma 1.3, M has a valuation w which is coarser than both v and v_p. Moreover $\overline{M}_w$ is algebraically closed. By Lemma 1.5, $G(M) = G(\overline{M}_w) = 1$.

Conclude from this contradiction that v is coarser than v_p. ∎

Lemma 1.7: *Let L be a Henselian* PpC *field of characteristic 0. If an algebraic extension F of L satisfies $\overline{L}F = \widetilde{L}$ for each p–adic closure $\overline{L}$ of L, then $F = \widetilde{L}$. In particular $\tilde{\mathbb{Q}}L = \widetilde{L}$.*

Proof: By Proposition D, F is PpC. Since F has no p–adic closures, it is PAC. As an algebraic extension of a Henselian field, F is Henselian. Conclude from Proposition A that $F = \widetilde{L}$.

Finally note that $\tilde{\mathbb{Q}}\overline{L} = \widetilde{L}$ for each p–adic closure $\overline{L}$ of L [HJ, Cor. 6.6]. So, $\tilde{\mathbb{Q}}L$ satisfies the above condition on F and is therefore algebraically closed. ∎

The following Lemma is well known for finite groups [FJ, Lemma 12.4].

Lemma 1.8: *Let H be a proper closed subgroup of a profinite group G. Then $\bigcup_{x\in G} H^x$ is a proper subset of G.*

Proof: Choose an epimorphism φ of G on a finite group $\overline{G}$ such that $\overline{H} = \varphi(H)$ is a proper subgroup of $\overline{G}$. For $x \in G$ let $\bar{x} = \varphi(x)$. Then there exists $g \in G$ such that $\bar{g} \notin \bigcup_{\bar{x}\in\overline{G}} \overline{H}^{\bar{x}}$. Hence $g \notin \bigcup_{x\in G} H^x$. ∎

The following result is proved in a different way by Haran and Lubotzky [HL, Lemma 5].

Lemma 1.9: *Let E be a p–adic closure of $\mathbb{Q}$ and let F be a q–adic closure of $\mathbb{Q}$. If $E \neq F$, then $EF = \tilde{\mathbb{Q}}$.*

Proof: The field EF is Henselian with respect to the unique extension of the p–adic valuation of E and also with respect to the unique extension of the q-adic valuation of F. If $EF \neq \tilde{\mathbb{Q}}$, then, by Proposition 1.3, the two valuations are equivalent. Denote the unique topology of EF which they define by T. As both E and F are the closures of $\mathbb{Q}$ in EF with respect to T they must coincide.

Conclude from this contradiction that $EF = \tilde{\mathbb{Q}}$. ∎

We refer to a field K of characteristic 0 as **algebraic** if it is algebraic over $\mathbb{Q}$.

LEMMA 1.10: *Let K be an algebraic field with a unique p–adic valuation v which is not p–adically closed. Then K has an algebraic extension F which is not algebraically closed such that $\overline{K}F = \tilde{\mathbb{Q}}$ for each p–adic closure $\overline{K}$ of K.*

Proof: Choose a p–adic closure E of K. By assumption $E \neq K$. Since K is algebraic and v is unique, each p–adic closure $\overline{K}$ of K is isomorphic to E over K. Hence, by Lemma 1.8, there exists $\sigma \in G(K)$ such that $\sigma \notin G(\overline{K})$ for each p–adic closure $\overline{K}$ of K. Let $F = \tilde{\mathbb{Q}}(\sigma)$. As an algebraic extension of $\overline{K}$, the field $\overline{K}F$ is Henselian. On the other hand $\overline{K}F$ is a Galois extension of F, since $G(F)$ is abelian. If $\overline{K}F \neq \tilde{\mathbb{Q}}$, then, by Proposition 1.3, F would be Henselian. Hence F would contain a q–adic closure L of $\tilde{\mathbb{Q}}$ for some prime number q. The choice of σ would imply that $L \neq \overline{K}$. Hence $\overline{K}L = \tilde{\mathbb{Q}}$ (Lemma 1.9). Conclude that $\overline{K}F = \tilde{\mathbb{Q}}$, a contradiction. ∎

For a positive integer n we denote a primitive root of 1 of order n by ζ_n.

