ON STABLE FIELDS IN POSITIVE CHARACTERISTIC[1)]

by

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1) This work was partially supported by a grant from the G.I.F, the German Israeli Foundation for Scientific Research and Development.

2) This work was partially done while the second author enjoyed the hospsitality of the Institute for Advanced Study at Princeton, New Jersey.

Introduction

Recall that a field extension F/K is **regular** if F is linearly disjoint from $\widetilde{K}$ over K ($\widetilde{K}$ is the algebraic closure of K.) A regular field extension F/K is called **stable** if it has a separating transcendence base T such that the Galois hull $\widehat{F}$ of $F/K(T)$ is regular over K (or, alternatively, such that $\mathcal{G}(\widehat{F}/K(T)) \cong \mathcal{G}(\widetilde{K}\widehat{F}/\widetilde{K}(T))$. In this case we shall also say that F is **stable** over K and call T a **stabilizing base** of F/K.

A field K is **stable in dimension** r if every finitely generated regular extension F/K of transcendence degree r is stable. If this holds for each r we say that K is **stable**.

A field K is **pseudo algebraically closed** (**PAC**) if every absolutely irreducible nonempty variety V defined over K has a K-rational point.

It is proved in [FJ1] that every PAC field is stable. This theorem yields a strong approximation property for a large class of PAC fields:

Proposition A ([FJ1, Thm. 5.4]): *Let K be a countable Hilbertian field and let v be a valuation of K. Then for each positive integer e, for almost all $(\sigma_1, \ldots, \sigma_e) \in G(K)^e$, for every absolutely irreducible variety V defined over $\widetilde{K}(\sigma_1, \ldots, \sigma_e)$ and for every absolute value w of $\widetilde{K}$ that extends v the set $V(\widetilde{K}(\sigma_1, \ldots, \sigma_e))$ is w-dense in $V(\widetilde{K})$.*

Here $G(K)$ is the absolute Galois group of K and $\widetilde{K}(\sigma_1, \ldots, \sigma_e)$ is the fixed field in $\widetilde{K}$ of $(\sigma_1, \ldots, \sigma_e)$.

For arbitrary fields [FJ2, Lemma 1.1] reduces the stability of an infinite field K to stability in dimension 1:

Lemma B: *A sufficient condition for an infinite field K to be stable is that every extension L of K is stable in dimension 1.*

Then [FJ2, Lemma 2.1] gives a geometrical criterion for a field K to be stable in dimension 1:

Lemma C: *Let F be a function field of one variable over a field K. Suppose that F has a projective plane model Γ of degree n and there exists a K-rational point $O \in \mathbb{P}^2 - \Gamma$ such that*

(1) *every line that passes through O cuts Γ in at least $n-1$ points.*

Then F is stable over K.

If $\mathrm{char}(K) = 0$, then [FJ2, Lemmas 3.2 and 3.3] shows how to construct a plane model Γ with property (1). Then [FJ2] concludes from Lemma C that

THEOREM D ([FJ2, Thm. 3.4]): *Every field of characteristic 0 is stable.*

If $\mathbb{P}^1$ is a model of F/K, then we may take Γ in Lemma C to be $\mathbb{P}^1$. In general [FJ2, Lemma 3.2] proves that any **node curve** (i.e., a curve with only nodes as singularities) satisfies (1). The proof of [FJ2, Lemma 3.2] depends upon three properties that each irreducible plane curve in characteristic 0 which is not a line has:

(2a) Γ has only finitely many inflection points.

(2b) Γ has only finitely many **double tangents** (i.e., tangents in at least two distinct points).

(2c) Only finitely many tangents to Γ pass through each point of $\mathbb{P}^2$ (i.e., Γ has no strange points).

In arbitrary characteristic we say that a plane curve is **common** if it is a line or it satisfies Condition (2). The following theorem supplies a plane node (not necessarily common) model Γ over each infinite field.

PROPOSITION E ([A, Appendix II]): *Let Δ be a smooth absolutely irreducible curve in $\mathbb{P}^3$ defined over an infinite field K. Then the plane at infinity of $\mathbb{P}^3$ has a nonempty K-open subset U such that for each $\mathbf{o} \in U$ the projection Γ of Δ from $\mathbf{o}$ on $\mathbb{P}^2$ is a node curve.*

Although Abhyankar [A] also eliminates strange points from Γ, there seems to be no reference in the literature for a construction that will also include (2a) and (2b). The aim of the present work is to fill up this gap and to prove that if K is an infinite perfect field, then each function field F/K of one variable has a plane common node model Γ. This guaranties that the following theorem is true.

THEOREM F: *Every conservative function field F of one variable over an infinite field K is stable.*

As usual, F is conservative if its genus does not change under constant field extensions. Theorem F implies the following generalization of [FJ2, Thm. 4.4]:

THEOREM G: *Let K be a countable perfect separably Hilbertian field. Then K has a Galois extension N with the following properties:*

(3a) *$\mathcal{G}(N/K)$ is isomorphic to the direct product of infinitely many finite groups,*

(3b) *N is a Hilbertian field,*

(3c) *N is a PAC field, and*

(3d) *N contains no field of the form $\widetilde{K}(\sigma_1, \ldots, \sigma_e)$, with $\sigma_1, \ldots, \sigma_e \in G(K)^e$.*

We carry out the construction of a plane common node model Γ for a conservative function field F/K in several steps:

(4a) If the genus of F/K is 0 distinguish between two cases: either F/K has a prime divisor of degree 1 or it does not have one. In the first case F/K is rational and therefore choose Γ to be $\mathbb{P}^1$. In the latter case F/K has a conic section as a model. If $\mathrm{char}(K) \neq 2$, choose Γ to be this conic section. If $\mathrm{char}(K) = 2$ the conic section has a strange point. Birationally transform it to a common node curve of degree 4 with the explicit defining equation (5) of Section 7. Assume from now on that the genus of F/K is positive.

(4b) Choose $x_1, y_1 \in F$ such that $F = K(x_1, y_1)$ and F is separable over $K(x_1)$. If $\mathrm{char}(K) \neq 2$, choose $u \in K$ such that the curve Γ_u generated by $(x_1, y_1 + ux_1^2)$ has only finitely many inflection points (Lemma 3.3). Then Γ_u has only finitely many double tangents (Proposition 4.5). If $\mathrm{char}(K) = 2$ make sure that Γ_u has only finitely many double tangents by choosing u more carefully (Lemma 6.1).

(4c) Let $x_0{:}x_1{:}x_2$ be generic homogeneous point for the common model Γ_u of F/K. Use an appropriate very ample divisor to construct a smooth projective model Δ of F/K with generic point $x_0{:}x_1{:}x_2{:}\cdots{:}x_n$ and prove that it has only finitely many "inflection points" (see below) and only finitely many double tangents (Proposition 8.3).

(4d) Choose a noninflection point $\mathbf{a}$ of $\Delta(\widetilde{K})$ such that the tangent through $\mathbf{a}$ is not double and it intersects only finitely many tangents to Δ (Lemma 10.1).

(4e) Choose a point $\mathbf{o}$ in $\mathbb{P}^n(K)$ not on Δ such that the projection π from $\mathbf{o}$ into $\mathbb{P}^{n-1}$ maps Δ onto a model Λ of F/K in $\mathbb{P}^{n-1}$ such that $\pi(\mathbf{a})$ is a simple noninflection point of Λ with a nondouble tangent (Lemma 10.1).

(4f) If $n \geq 4$, then choose $\mathbf{o}$ such that in addition Λ is smooth and the tangent to Λ at $\pi(\mathbf{a})$ intersects only finitely many tangents of Λ. Then repeat step (4e).

(4g) If $n = 3$, then choose $\mathbf{o}$ such that in addition the only singular points of Λ are nodes and the tangent at $\pi(\mathbf{a})$ goes through no singular point. Apply Corollary 3.2 and Lemma 4.3 to conclude that $\Gamma = \Lambda$ has only finitely many inflection points and only finitely many double tangents. By Lemma 5.4, Γ has no strange points. So Γ is the desired plane model for F/K.

We use F.K. Schmidt's modified derivatives to define inflection points in higher dimensional space (Section 2). Thus, if $\mathbf{x} = x_0{:}\cdots{:}x_n$ is a generic point for Δ with $x_i \in F$ and P is a prime divisor of F/K with center $\mathbf{b}$ at Δ, then $\mathbf{b}$ is an **inflection** point if $\mathrm{rank}(\mathbf{x}(P)\ \ \mathbf{x}'(P)\ \ \mathbf{x}^{[2]}(P)) < 3$. Here $\mathbf{x}^{[2]} = x_0^{[2]}{:}\cdots{:}x_n^{[2]}$ and $x_i^{[2]}$ is the modified derivative of F/K of order 2.

In analogy to the one dimensional case we call an algebraic function field F/K of several variables **conservative** if it has a model V which is normal over $\widetilde{K}$. The intersection with a hyper plane technique reduces the stability question of F/K to the one dimensional case, which is covered by Theorem F.

THEOREM H: *Let F be a finitely generated regular conservative extension of an infinite field K. Then F/K is a stable extension.*

COROLLARY I: *Each infinite perfect field K is stable.*

1. Basic concepts of plane curves.

The main tool to analyze a plane curve is its intersection multiplicity with a line. So we first fix our convention for curves and expose about intersection multiplicity and related concepts.

We consider the affine plane $\mathring{A}^2$ as a subspace of the projective plane $\mathbb{P}^2$. A point P of $\mathring{A}^2$ can be given either by its affine coordinates (x, y) or by its homogeneous coordinates $x_0{:}x_1{:}x_2$, where $x = x_1/x_0$, $y = x_2/x_0$ and $x_0 \neq 0$. All elements belong to some universal extension Ω of a fixed basic field K. As usual Ω is an algebraically closed extension of K of infinite transcendence degree. Refer to the points of $\mathring{A}^2$ as **finite**. Each point of $\mathbb{P}^2 - \mathring{A}^2$ is **infinite** and has the form $0{:}x_1{:}x_2$.

By a **plane curve** we always mean an absolutely irreducible projective curve Γ defined over K. It is the set of all points $x_0{:}x_1{:}x_2$ that satisfy $f(x_0, x_1, x_2) = 0$ for some absolutely irreducible homogeneous polynomial $f \in K[X_0, X_1, X_2]$. The **degree** of Γ is $\deg(f)$. The **finite part** of Γ is the set of all points (x, y) such that $f(1, x, y) = 0$.

Conversely let $g \in K[X, Y]$ be an absolutely irreducible polynomial of degree d. Then the equation $g(X, Y) = 0$ defines the finite part of a plane curve Γ of degree d. The corresponding homogeneous polynomial is $f(X_0, X_1, X_2) = X_0^d g(X_1/X_0, X_2/X_0)$.

A point (x, y) of Γ is **generic** if $F = K(x, y)$ has transcendence degree 1 over K. In this case F is the **function field** of Γ, and Γ is a **plane model** of F/K. In homogeneous terms we write $F = K(x_0{:}x_1{:}x_2)$. A point (a, b) of Γ is **algebraic** if $a, b \in \widetilde{K}$. The **local ring** of Γ at (a, b) (over $\widetilde{K}$) is

$$\mathcal{O}_{\Gamma,(a,b)} = \Big\{\frac{p(x, y)}{q(x, y)} \mid p, q \in \widetilde{K}[X, Y] \text{ and } q(a, b) \neq 0\Big\}.$$

In general it is important to note that while the curves we consider and the birational transformations we perform are all defined over the basic field K, the geometrical study of these objects is carried out over $\widetilde{K}$.

The point (a, b) is **simple** if $\frac{\partial g}{\partial X}(a, b) \neq 0$ or $\frac{\partial g}{\partial Y}(a, b) \neq 0$. The corresponding homogeneous condition is that at least one of the expressions $\frac{\partial f}{\partial X_i}(a_0, a_1, a_2)$, $i = 0, 1, 2$, is nonzero. Thus Γ has only finitely many singular points. A necessary and sufficient condition for (a, b) to be simple is that $\mathcal{O}_{\Gamma,(a,b)}$ is integrally closed [FJ3, Lemma 4.3]. In this case $\mathcal{O}_{\Gamma,(a,b)}$ is a discrete valuation ring. Let t be a prime element of $\mathcal{O}_{\Gamma,(a,b)}$ (also called a **local parameter** of Γ at (a,b)). Then the completion of $\mathcal{O}_{\Gamma,(a,b)}$ is the ring of power series $\widetilde{K}[[t]]$ over $\widetilde{K}$. In particular x and y has unique expansions as power series in t with coefficients in $\widetilde{K}$:

$$x = a + \alpha_1 t + \alpha_2 t^2 + \cdots \tag{1a}$$

$$y = b + \beta_1 t + \beta_2 t^2 + \cdots . \tag{1b}$$

The expansion in homogeneous coordinates takes the form

$$x_0 = a_0 + \alpha_{0,1} t + \alpha_{0,2} t^2 + \cdots \tag{2a}$$

$$x_1 = a_1 + \alpha_{1,1} t + \alpha_{1,2} t^2 + \cdots \tag{2b}$$

$$x_2 = a_2 + \alpha_{2,1} t + \alpha_{2,2} t^2 + \cdots . \tag{2c}$$

Here the expansion is unique up to a product with a common power series with nonzero constant term. Denote the canonical valuation of $\widetilde{K}[[t]]$ by ord_t.

A line L (in $\mathbb{P}^2$) is defined by an equation $c_0X_0 + c_1X_1 + c_2X_2 = 0$ where at least one c_i is not 0. The affine part of L is given by $c_0 + c_1X + c_2Y = 0$. Thus $X_0 = 0$ defines the **line at infinity**.

Suppose that $\mathbf{a}{=}a_0{:}a_1{:}a_2$ and $\mathbf{b} = b_0{:}b_1{:}b_2$ are two distinct points of a line L. An arbitrary point of L has the form $\mathbf{a}t_0 + \mathbf{b}t_1$. Expand $f(\mathbf{a}+\mathbf{b}t)$ around $\mathbf{a}$:

$$f(\mathbf{a}+\mathbf{b}t) = f(\mathbf{a}) + \sum_{i=0}^{2} \frac{\partial f}{\partial X_i}(\mathbf{a}) b_i t + f_2(\mathbf{a},\mathbf{b}) t^2 + \cdots , \tag{3}$$

where $f_2 \in K[\mathbf{X},\mathbf{Y}]$ is a form of degree 2 in Y and a form of degree $\deg(f) - 2$ in X, etc. The **intersection multiplicity**, $\mathrm{i}(\Gamma, L; \mathbf{a})$, is the highest power of t that divides the right hand side of (3). It does not depend on $\mathbf{b}$.

A convenient way to compute the intersection multiplicity at a simple point $\mathbf{a}$ is to use the power series expansion. In the notation of (2) we have:

$$\mathrm{i}(\Gamma, L; \mathbf{a}) = \mathrm{ord}_t(c_0x_0 + c_1x_1 + c_2x_2) \tag{4}$$

(e.g., [S, p. 93]).

In affine coordinates a line L that goes through a simple point (a, b) of Γ has an equation $c_1(X-a)+c_2(Y-b)=0$. In the notation of (1) we have for (a, b) simple:

$$\mathrm{i}(\Gamma, L; (a,b)) = \mathrm{ord}_t(c_1(x-a)+c_2(y-b)). \tag{5}$$

If $\mathbf{a}$ is a simple point of Γ, then the line

$$\sum_{i=0}^{2} \frac{\partial f}{\partial X_i}(\mathbf{a})X_i = 0$$

intersects Γ at $\mathbf{a}$ with multiplicity at least 2. This is the **tangent** to Γ at $\mathbf{a}$. We denote it by $T_{\Gamma,\mathbf{a}}$. If $T_{\Gamma,\mathbf{a}} = T_{\Gamma,\mathbf{a}'}$ for another simple point $\mathbf{a}'$ of Γ, then $T_{\Gamma,\mathbf{a}}$ is a **double tangent**. The affine equation for the tangent to Γ through (a, b) is

$$\frac{\partial g}{\partial X}(a,b)(X-a) + \frac{\partial g}{\partial Y}(a,b)(Y-b) = 0.$$

The simple point $\mathbf{a}$ is an **inflection point** if $\mathrm{i}(\Gamma, T_{\Gamma,\mathbf{a}}; \mathbf{a}) \geq 3$. Observe that if Γ is a line, then $\Gamma = T_{\Gamma,\mathbf{a}}$ and $\mathrm{i}(\Gamma, T_{\Gamma,\mathbf{a}}; \mathbf{a}) = \infty$. Thus each point of a line is an inflection point. This is the usual convention in algebraic geometry (e.g., [S. p. 67]). The standard criterion for a simple point to be an inflection point is usually proved only in characteristic 0. However the proof is valid if $\mathrm{char}(K) \neq 2$ (e.g., [S, pp. 67–68]):

(6) Suppose that $\mathrm{char}(K) \neq 2$ and that $\deg(\Gamma) \geq 2$. Then a simple point $\mathbf{a}$ of Γ is a point of inflection if and only if it lies on the **Hessian curve** of Γ:

$$\det\left(\frac{\partial^2 f}{\partial X_i \partial X_j}\right) = 0.$$

In particular if the Hessian is not identically zero then Γ has only finitely many inflection points.

LEMMA 1.1: *Let (a, b) be a simple algebraic point of Γ. Then, in the notation of (1), (a,b) is an inflection point if and only if*

$$\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = 0. \tag{7}$$

Similarly, in homogeneous terms, a simple point $a_0{:}a_1{:}a_2 = \alpha_{00}{:}\alpha_{01}{:}\alpha_{02}$ of Γ is an inflection point if and only if

$$\det(\alpha_{ij})_{0 \leq i,j \leq 2} = 0. \tag{8}$$

Proof: We prove only the affine version of the lemma. Suppose first that (a, b) is an inflection point of Γ. Let $c_1 = \frac{\partial g}{\partial X}(a, b)$ and $c_2 = \frac{\partial g}{\partial Y}(a, b)$. By (5), $c_1(x-a)+c_2(y-b) \cong 0 \mod t^3$. Hence, by (1)

$$c_1\alpha_1 + c_2\beta_1 = 0 \tag{9a}$$

$$c_1\alpha_2 + c_2\beta_2 = 0. \tag{9b}$$

Since $c_1 \neq 0$ or $c_2 \neq 0$, (7) holds.