LEMMA 1.11: *Let K be an algebraic field with distinct p–adic valuations v_1 and v_2. Then K has an algebraic extension F which is not algebraically closed such that $\overline{K}F = \tilde{\mathbb{Q}}$ for each p–adic closure $\overline{K}$ of K.*

Proof: Let $\overline{K}_i$ be a p–adic closure of K with respect to v_i, $i = 1, 2$. Choose a prime $q \neq p$. Then $\overline{L}_i = \overline{K}_i(\zeta_q)$ satisfies $(\overline{L}_i^\times : (\overline{L}_i^\times)^q) = q^2$ [N, p. 41]. Choose a system of generators A_i for $\overline{L}_i^\times$ modulo $(\overline{L}_i^\times)^q$ of q^2 elements.

Extend the valuation v_i to a valuation v_i' of $L = K(\zeta_q)$. Then L is v_i'–dense in $\overline{L}_i$. Moreover, the valuations v_1' and v_2' are distinct, of rank 1, and therefore independent. Hence, for each $(a_1, a_2) \in A_1 \times A_2$ there exists $x = x(a_1, a_2)$ in L such that

$$v_i'(x - a_i) > 2v_i'(a_i) \qquad i = 1, 2 \tag{1}$$

[R, p. 135]. It follows from Netwon's Lemma that the equation $a_i Z^q - x = 0$ is solvable in $\overline{L}_i$. Thus $a_i(\overline{L}_i^\times)^q = x(\overline{L}_i^\times)^q$. If $(a_1', a_2') \in A_1 \times A_2$, $x' = x(a_1', a_2')$ and $x' \in x(L^\times)^q$, then $a_i' \in a_i(\overline{L}_i^\times)^q$, and therefore $a_i' = a_i$ for $i = 1, 2$. It follows that the elements $x(a_1, a_2)$ represent q^4 distinct congruence classes of $L^\times$ modulo $(L^\times)^q$.

The field $E = L(\zeta_{q^n}, \sqrt[q^n]{p} \mid n = 1, 2, 3, \ldots)$ is a procyclic extension of $L' = L(\zeta_{q^n} \mid n = 1, 2, 3, \ldots)$ whose order is a q–power (possibly infinite). Also L'/L is a procyclic group whose order is a q–power (possibly infinite). Also E/L is a Galois extension. Hence $\mathcal{G}(E/L)$ is a pro–q group whose rank is at most 2. In particular L has at most $q+1$ extensions of rank q which are contained in E.

Since $\zeta_q \in L$ we can, by Kummer's theory and by the preceeding paragraph, choose $c \in L$ such that $M = L(\sqrt[q]{c})$ is a cyclic extension of L of degree q which is not contained in E, and which therefore satisfies $M \cap E = L$.

By Zorn's Lemma, E has a maximal extension F such that $M \cap F = L$. In terms of Galois theory, $G(F)$ is a minimal closed subgroup of $G(L)$ which the restriction map maps onto $\mathcal{G}(M/L)$. Hence $G(F)$ is the universal Frattini cover of $\mathcal{G}(M/L)$, which is $\mathbb{Z}_q$ [FJ, Example 20.39].

Let now $\overline{K}$ be a p–adic closure of K and let w be the unique valuation of $\overline{K}F$. For each finite extension N of $\overline{K}F$ we have $[N : \overline{K}F] = ef$ where e is the ramification index and f is the residue degree of the extension [R, p. 136]. By the preceeding paragraph, these two numbers are q–powers. On the other hand, $\overline{K}F$ contains all the elements $\sqrt[q^n]{p}$. Hence $w((\overline{K}F)^\times)$, as a subgroup of $\mathbb{Q}$, is q–divisible. Thus $e = 1$. Also, the residue field of $\overline{K}F$ contains all the roots of unity ζ_{q^n} and hence also the maximal q–extension of $\mathbb{F}_p$. Hence q does not divide f . Thus $f = 1$ and $N = \overline{K}F$. Conclude that $\overline{K}F = \tilde{\mathbb{Q}}$.
∎

We combine Lemmas 1.10 and 1.11 together:

PROPOSITION 1.12: *Let K be a proper subfield of the p–adic closure $\mathbb{Q}_{p,\mathrm{alg}}$ of $\mathbb{Q}$. Then K has an algebraic extension, different from $\tilde{\mathbb{Q}}$ such that $\overline{K}F = \tilde{\mathbb{Q}}$ for each p–adic closure $\overline{K}$ of K.*

Proof: $\mathbb{Q}_{p,\mathrm{alg}}$ induces a p–adic valuation v on K. If v is the unique p–adic valuation of K use Lemma 1.10, otherwise use Lemma 1.11. ∎

We are now ready to prove the main result of this note.

Theorem 1.13: *Let L an algebraic extension of a* PpC *field K. Suppose that L is Henselian with respect to a valuation v. Then L is separably closed or it contains a p–adic closure $\overline{K}$ of K and v is coarser than the unique extension of the p–adic valuation of $\overline{K}$ to L.*

Proof: Let $\tilde{v}$ be the unique extension of v to $\widetilde{K}$. Then the decomposition field, L_0, of $\tilde{v}$ is contained in L. Hence, it suffices to prove that L_0 is separably closed or p–adically closed. So, we replace L by L_0 if necessary to assume that (L, v) is the Henselization of $(K, \mathrm{res}_K v)$ and prove that L is separably closed or p–adically closed.

To that end, consider a p–adic closure $\overline{K}$ of K and let v_p be the p–adic valuation of $\overline{K}$. Then $(\overline{K}, v_p)$ is the Henselization of $(K, \mathrm{res}_K v_p)$ [J, Thm. 10.8]. In other words, $\overline{K}$ is the decomposition field of the unique extension $\tilde{v}_p$ of v_p to $\widetilde{K}$.
Claim: $L \subseteq \overline{K}$ or $L\overline{K} = \widetilde{K}$.

Indeed, suppose that $M = L\overline{K} \neq \widetilde{K}$. Let v_M (resp., $v_{p,M}$) be the unique extension of v (resp., v_p) to M. Then (M, v_M) is Henselian. By Lemma 1.6, v_M is coarser than $v_{p,M}$. Hence, by Lemma 1.2, $L \subseteq \overline{K}$, and the claim has been proved.

It follows from the claim by Proposition D that L is PpC. If L has no p–adic closure, then L is PAC and Henselian. By Proposition A, L is separably closed and our theorem holds. Otherwise L has a p–adic closure $\overline{L}$. Hence $L_0 = \tilde{\mathbb{Q}} \cap L$ is contained in the p–adically closed field $\overline{L}_0 = \tilde{\mathbb{Q}} \cap \overline{L}$. By Lemma 1.7, $\tilde{\mathbb{Q}}L = \widetilde{L}$. Hence, the restriction map res: $G(L) \to G(L_0)$ is an isomorphism. If $L \neq \overline{L}$, then $L_0 \neq \overline{L}_0$. Hence, by Proposition 1.12, L_0 has an algebraic extension F_0 such that $F_0 \neq \tilde{\mathbb{Q}}$ and $L_0' F_0 = \tilde{\mathbb{Q}}$ for each p–adic closure L_0' of L_0. The field $F = LF_0$ is an algebraic extension of L and $F \neq \widetilde{L}$. If L' is a p–adic closure of L, then $L_0' = \tilde{\mathbb{Q}} \cap L$ is a p–adic closure of L_0 and $L_0' L = L'$. Hence $L'F = \widetilde{L}$. By Lemma 1.7, $F = \widetilde{L}$. This contradiction proves that $L = \overline{L}$ is p–adically closed. ∎

The proof of Theorem 1.13 actually gives:

Theorem 1.14: *Let K be a* PpC *field and let v be a valuation of K. Then the Henselization of K with respect to v is either algebraically closed or p–adically closed and v is coarser than a p–adic valuation of K.*

Corollary 1.15: *If K is* PpC *but neither algebraically closed nor p–adically closed, then K is not Henselian.*

2. Compositum of p-adically closed fields.

One of the distinguished properties of PpC fields is that the compositum of any distinct p–adic closures is algebraically closed. This is a consequence of [HJ, Lemma 4.5(b)]. We have noticed (Lemma 1.9) that the same statement also holds for algebraic fields. In this section we give an example which proves that this fails to be true for p–adic closures of an arbitrary field.

We start with a result which is a special case of Pop's theorem [Po, Thm. E9]. However, since its proof is elementary, in particular, unlike Pop's proof, it does not use cohomology, we include it here.

Proposition 2.1: *Let K be Henselian field with respect to a p-adic valuation v. Suppose that L is an algebraic extension of K such that $\tilde{\mathbb{Q}} \cap L = \mathbb{Q}_{p,\mathrm{alg}}$ and $\tilde{\mathbb{Q}} L = \widetilde{K}$. Then L is p-adically closed.*

Proof: We use results of Prestel and Roquette [PR]. However, to avoid conflict in terminology we use "$\mathfrak{p}$-valuation" for what they call "p-adic valuation".