Conversely if (7) is true, then there exist c_1 and c_2, not both zero such that (9) holds. Then the line L defined by $c_1(X - a) + c_2(Y - b) = 0$ is the tangent to Γ at (a, b) and it intersects Γ at (a, b) with multiplicity at least 3. Conclude that (a, b) is an inflection point of Γ. ∎

Finally observe that if $\mathbf{a}$ is a singular point of Γ, then every line L through $\mathbf{a}$ intersects Γ at $\mathbf{a}$ with multiplicity at least 2. If, in the notation of (3), f_2 is not identically zero, then the intersection multiplicity is exactly 2 except when L is defined by one of the two linear factors of $f_2(X, Y)$. In the latter case L is called a tangent to Γ at $\mathbf{a}$. If Γ has two distinct tangents at $\mathbf{a}$, then $\mathbf{a}$ is a **node** of Γ.

André Weil [W] has established an alternative way to define intersection multiplicities. Let u_0, u_1, u_2 be algebraically independent elements over K and denote the line $u_1X + u_2Y = u_0$ by $L_{\mathbf{u}}$. Then $\Gamma \cap L_{\mathbf{u}}$ consists of d distinct points $\mathbf{p}_1, \ldots, \mathbf{p}_d$, where $d = \deg(\Gamma)$. These points are separably algebraic and are conjugate to each other over $K(\mathbf{u})$ [W, p. 118]. If w_0, w_1, w_2 are algebraic elements over K, then the K-specialization $\mathbf{u} \to \mathbf{w}$ extends to a K-specialization $(\mathbf{u}, \mathbf{p}_1, \ldots, \mathbf{p}_d) \to (\mathbf{w}, \mathbf{q}_1, \ldots, \mathbf{q}_d)$. Suppose that the line $L_{\mathbf{w}}$ defined by $w_1X + w_2Y = w_0$ does not coincide with Γ. Then, each point of $\Gamma \cap L_{\mathbf{w}}$ is, in Weil's terminology, a proper point of intersection of Γ and $L_{\mathbf{w}}$. Therefore theorem 2 on page 119 of [W] asserts that $\Gamma \cap L_{\mathbf{w}} = \{\mathbf{q}_1, \ldots, \mathbf{q}_d\}$. Moreover, for each $\mathbf{q} \in \{\mathbf{q}_1, \ldots, \mathbf{q}_d\}$ the intersection multiplicity of Γ and $L_{\mathbf{w}}$ at $\mathbf{q}$ à la Weil is

$$\mathrm{j}(\Gamma, L_{\mathbf{w}}; \mathbf{q}) = \#\{i \mid 1 \leq i \leq d, \;\; \mathbf{q}_i = \mathbf{q}\}.$$

LEMMA 1.2: *Let Γ be a plane curve and let L be a line which does not coincide with Γ. Then $\mathrm{j}(\Gamma, L; \mathbf{q}) = \mathrm{i}(\Gamma, L; \mathbf{q})$ for each point $\mathbf{q} \in \Gamma \cap L$.*

Proof: After a linear projective transformation we may assume that $\mathbf{q}$ is the origin and that L is the X-axis: $Y = 0$. In the above notation let $v = u_0/u_2$ and $u = -u_1/u_2$. Suppose as above that Γ is defined by an absolutely irreducible equation $g(X,Y) = 0$ of degree d. Consider the solutions $x_1, \ldots, x_d$ of the equation $g(X, uX + v) = 0$ in the algebraic closure of $K(u,v)$ and put $y_i = ux_i + v, \quad i = 1, \ldots, d$. Then $\mathbf{p}_i$ above can be taken to be $(x_i, y_i), \ i = 1, \ldots, d$. As the $\mathbf{p}_i$'s are distinct, separable and conjugate over $K(u_0, u_1, u_2)$, so are the x_i's over $K(u,v)$. Hence $g(X, uX + v)$ is an irreducible polynomial over $K(u,v)$. Let d' be the degree of $g(X,0)$. Factor $g(X,0)$ as

$$g(X,0) = c \prod_{i=1}^{d'} (X - a_i), \tag{11}$$

for some $d' \le d$ and $c \ne 0$. Any extension of $(u,v) \to (0,0)$ to a specialization of $(u, v, x_1, \ldots, x_d)$ can be renumerated to have the form $(0, 0, a_1, \ldots, a_{d'}, \infty, \ldots, \infty)$ [W, p. 34, Proposition 9]. In particular $\mathrm{j}(\Gamma, L; (0,0))$ is the number of the i's such that $a_i = 0$, i.e., the degree of the highest power of X that divides $g(X,0)$. But this is also $\mathrm{i}(\Gamma, L; (0,0))$ [S, p. 33]. Conclude that $\mathrm{j}(\Gamma, L; (0,0)) = \mathrm{i}(\Gamma, L; (0,0))$. ∎

2. Derivatives of higher order.

Consider a **function field of one variable** F/K (i.e., a finitely generated regular extension of transcendence degree 1). Let t be a separating transcendental element for F/K and embed F in the field of power series $\widetilde{K}((t))$. The formal derivation of power series $(\sum_{i=m}^{\infty} \alpha_i t^i)' = \sum_{i=m}^{\infty} i\alpha_i t^{i-1}$ defines a unique derivation of F such that $t' = 1$ and $\alpha' = 0$ for each $\alpha \in K$. Iteration of derivation leads to derivatives of higher order: $x^{(n)} = \left(x^{(n-1)}\right)'$. The efficiency of higer derivatives for the study of F/K and its models in characteristic $p > 0$ is however limited by the vanishing of the derivative of order p. For example, in characteristic 2 we have $(t^3)^{(2)} = 2 \cdot 3t = 0$.

F.K. Schmidt modifies in [HS] the definition of the second derivative by "dividing" 2 out of the last relation: $(t^3)^{[2]} = 3t$. More generally he defines for each nonnegative integer k a **derivative of order k with respect to t**. It is a function from F to F. We denote its value at an element x of F by $D^k x/Dt^k$ or by $x^{[k]}$, if it is clear from the context what t is. It satisfies the following rules:

(1a) The zero and the first derivatives: $x^{[0]} = x$, $x^{[1]} = x'$.

(1b) Vanishing on K: $a^{[k]} = 0$ for $a \in K$ and $k \geq 1$.

(1c) Linearity: $(x+y)^{[k]} = x^{[k]} + y^{[k]}$.

(1d) Rule for multiplication: $(xy)^{[k]} = \sum_{i=0}^{k} x^{[k-i]} y^{[i]}$; in particular $(ax)^{[k]} = ax^{[k]}$ for $a \in K$.

(1e) Derivatives at t: $t^{[0]} = t$, $t^{[1]} = 1$ and $t^{[k]} = 0$ for $k \geq 2$.

(1f) Powers of t: $(t^m)^{[k]} = \binom{m}{k} t^{m-k}$ for $k \geq 0$ and each integer m; in particular: $(t^{-1})^{[k]} = (-1)^k t^{-(k+1)}$.

(1g) Extension to power series: The system of higher derivatives can be extended to $\widetilde{K}((t))$: $(\sum_{m=r}^{\infty} \alpha_m t^m)^{[k]} = \sum_{m=r}^{\infty} \binom{m}{k} \alpha_m t^{m-k}$. In particular, if $x = \sum_{i=0}^{\infty} \alpha_i t^i$, then $x^{[k]} \cong \alpha_k \bmod t$ for each $k \geq 0$.

(1h) The chain rule: If u is another separating transcendental element for F/K, then

$$\frac{D^k x}{Du^k} = \sum_{j=1}^{k} \frac{D^j x}{Dt^j} g_{kj} \left(\frac{Dt}{Du}, \dots, \frac{D^{k-j+1} t}{Du^{k-j+1}} \right),$$

where $g_{kj} \in \mathbb{Z}[X_1, \dots, X_{k-j+1}]$ are universal polynomials. In particular

$$\frac{Dx}{Du} = \frac{Dx}{Dt} \cdot \frac{Dt}{Du} \quad \text{and} \quad \frac{D^2 x}{Du^2} = \frac{Dx}{Dt} \left(\frac{D^2 t}{Du^2} \right) + \frac{D^2 x}{Dt^2} \left(\frac{Dt}{Du} \right)^2.$$

F.K. Schmidt approaches the higher derivatives in two ways. The second one which starts on page 223 of [HS] is more elementary and more general. The map $a \mapsto a$ for $a \in K$ and $t \mapsto t+Y$ uniquely extends to a ring homomorphism $\delta = \delta_t \colon K[t] \to K[t][[Y]]$. In particular

$$\delta x \cong x \bmod Y \tag{2}$$

for each $x \in K[t]$. For $x \neq 0$ this implies that δx is a unit of $K(t)[[Y]]$. Hence δ uniquely extends to a ring homomorphism $\delta \colon K(t) \to K(t)[[Y]]$ such that (2) holds for each $x \in K((t))$. Next choose a primitive element z for the extension $F/K(t)$ and let $f = \mathrm{irr}(z, K(t))$. Then $f(z) = 0$ and $f'(z) \neq 0$. By (2), $(\delta f)(Z) \cong f(Z) \bmod Y$ and $(\delta f')(Z) \cong f'(Z) \bmod Y$. Hence $(\delta f)(z) \cong 0 \bmod Y$ and $(\delta f')(z) \not\cong 0 \bmod Y$. By Hensel's Lemma there exists $\delta z \in F[[Y]]$ such that $(\delta f)(\delta z) = 0$ and $\delta z \cong z \bmod Y$. It follows that δ uniquely extends to a ring homomorphism $\delta\colon F \to F[[Y]]$ such that (2) holds for each $x \in F$. Now define $x^{[k]}$ to be the coefficient of Y^k in the expansion of δx:

$$\delta x = \sum_{k=0}^{\infty} x^{[k]} Y^k. \tag{3}$$

The additivity and multiplicativity of δ, respectively imply the rules (1c) and (1d). Rule (1f) follows from (1d) by induction on m. Since $t^{[1]} = 1 = t'$, the rule $x^{[1]} = x'$ follows from the uniqueness of the derivative of F over K under the condition $t' = 1$ [L1, p. 184]. The rule $x^{[0]} = x$ is a reinterpretation of (2). Since the "kth derivative" on $\widetilde{K}((t))$ given in (1g) satisfies (1a)-(1f) for all $x \in \widetilde{K}((t))$ and since it coincides with the map $x \mapsto x^{[k]}$ on $K[t]$ the uniqueness of the extension implies that it also coincides with the map $x \mapsto x^{[k]}$ on F.

Finally to prove the chain rule (1h) rewrite (3) for u:

$$\delta_u x = \sum_{k=0}^{\infty} \frac{D^k x}{Du^k} Y^k. \tag{4}$$

Since u is a separating element for F/K, $Du/Dt \neq 0$ ([L1, p. 186] and (1a)). Therefore

$$\delta u - u = \sum_{k=1}^{\infty} \frac{D^k u}{Dt^k} Y^k = h(Y) \tag{5}$$

is an invertible power series. Let $\varepsilon\colon F \to F[[Y]]$ be the homomorphism:

$$\varepsilon x = \sum_{k=0}^{\infty} \frac{D^k x}{Dt^k} (h^{-1}(Y))^k. \tag{6}$$

Obviously $\varepsilon a = a$ for each $a \in K$ and $\varepsilon u = u + h(h^{-1}(Y)) = u + Y$. Since δ_u is the unique homomorphism with these properties $\varepsilon = \delta_u$. Now compute the coefficients of $h^{-1}(Y)$ in terms of $D^k u/Dt^k$ from (5), substitute in (6) and compare the coefficients of Y^k on the right sides of (4) and (6) to obtain (1h).

Each element x of $\widetilde{K}F$ (and therefore each higher derivative of x) can be viewed as a function from the space of all prime divisors of $\widetilde{K}F/\widetilde{K}$ into $\widetilde{K} \cup \{\infty\}$. We denote the value of x at a prime divisor P by $x(P)$. Let v be the valuation of F associated with P. For elements $x_0, \ldots, x_n \in \widetilde{K}F$, not all zero, choose $u \in \widetilde{K}F$ such that each $(ux_i)(P)$ is finite and at least one of them is nonzero. Then $v(u^{-1}) = \min_{0 \le i \le n}\{v(x_i)\}$. Hence, if w is another element of F such that each $(wx_i)(P)$ is finite and at least one of them is nonzero, then $v(u) = v(w)$. It follows for $\mathbf{x} = x_0{:}\cdots{:}x_n$ that $(w\mathbf{x})(P) = (wu^{-1})(P)\cdot(u\mathbf{x})(P)$. Thus we may define $\mathbf{x}(P)$ as the point of $\mathbb{P}^n(\widetilde{K})$ with homogeneous coordinates $(u\mathbf{x})(P)$ for each u as above. This point is the **center** of P at Γ.

Now apply (1g) to a local parameter t at P to conclude that $(ux_i)^{[k]}(P)$ is finite (possibly 0) for each i. However the $(n+1)$-tuples $(u\mathbf{x})^{[k]}(P)$ for various u are in general nonproportional to one another. Nevertheless in expressions involving $(u\mathbf{x})^{[k]}(P)$ which do not depend on u, we write $\mathbf{x}^{[k]}(P)$ for $(u\mathbf{x})^{[k]}(P)$. This is the case for (8) below.

Lemma 2.1: *Let $k_1, \ldots, k_r$ be nonnegative integers. Let $P_1, \ldots, P_r$ be prime divisors of F/K. For each j consider $\mathbf{x}^{[k_j]}(P_j)$ as a column of height $n+1$. Suppose that for each j, $1 \le j \le r$,*

(7) $$\{(0, P_j), \ldots, (k_j, P_j)\} \subseteq \{(k_1, P_1), \ldots, (k_r, P_r)\}.$$

Then

(8) $$\operatorname{rank}(\mathbf{x}^{[k_1]}(P_1)\ \ \mathbf{x}^{[k_2]}(P_2)\ \ \cdots\ \ \mathbf{x}^{[k_r]}(P_r)\ )$$

does not depend on the homogeneous coordinates of $\mathbf{x}$ used to define $\mathbf{x}^{[k_j]}(P_j)$, $j = 1, \ldots, r$, nor does it depend on t.

Proof: Suppose without loss that for each j every $x_i(P_j)$ is finite and at least one of them is nonzero. Then (8) is the dimension of the subspace V of $\widetilde{K}^{n+1}$ spanned by the columns of the matrix

(9) $$(\mathbf{x}^{[k_1]}(P)\ \ \mathbf{x}^{[k_2]}(P_2)\ \ \cdots\ \ \mathbf{x}^{[k_r]}(P_r)).$$

By symmetry it suffices to consider $u \in F$ such that $(u\mathbf{x})(P_1)$ is a homogeneous coordinate system for the point $\mathbf{x}(P_1)$ of $\mathbb{P}^n(\widetilde{K})$ and to prove that, as an element of $\widetilde{K}^{n+1}$, $(u\mathbf{x})^{[k_1]}(P_1)$ belongs to V.

Our assumption implies that both $u(P_1)$ and $\mathbf{x}(P_1)$ are finite. By (1d)

$$(u\mathbf{x})^{[k_1]}(P_1) = \sum_{i=0}^{k_1} u^{[k_1-i]}(P_1)\mathbf{x}^{[i]}(P_1). \tag{10}$$

Conclude from (7) that the right hand side and therefore the left hand side of (10) belongs to V.

To prove the independence of (8) from t let u be another separating transcendental element for F/K and consider the matrix

$$\left(\frac{D^{[k_1]}\mathbf{x}}{du^{k_1}}(P_1)\ \ \frac{D^{[k_2]}\mathbf{x}}{du^{k_2}}(P_2)\ \ \cdots\ \ \frac{D^{[k_1]}\mathbf{x}}{du^{k_1}}(P_1)\right). \tag{11}$$

Again, it suffices to show that each of the columns of this matrix belongs to V. This is however a consequence of (1h). ∎

Remark 2.2: Obviously the above proof works also for generic P_j's. So, if for each j,

$$\{0, \ldots, k_j\} \subseteq \{k_1, \ldots, k_r\}, \tag{12}$$

then

$$\mathrm{rank}(\mathbf{x}^{[k_1]}\ \ \mathbf{x}^{[k_2]}\ \ \cdots\ \ \mathbf{x}^{[k_r]}) \tag{13}$$

does not depend on the homogeneous coordinates of $\mathbf{x}$, nor does it depends on t.

3. Inflection points.

A plane curve defined over a field K of characteristic 0 has only finitely many inflection points. In contrast, if $\mathrm{char}(K) = p$ and $p \neq 0, 2$, then the curve Γ defined over K by $X_0^{p+1} + X_1^{p+1} + X_2^{p+1} = 0$ has no singular points. However the Hessian of Γ is identically 0. Hence, by (6) of Section 1, every point of Γ is an inflection point. Fortunately it is easy to find a birational transformation of Γ that maps Γ onto a curve with only finitely many inflection points.

Lemma 3.1: *Let Γ be a plane curve with function field F. Let $\mathbf{x} = x_0{:}x_1{:}x_2$ be a generic point of Γ with coordinates x_i in F. Suppose that P is a prime divisor of $\widetilde{K}F/\widetilde{K}$ whose center $\mathbf{a}$ is a simple point of Γ. Then $\mathbf{a}$ is a noninflection point of Γ if and only if* $\mathrm{rank}(\mathbf{x}(P)\ \ \mathbf{x}'(P)\ \ \mathbf{x}^{[2]}(P)) = 3$.