First note that $\tilde{\mathbb{Q}} \cap K$ is Henselian and contained in $\mathbb{Q}_{p,\mathrm{alg}}$. Hence $\tilde{\mathbb{Q}} \cap K = \mathbb{Q}_{p,\mathrm{alg}}$. Now let L_0 be a finite extension of K contained in L. Then L_0 is a Henselian $\mathfrak{p}$-valued field with respect to the unique extension of v to L_0. Since $\mathbb{Q}_{p,\mathrm{alg}}$ is algebraically closed in L_0, both fields have the same residue fields [PR, p. 39, Lemma 3.5(i)] and $\mathbb{Q}_{p,\mathrm{alg}}$ contains a prime element of L_0. Thus $\mathbb{Q}_{p,\mathrm{alg}}$ and L_0 have the same p-rank, and therefore L_0 is p–adic.

Now let L_0 range over all finite extensions of K in L to conclude that the unique extension of v to L is p-adic.

Each finite proper extension of L is of the form $L(a)$, where $a \in \tilde{\mathbb{Q}} - \mathbb{Q}_{p,\mathrm{alg}}$ and therefore of $\mathfrak{p}$-rank greater than 1. This means that L is p-adically closed. ∎

Example 2.2: *Distinct p-adic closures of a field whose compositum is not algebraically closed.* Consider the field $K = \mathbb{Q}_p((t))$ of formal power series in t over $\mathbb{Q}_p$. It is Henselian with respect to the valuation w having $\mathbb{Q}_p[[t]]$ as its valuation ring. A finer valuation v of K has

$$O_p = \{\sum_{i=0}^{\infty} a_i t^i |\ a_i \in \mathbb{Q}_p,\ a_0 \in \mathbb{Z}_p\}$$

as its valuation ring. It is a p-adic valuation. In the notation of Section 1, $O_{\bar{v}} = \mathbb{Z}_p$. In particular v is Henselian (Lemma 1.2).

Choose two sequences, $\alpha_1, \alpha_2, \alpha_3, \ldots$ and $\beta_1, \beta_2, \beta_3, \ldots$, of elements of $\widetilde{K}$ such that $\alpha_n^n = \beta_n^n = t$, $\alpha_{mn}^m = \alpha_n$, $\beta_{mn}^m = \beta_n$, and $\alpha_n \neq \beta_n$ for every $n > 1$. Let $\overline{K}_1 = \bigcup_{n=1}^{\infty} K(\alpha_n)$, $\overline{K}_2 = \bigcup_{n=1}^{\infty} K(\beta_n)$, $K_{\mathrm{cycl}} = K(\zeta_n |\ n = 1, 2, 3, \ldots)$ (ζ_n is a primitive root of 1 of order n), and $N = \tilde{\mathbb{Q}}K$. Then $\tilde{\mathbb{Q}} \cap \overline{K}_i = \mathbb{Q}_{p,\mathrm{alg}}$ and $\tilde{\mathbb{Q}}\overline{K}_i = \widetilde{K}$ [GJ, Prop. 4.1]. By Proposition 2.2, $\overline{K}_i$ is p-adically closed, $i = 1, 2$.

Also, as $G(N) \cong \hat{\mathbb{Z}}$, $K(\alpha_n)$ is the unique extension of K of degree n which is contained in $\overline{K}_1$ and $K(\beta_n)$ is the unique extension of K of degree n which is contained in $\overline{K}_2$. If $K(\alpha_q) = K(\beta_q)$ for some prime number q, then $\zeta_q \in K_{\mathrm{cycl}} \cap K(\alpha_q) = K$. Since $\mathbb{Q}_p$ is algebraically closed in K, we have $\zeta_q \in \mathbb{Q}_p$. Hence $q | p-1$. We may therefore take $q > p-1$ and conclude that $K(\alpha_q) \neq K(\beta_q)$ and therefore $\overline{K}_1 \neq \overline{K}_2$.

In addition,

$$K(\alpha_n, \beta_n) \subseteq K(\alpha_n, \zeta_n) \subseteq K_{\mathrm{cycl}}(\alpha_n).$$

Since $\mathbb{Q}_{p,\mathrm{cycl}} \subset \tilde{\mathbb{Q}}_p$, we have

$$\overline{K}_1\overline{K}_2 \subseteq K_{\mathrm{cycl}}\overline{K}_1 = \mathbb{Q}_{p,\mathrm{cycl}}\overline{K}_1 \subset \widetilde{K}.$$

Hence $\overline{K}_1\overline{K}_2$ is not algebraically closed. ∎

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