Proof: By Lemma 2.1 we may compute the rank with respect to a local parameter t of Γ at $\mathbf{a}$. Apply (1g) of Section 2 on the power series expansions (2) of Section 1 to conclude that

$$(\mathbf{x}(P)\ \ \mathbf{x}'(P)\ \ \mathbf{x}^{[2]}(P)) = \begin{pmatrix} a_0 & \alpha_{01} & \alpha_{02} \\ a_1 & \alpha_{11} & \alpha_{12} \\ a_2 & \alpha_{21} & \alpha_{22} \end{pmatrix}.$$

Thus, our assertion is just a reinterpretation of Lemma 1.1. ∎

Corollary 3.2: *In the notation of Lemma 3.1, if Γ has a simple noninflection point, then Γ has only finitely many inflection points.*

Proof: By Lemma 3.1, $\det(\mathbf{x}(P)\ \ \mathbf{x}'(P)\ \ \mathbf{x}^{[2]}(P)) \neq 0$, hence $\det(\mathbf{x}\ \ \mathbf{x}'\ \ \mathbf{x}^{[2]}) \neq 0$. Therefore

$$\det(\mathbf{x}(Q)\ \ \mathbf{x}'(Q)\ \ x^{[2]}(Q)) = 0$$

for only finitely many prime divisors Q of $\widetilde{K}F/\widetilde{K}$. Conclude from Lemma 3.1 that Γ has only finitely many inflection points. ∎

Lemma 3.3: *Let Γ be a plane curve defined by the absolutely irreducible equation $g(X, Y) = 0$. Suppose that $\partial g/\partial Y \neq 0$. Then there exists $u_0 \in \widetilde{K}$ such that for each $u \neq u_0$ the birational transformation*

$$x' = x \qquad \text{and} \qquad y' = y + ux^2 \tag{1}$$

maps Γ *onto a plane curve* Γ_u *with only finitely many inflection points.*

Proof: Let (x, y) be a generic point of Γ. Our assumptions imply that $\partial g/\partial Y$ does not vanish on Γ. Choose a simple algebraic point (a, b) of Γ such that $\frac{\partial g}{\partial Y}(a, b) \neq 0$. Let P be a prime divisor of $\widetilde{K}F/\widetilde{K}$ with center (a, b) and choose a local parameter t for P. Then $x - a$ and $y - b$ generate the unique maximal ideal of $O_{\Gamma,(a,b)}$. Since t lies in this ideal $\mathrm{ord}_t(x - a) = 1$ or $\mathrm{ord}_t(y - b) = 1$. If $\mathrm{ord}_t(x - a) > 1$ and $\mathrm{ord}_t(y - b) = 1$, then, $\mathbf{x}'(P) = 0$ and $\mathbf{y}'(P) \neq 0$ ((1g) of Section 2). Now take the derivative of $g(x, y) = 0$ at P to contradict the choice of (a, b):

$$\frac{\partial g}{\partial X}(a, b)\mathbf{x}'(P) + \frac{\partial g}{\partial Y}(a, b)\mathbf{y}'(P) = 0.$$

Thus we may take t to be $x - a$. In particular, in the expansion (1a) of Section 1 we may assume that $\alpha_1 = 1$ and $\alpha_2 = 0$.

Let $a' = a$ and $b' = b + ua^2$. Obviously the transformation (1) is biregular at (a, b), i.e., $O_{\Gamma,(a,b)} = O_{\Gamma_u,(a',b')}$. In particular (a', b') is a simple point of Γ_u and t is also a local parameter of Γ_u at (a', b').

Substitute (1) of Section 1 in (1) to obtain

$$x' \cong a' + t \bmod t^3$$
$$y' \cong b' + \beta_1' t + \beta_2' t^2 \bmod t^3$$

with $\beta_1' = \beta_1 + 2au$ and $\beta_2' = \beta_2 + u$. The corresponding determinant is $\beta_2 + u$. If $u \neq -\beta_2$, then the determinant does not vanish. By Lemma 1.1, (a', b') is a noninflection point of Γ_u.

Conclude from Lemma 3.2 that Γ_u has only finitely many inflection points. ∎

4. Double tangents.

In Section 3 we have shown how to transform a plane curve Γ onto a plane curve Γ' with only finitely many inflection points. It turns out that if $\mathrm{char}(K) \neq 2$, then Γ' has in addition only finitely many double tangents. The main tool to prove this statement is the dual curve. It is however also useful in characteristic 2.

Let Γ be a plane curve defined over a field K by the homogeneous absolutely irreducible equation $f(X_0, X_1, X_2) = 0$ of degree $d \geq 2$. Let $\mathbf{x} = x_0{:}x_1{:}x_2$ be a generic point of Γ over K. Then $x_i \neq 0$ for $i = 0, 1, 2$. At least one of the following expressions is nonzero:

$$x_i^* = \frac{\partial f}{\partial X_i}(\mathbf{x}), \qquad i = 0, 1, 2, \tag{1}$$

and $K(x_0^*{:}x_1^*{:}x_2^*)$ is a subfield of $K(\Gamma) = K(x_0{:}x_1{:}x_2)$.

Lemma 4.1: *The transcendence degree of $K(x_0^*{:}x_1^*{:}x_2^*)$ over K is 1.*

Proof: As in Section 1 let $g(X, Y) = f(1, X, Y)$. Then

$$f(X_0, X_1, X_2) = X_0^d g(X_1/X_0, X_2/X_0)$$

and with $x = x_1/x_0$, $y = x_2/x_0$, $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial X}(x, y)$ and $\frac{\partial g}{\partial y} = \frac{\partial g}{\partial Y}(x, y)$ we have

$$x_0^* = -x_0^{d-1}(x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y}), \qquad x_1^* = x_0^{d-1}\frac{\partial g}{\partial x} \qquad \text{and} \qquad x_2^* = x_0^{d-1}\frac{\partial g}{\partial y}. \tag{2}$$

Assume without loss that $\frac{\partial g}{\partial y} \neq 0$. Then

$$x_0^* : x_1^* : x_2^* = -x\frac{\partial g/\partial x}{\partial g/\partial y} - y : \frac{\partial g/\partial x}{\partial g/\partial y} : 1\,. \tag{3}$$

If both $\frac{\partial g/\partial x}{\partial g/\partial y} = a$ and $\frac{\partial g/\partial x}{\partial g/\partial y}x + y = c$ are algebraic, then $ax + y = c$ and Γ is a line, a contradiction. Conclude that the point (3) is transcendental over K. ∎

The curve Γ^* that $\mathbf{x}^* = x_0^*{:}x_1^*{:}x_2^*$ generates over K is the **dual curve** of Γ. For each $\mathbf{a} \in \Gamma$ let

$$a_i^* = \frac{\partial f}{\partial X_i}(\mathbf{a}), \qquad i = 0, 1, 2,$$

and $\mathbf{a}^* = a_0^*{:}a_1^*{:}a_2^*$. The rational map $\tau\colon \Gamma \to \Gamma^*$ given by $\tau(\mathbf{a}) = \mathbf{a}^*$ is defined at each simple point of Γ. Another useful point of view is to consider Γ^* as a subset of the dual projective plane $(\mathbb{P}^2)^*$. Each point $\mathbf{a}^*$ of Γ^* bijectively corresponds to the tangent $T_{\Gamma,\mathbf{a}}$: $a_0^*X_0 + a_1^*X_1 + a_2^*X_2 = 0$.

LEMMA 4.2: *The following conditions on a plane curve* Γ *which is not a line are equivalent:*

(a) Γ *has only finitely many double tangents,*

(b) $K(\Gamma)/K(\Gamma^*)$ *is a purely inseparable extension, and*

(c) *no double tangent goes through a generic point of* Γ.

Proof: Condition (a) means, in the above notation, that $\tau^{-1}(\tau(\mathbf{a})) = \{\mathbf{a}\}$ for almost all $\mathbf{a} \in \Gamma$. Condition (c) means that $\tau^{-1}(\tau(\mathbf{x})) = \{\mathbf{x}\}$ for a generic point $\mathbf{x}$ of Γ. By [L1, p. 90], both conditions are equivalent to (b). ∎

LEMMA 4.3: *If* Γ *has a simple noninflection point* $\mathbf{a}$ *such that* $T_{\Gamma,\mathbf{a}}$ *is not a double tangent nor does it go through a singular point of* Γ, *then* Γ *has only finitely many double tangents.*

Proof: Assume that Γ has infinitely many double tangents. Then, for $\mathbf{x}$ generic, $T_{\Gamma,\mathbf{x}}$ is a double tangent (Lemma 4.2). Let u_0, u_1, u_2 be algebraically independent elements over K. Denote the line $u_0X_0 + u_1X_1 + u_2X_2 = 0$ by $L_{\mathbf{u}}$. Let $\Gamma \cap L_{\mathbf{u}} = \{\mathbf{y}_1, \ldots, \mathbf{y}_d\}$, where $d = \deg(\Gamma)$. Consider the specialization $(\mathbf{u}, \mathbf{y}_1, \ldots, \mathbf{y}_d) \longrightarrow (\mathbf{x}^*, \mathbf{x}_1, \ldots, \mathbf{x}_d)$. By Lemma 1.2, there exist two pairs of integers between 1 and d, say $(1, 2)$ and $(3, 4)$, such that $\mathbf{x} = \mathbf{x}_1 = \mathbf{x}_2$ and $\mathbf{x}_3 = \mathbf{x}_4$ (here we use the above notation). Next extend $(\mathbf{x}^*, \mathbf{x}) \longrightarrow (\mathbf{a}^*, \mathbf{a})$ to a specialization $(\mathbf{x}^*, \mathbf{x}_1, \ldots, \mathbf{x}_d) \longrightarrow (\mathbf{a}^*, \mathbf{a}_1, \ldots, \mathbf{a}_d)$, where, necessarily $\mathbf{a} = \mathbf{a}_1 = \mathbf{a}_2$ and $\mathbf{a}_3 = \mathbf{a}_4$. Hence $(\mathbf{u}, \mathbf{y}_1, \ldots, \mathbf{y}_d) \longrightarrow (\mathbf{a}^*, \mathbf{a}_1, \ldots, \mathbf{a}_d)$ is also a specialization. Since $\mathbf{a}$ is a noninflection point Lemma 1.2 implies that $\mathbf{a} \neq \mathbf{a}_4$. As $T_{\Gamma,\mathbf{a}}$ intersects Γ at $\mathbf{a}_4$ with multiplicity at least 2 and as $\mathbf{a}_4$ is simple (by assumption), $T_{\Gamma,\mathbf{a}}$ tangents Γ at $\mathbf{a}_4$. Conclude that $T_{\Gamma,\mathbf{a}}$ is a double tangent. This contradiction proves that Γ has only finitely many double tangents. ∎

Here is a sufficient condition for a simple point of Γ to be an inflection points in characteristic $\neq 2$.

LEMMA 4.4: *Suppose that* $\mathrm{char}(K) = p \neq 2$. *Let* t *be a local parameter of* Γ *at a simple point* $\mathbf{a}$. *In the above notation and assumptions, if for* $p \neq 0$ *there exist* $u_1, u_2 \in \widetilde{K}((t))$ *such that* $x_i^* = x_0^* u_i^p$, $i = 1, 2$, *(resp., for* $p = 0$ *there exist* $c_1, c_2 \in \widetilde{K}$ *such that* $x_i^* = x_0^* c_i$, $i = 1, 2$*), then* $\mathbf{a}$ *is an inflection point of* K.

Proof: We carry out the proof only for $p \neq 0$. By (2), $\frac{\partial g}{\partial y} u_1^p = \frac{\partial g}{\partial x} u_2^p$. Consider the relation $g(x, y) = 0$ as an identity in t and take its derivative:

$$\frac{\partial g}{\partial x} x' + \frac{\partial g}{\partial y} y' = 0.$$

Thus

$$x' u_1^p + y' u_2^p = 0. \tag{4}$$

Take the derivative of (1) of Section 1:

$$x' = \alpha_1 + 2\alpha_2 t + 3\alpha_3 t^2 + \cdots \tag{5a}$$

$$y' = \beta_1 + 2\beta_2 t + 3\beta_3 t^2 + \cdots . \tag{5b}$$

Assume without loss that $\mathrm{ord}_t(u_2) \geq \mathrm{ord}_t(u_1)$. Then $u = u_2/u_1$ has nonnegative order, which means

$$u^p = \delta_0 + \delta_1 t^p + \delta_2 t^{2p} + \cdots \cong \delta_0 \bmod t^2,$$

with $\delta_i \in \widetilde{K}$. Also, (4) simplifies to $x' + y' u^p = 0$. By (5)

$$\alpha_1 + 2\alpha_2 t \cong -(\beta_1 + 2\beta_2 t)\delta_0 \bmod t^2.$$

Since $2 \neq 0$ in K conclude that $\alpha_1 = -\beta_1 \delta_0$ and $\alpha_2 = -\beta_2 \delta_0$. Hence

$$\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = 0 \ .$$

By Lemma 1.1, (a, b) is an inflection point of Γ. ∎

In characteristic 0 each curve which is not a line is birationally equivalent to its dual [Wa, p. 67]. A slight modification in v.d. Waerden's proof shows that under a certain restriction the same statement holds in each characteristic $\neq 2$.

PROPOSITION 4.5: *Let Γ be a plane curve defined over a field K of characteristic $p \neq 2$. Suppose that Γ has only finitely many inflection points. Then Γ is birationally equivalent to its dual curve Γ^*. In particular Γ has only finitely many double tangents.*

Proof: The second statement follows from the first one and from Lemma 4.2. So in the above notation it suffices to prove that $K(\Gamma) = K(\Gamma^*)$.

Let $a_0{:}a_1{:}a_2$ be a simple noninflection point of Γ and let t be a local parameter of Γ at $\mathbf{a}$. Expand x_0, x_1 and x_2 as power series in t. Then consider the equality

$$f(x_0, x_1, x_2) = 0 \tag{6}$$

as an identity in t. Take the derivative of (6) with respect to t:

$$x_0^* x_0' + x_1^* x_1' + x_2^* x_2' = 0. \tag{7a}$$

Now write down the relation which expresses that $\mathbf{x}$ lies on $T_{\Gamma,\mathbf{x}}$:

$$x_0^* x_0 + x_1^* x_1 + x_2^* x_2 = 0. \tag{7b}$$

Next use (1) to consider x_0^*, x_1^* and x_2^* as power series in t. Take the derivative of (7b) with respect to t and use (7a):

$$(x_0^*)' x_0 + (x_1^*)' x_1 + (x_2^*)' x_2 = 0. \tag{8}$$

Consider (7b) and (8) as a system of linear homogeneous equations in x_0, x_1 and x_2. The coefficients matrix of this system is

$$\begin{pmatrix} x_0^* & x_1^* & x_2^* \\ (x_0^*)' & (x_1^*)' & (x_2^*)' \end{pmatrix}. \tag{9}$$

If the rank of (9) is 1, and, say $x_0^* \neq 0$, then

$$\left(\frac{x_i^*}{x_0^*}\right)' = \frac{(x_i^*)' x_0^* - (x_0^*)' x_i^*}{(x_0^*)^2} = 0, \qquad i = 1, 2.$$

Hence $x_i^* = x_0^* u_i^p$ for some $u_i \in \widetilde{K}((t))$, $i = 1, 2$ if $p \neq 0$, or $x_i^* = x_0^* c_i$ for some $u_i \in \widetilde{K}$, $i = 1, 2$, if $p = 0$. By Lemma 4.4, $\mathbf{a}$ is an inflection point. This contradiction to the choice of $\mathbf{a}$ proves that the rank of (9) is 2.

Let $h(X_0, X_1, X_2) = 0$ be an absolutely irreducible homogeneous equation for Γ^* over K. In analogy to (1) let

$$x_i^{**} = \frac{\partial h}{\partial X_i}(x^*), \qquad i = 0, 1, 2.$$

Then write for $\mathbf{x}^*$ the relations that correspond to (7):

$$x_0^{**}(x_0^*)' + x_1^{**}(x_1^*)' + x_2^{**}(x_2^*)' = 0$$
$$x_0^{**}x_0^* + x_1^{**}x_1^* + x_2^{**}x_2^* = 0.$$

It follows that $x_0^{**}{:}x_1^{**}{:}x_2^{**}$ coincides with the unique solution of the system (7b)&(8). That is, $x_0^{**}{:}x_1^{**}{:}x_2^{**} = x_0{:}x_1{:}x_2$. Thus Γ^{**} is well defined and coincides with Γ. In particular $K(\Gamma^{**}) = K(\Gamma)$. Finally, the inclusion $K(\Gamma^{**}) \subseteq K(\Gamma^*) \subseteq K(\Gamma)$ implies that $K(\Gamma^*) = K(\Gamma)$. ∎

Example 4.6: *A plane curve with finitely many inflection points and infinitely many double tangents in characteristic 2.* Let K be a field of characteristic 2. Consider the plane curve defined by the homogeneous polynomial

$$f(X_0, X_1, X_2) = X_0^2X_1^2 + X_0^2X_2^2 + X_0X_1^2X_2 + X_1^4.$$

The corresponding nonhomogeneous polynomial is

$$g(X, Y) = X^2 + Y^2 + X^2Y + X^4.$$

Let (x, y) be a generic point of Γ over K. In the notation of (1)

$$x_0^* : x_1^* : x_2^* = x^2y : 0 : x^2.$$

Thus $x_0^*/x_2^* = y$ and $x_1^*/x_2^* = 0$. Hence $K(\Gamma) = K(x, y)$ and $K(\Gamma^*) = K(y)$. Note that $K(x, y)/K(x^2, y)$ is a purely inseparable quadratic extension while $K(x^2, y)/K(y)$ is a separable quadratic extension. By Lemma 4.2, Γ has infinitely many double tangents.

We show that Γ has only finitely many inflection points. Let $\mathbf{a} = 1{:}a_1{:}a_2$ be a simple point of Γ such that $a_1 \neq 0$ and $a_2 \neq 1$. Then $T_{\Gamma,\mathbf{a}}$ is the line $X_2 = a_2X_0$. Another point that lies on this tangent is $\mathbf{b} = 1{:}0{:}a_2$. Thus

$$\begin{aligned} f(\mathbf{a} + \mathbf{b}t) &= (1 + t^2)a_1^2 + (1 + t^4)a_2^2 + (1 + t^2)a_1^2a_2 + a_1^4 \\ &= a_1^2(1 + a_2)t^2 + a_2^2t^4, \end{aligned}$$

and the coefficient of t^2 is nonzero. Hence $\mathrm{i}(\Gamma, T_{\Gamma,\mathbf{a}}; \mathbf{a}) = 2$. Conclude that $\mathbf{a}$ is a noninflection point of Γ. By Lemma 3.1, Γ has only finitely many inflection points. ∎

5. Strange points.

A point $\mathbf{p} \in \mathbb{P}^2$ is **strange** with respect to a curve Γ if infinitely many tangents to Γ go through $\mathbf{p}$ (We also say that $\mathbf{p}$ is a strange point of Γ.) For example if $\mathrm{char}(K) = p \neq 0$ and Γ is defined by

$$X_0^{p-1} X_2 + X_1^p = 0$$

and $a_0{:}a_1{:}a_2 \in \Gamma$ with $a_0 \neq 0$, then $\mathbf{a}$ is simple and the equation for $T_{\Gamma,\mathbf{a}}$ is $a_0 X_2 = a_2 X_0$. Thus, all tangents to Γ at finite points go through $(0,1,0)$. So $(0,1,0)$ is a strange point of Γ.

LEMMA 5.1: *Let Γ be a plane curve with homogeneous equation $f(X_0, X_1, X_2) = 0$ and generic point $x_0{:}x_1{:}x_2$ (resp. nonhomogeneous equation $g(X, Y) = 0$ and generic point (x, y)) over K. Suppose that Γ is not a line. Then Γ has at most one strange point, which must be K-rational. A necessary and sufficient condition for Γ to have a strange point is that $\frac{\partial f}{\partial X_0}$, $\frac{\partial f}{\partial X_1}$ and $\frac{\partial f}{\partial X_2}$ (resp., $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$) are linearly dependent over $\widetilde{K}$ (or, equivalently, over K). A point $p_0{:}p_1{:}p_2$ is strange if and only if it satisfies (2) or (3) below.*

Proof: Suppose that $\mathbf{p} = p_0{:}p_1{:}p_2$ is a strange point with respect to Γ. Then infinitely many points of Γ lie on the curve

$$\frac{\partial f}{\partial X_0} p_0 + \frac{\partial f}{\partial X_1} p_1 + \frac{\partial f}{\partial X_2} p_2 = 0.$$

Since Γ is absolutely irreducible each point of Γ lies on this curve. In other words, each tangent of Γ goes through $\mathbf{p}$. In particular if $x_0{:}x_1{:}x_2$ is a homogeneous generic point for Γ, then

$$\frac{\partial f}{\partial x_0} p_0 + \frac{\partial f}{\partial x_1} p_1 + \frac{\partial f}{\partial x_2} p_2 = 0. \tag{1}$$

Since $f(X_0, X_1, X_2)$ is irreducible over $\widetilde{K}$ it divides $\frac{\partial f}{\partial X_0} p_0 + \frac{\partial f}{\partial X_1} p_1 + \frac{\partial f}{\partial X_2} p_2$. But the degree of the latter polynomial is less than $\deg(f)$. Hence

$$\frac{\partial f}{\partial X_0} p_0 + \frac{\partial f}{\partial X_1} p_1 + \frac{\partial f}{\partial X_2} p_2 = 0. \tag{2}$$

By (2) of Section 4, condition (1) means in affine terms that

$$-(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y})p_0+\frac{\partial f}{\partial x}p_1+\frac{\partial f}{\partial y}p_2=0, \tag{3}$$

i.e., $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are linearly dependent over $\widetilde{K}$.

Let T be a tangent to Γ and let $\mathbf{a}$ be a simple point of Γ that does not lie on T. Then $T_{\Gamma,\mathbf{a}}$ is different from T and goes through $\mathbf{p}$. Hence $\mathbf{p}$ is uniquely determined. Since $K(x,y)$ is linearly disjoint from $\widetilde{K}$ over K, the elements $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are linearly dependent over K. Thus p must be K-rational.

Conversely, if Γ is not a line and $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are linearly dependent over $\widetilde{K}$, then they are linearly dependent over K and there exists a point $\mathbf{p}$ in $\mathbb{P}^2(K)$ such that (2) holds. This point is strange with respect to Γ. ∎

Remark 5.2: *Inflection points and strange points.* The above proof shows that if Γ is not a line, then a necessary and sufficient condition for Γ to have a strange point is that the dual curve Γ^* is a line. Proposition 4.5 therefore shows that if $\mathrm{char}(K)\neq 2$, $K(\Gamma)$ is nonrational function field and Γ has only finitely many inflection points, then Γ has no strange points. In particular strange points may appear only in positive characteristic.

∎

Remark 5.3: *Infinite strange points.* Similar (but simpler) arguments to those given in the proof of Lemma 5.1 give some details about the existence of infinite strange points:

(a) 0:0:1 is not a strange point of $\Gamma \leftrightarrow \frac{\partial f}{\partial y}\neq 0$.
(b) 0:1:0 is not a strange point of $\Gamma \leftrightarrow \frac{\partial f}{\partial x}\neq 0$.
(c) Γ has no infinite strange point $\leftrightarrow \frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are linearly independent over $\widetilde{K}$.

Lemma 5.4: *Let Γ be a plane curve defined over a field K. Suppose that $K(\Gamma)/K$ is a conservative function field of one variable and of positive genus and that Γ has only finitely many double tangents. Then Γ has no strange point.*

Proof: Let Γ^* be the dual curve to Γ. By Lemma 4.2, $K(\Gamma)/K(\Gamma^*)$ is a purely inseparable extension. If Γ had a strange point, then $K(\Gamma^*)=K(t)$ for some transcendental element t over K (Remark 5.2). Let q be a power of $\mathrm{char}(K)$ such that

$\widetilde{K}(\Gamma)^q \subseteq \widetilde{K}(t)$. Then $\widetilde{K}(\Gamma) \subseteq \widetilde{K}(t^{1/q})$. Therefore the genus of $\widetilde{K}(\Gamma)/\widetilde{K}$ would be 0 (e.g., by the Riemann-Hurwitz genus formula). Since $K(\Gamma)/K$ is conservative its genus would also be 0. This contradiction proves that Γ has no strange point. ∎

6. Double tangents in characteristic 2.

Example 4.6 makes it necessary in characteristic 2 to give a substitute for Proposition 4.5. We show how to choose u such that the birational transformation (1) of Section 3 yields a curve with not only finitely many inflection points but also with only finitely many double tangents.

LEMMA 6.1: *Let Γ be a plane curve which is defined over a field K of characteristic 2 by the absolutely irreducible equation $g(X,Y)=0$. Suppose that Γ is not K-birationally equivalent to a line, and that no infinite point of $\mathbb{P}^2$ is strange with respect to Γ. Then K has a cofinite subset U such that for each $u \in U$ the birational transformation*

$$\varphi\colon \qquad X' = X, \qquad Y' = Y + uX^2 \tag{1}$$

maps Γ onto a curve Γ' with only finitely many double tangents.

Proof: The infinite points of Γ are algebraic. Hence the tangents to Γ at these points are defined over $\widetilde{K}$. Let (x,y) be a generic point of Γ over K. If one of these tangents went through (x,y), then each point of Γ would lie on this tangent. This would imply that Γ is a line, contrary to our assumption. Conclude that (x,y) lies on no tangent to Γ through an infinite point.

Consider the tangent $T_{\Gamma,(x,y)}$ of Γ at (x,y):

$$\frac{\partial g}{\partial x}(X-x) + \frac{\partial g}{\partial y}(Y-y) = 0.$$

Let S be the set of all simple points of Γ with tangents parallel to $T_{\Gamma,(x,y)}$. If S were infinite, then the common point at infinity to all of these tangents would be strange with respect to Γ. This contradiction to the assumption of the lemma shows that S is a finite set.

Let U the set of all $u \in K$ such that

$$\frac{\partial g}{\partial x}(a-x) + \frac{\partial g}{\partial y}(b-y+u(a^2-x^2)) \neq 0 \tag{2}$$

for each (a,b) in $S - \{(x,y)\}$. Since $\partial g/\partial y \neq 0$ (Remark 5.3(a)), U is cofinite in K.

Let $u \in U$. Then φ maps Γ onto a plane curve Γ'. The defining polynomial of Γ' is $g'(X',Y') = g(X', Y' - uX'^2)$. Also, φ maps (x,y) onto a generic point (x',y') with $x' = x$ and $y' = y + ux^2$. The partial derivatives satisfy the following relations:

$$\frac{\partial g'}{\partial x'} = \frac{\partial g}{\partial X}(x', y' - ux'^2) = \frac{\partial g}{\partial x} \quad \text{and} \quad \frac{\partial g'}{\partial y'} = \frac{\partial g}{\partial y}.$$

Hence, the tangent $T_{\Gamma',(x',y')}$ of Γ' at (x',y') has the form:

$$\frac{\partial g}{\partial x}(X'-x') + \frac{\partial g}{\partial y}(Y'-y') = 0. \tag{3}$$

By assumption Γ' is not a line, and, as before, no tangent to Γ' at an infinite point goes through (x',y').

Assume that $T_{\Gamma',(x',y')}$ is also a tangent to Γ' at another simple point. Then this point, say (a',b'), is finite (and generic). Let (a,b) be the point of Γ which φ maps onto (a',b'). Then $(x,y) \neq (a,b)$. We have

$$T_{\Gamma',(x',y')} = T_{\Gamma',(a',b')}. \tag{4}$$

In particular $\partial g/\partial b \neq 0$ and $T_{\Gamma',(x',y')}$ is also given by the equation

$$\frac{\partial g}{\partial a}(X'-a') + \frac{\partial g}{\partial b}(Y'-b') = 0. \tag{5}$$

By (3), (4) and (5)

$$\frac{\partial g}{\partial x} : \frac{\partial g}{\partial y} = \frac{\partial g}{\partial a} : \frac{\partial g}{\partial b}.$$

Hence $T_{\Gamma,(x,y)}$ is parallel to $T_{\Gamma,(a,b)}$ and therefore (2) holds for (a,b):

$$\frac{\partial f}{\partial x}(a'-x') + \frac{\partial f}{\partial y}(b'-y') \neq 0. \tag{6}$$

On the other hand (4) implies that $(a',b') \in T_{\Gamma',(x',y')}$. Substitute (a',b') in (3) to obtain (6) with left hand side equals to 0. This contradiction shows that $T_{\Gamma',(x',y')}$ is not a double tangent.

Conclude from Lemma 4.2 that Γ' has only finitely many double tangents. ∎

REMARK 6.2: The Proposition and its proof remain valid in arbitrary characteristic p if the transformation (1) is replaced by $X' = X,\ Y' = Y + uX^p$. ∎

7. Function fields of genus 0.

The combination of Remark 5.2 and Lemma 5.4 allows us not to worry about strange points if the genus of F/K is positive. In the special case where the genus of F/K is 0 this section gives an explicit plane node model for F/K with only finitely many inflection points, with only finitely many double tangents and with no strange point. We distinguish between two cases.

Case A: $\mathrm{char}(K) \neq 2$. There are two subcases:

Case A1: *F/K has a prime divisor of degree 1.* Then F is a rational function field [D, p. 50]. Any nonsingular conic section Γ with a rational point will be a model for F/K with the desired properties. Define for example Γ by the equation $Y^2 + X^2 + X = 0$. If (x, y) is a generic point of Γ, then $K(x, y) = K(y/x)$ is a rational function field. Since $\deg(\Gamma) = 2$, the curve Γ has no inflection point and no double tangent. The partial derivatives of the homogeneous polynomial $f(X_0, X_1, X_2) = X_2^2 + X_1^2 + X_1X_0$ for Γ are $\partial f/\partial X_0 = X_1$, $\partial f/\partial X_1 = X_0 + 2X_1$, and $\partial f/\partial X_2 = 2X_2$. As they are linearly independent Γ has no strange point (Lemma 5.1). Since their only common zero is $(0, 0, 0)$, Γ is nonsingular.

Case A2: *F/K has no prime divisor of degree 1.* Then $F = K(x, y)$, where (x, y) generates a conic section Γ over K with an absolutely irreducible defining polynomial

$$g(X, Y) = aX^2 + bXY + cY^2 + dX + eY + f \tag{1}$$

with coefficients in K and without a K-zero [E, p. 139]. Use a linear transformation on (X, Y) to rewrite g in the form $g(X, Y) = aX^2 + bY^2 + c$ with nonzero $a, b, c \in K$. As in Case A1, Γ has no inflection point, no double tangent, no singular point, and no strange point.

Case B: $\mathrm{char}(K) = 2$. Again there are two subcases.

Case B1: *F/K has a prime divisor of degree 1.* Then F is a rational function field but the conic sections of Case A1 will not work since they have strange point in characteristic 2. Therefore let Γ be the cubic: $Y^2 + XY + X^2 + X^3 = 0$. If (x, y) is a generic point of

Γ over K, then $K(x,y) = K(x/y)$. Hence Γ is a model for F/K. The tangent $T_{\Gamma,(x,y)}$ is defined by $(y+x^2)X + xY = x^3$. It intersects Γ twice in (x,y) and once in (x^2,y^2). So, (x,y) is a noninflection point of Γ. By Corollary 3.2, Γ has only finitely many inflection points. Since the degree of Γ is 3 it can not have double tangents (e.g., by Bezout's theorem). Finally, the only singular point of Γ is $(0,0)$ and it is a node.

Case B2: *F/K has no prime divisor of degree 1.* In particular F is nonrational. As in Case A2, $F = K(x,y)$ where (x,y) generates a conic section Γ over K: $g(X,Y) = 0$ with g given by (1).

Lemma 7.1: *$F = K(x,y)$ where*

$$y^2 + y = ax^2 + b, \tag{2}$$

with elements a and b of K such that $a \notin K^2$ and $b \neq 0$.

Proof: Compute from (1)

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = dx + ey, \qquad \frac{\partial g}{\partial x} = by + d \quad \text{and} \quad \frac{\partial g}{\partial y} = bx + e.$$

These elements are linearly dependent over K:

$$b(dx + ey) + e(by + d) + d(bx + e) = 0. \tag{3}$$

Indeed, since g is not a square in $\widetilde{K}[X,Y]$ at least one of the elements b, e or d is nonzero. The point $b{:}e{:}d$ is strange with respect to Γ (Lemma 5.1). Apply an appropriate projective transformation over K to transform Γ to a curve with 0:1:0 as its strange point. This brings g to the form $g(X,Y) = aX^2 + cY^2 + Y + f$. If $c = 0$, then $F = K(x)$, a contradiction. Hence $c \neq 0$, and we can first multiply g by c and then replace cY by Y to put g in the desired form.

Finally, if in (2) we had $a = a_1^2$ with $a_1 \in K$, then $F = K(y + a_1 x)$. If $b = 0$, then $F = K(y/x)$. Both conclusions contradict the assumption. So, $a \notin K^2$ and $b \neq 0$. ∎

A simple computation shows that the conic section (2) has no singular points. In order to eliminate its strange point we apply the birational transformation

$$x = \frac{v}{u(1+v)}, \qquad y = \frac{u+v}{u(1+v)} \tag{4}$$

whose inverse is

$$u = \frac{1+x+y}{x}\,, \qquad v = \frac{1+x+y}{x+y}\,.$$

So F/K has a a plane model Γ' with generic point (u,v) that satisfies the following absolutely irreducible equation of degree 4:

$$bu^2v^2 + u^2v + uv^2 + bu^2 + uv + sv^2 = 0 \tag{5}$$

with $b \neq 0$ and $s = a+1$. The corresponding homogeneous polynomial is

$$f(U_0,U_1,U_2) = bU_1^2U_2^2 + U_0U_1^2U_2 + U_0U_1U_2^2 + bU_0^2U_1^2 + U_0^2U_1U_2 + sU_0^2U_2^2.$$

Lemma 7.2: *The curve* Γ' *has three singular points,* 1:0:0, 0:1:0 *and* 0:0:1. *All three of them are nodes.*

Proof: Compute the partial derivatives of f:

$$\frac{\partial f}{\partial U_0} = U_1U_2(U_1+U_2), \quad \frac{\partial f}{\partial U_1} = U_0U_2(U_0+U_2), \quad \text{and} \quad \frac{\partial f}{\partial U_2} = U_0U_1(U_0+U_1). \tag{6}$$

The partial derivatives have four common zeros, 1:1:1 and the above mentioned three. However $f(1,1,1) = a \neq 0$. So Γ' has only three singular points.

To show that 1:0:0 is a node take a generic point $w_0{:}w_1{:}w_2$ of $\mathbb{P}^2$, a variable t and compute the coefficient of t^2 in $f(1+tw_0, tw_1, tw_2)$ (by (3) of Section 1). The same coefficient is obtained by substituting $U_0 = 1$, $U_1 = w_1$ and $U_2 = w_2$ in f and computing the quadratic term. The result is $bw_1^2 + w_1w_2 + sw_2^2$. This quadratic form is the product of two distinct linear forms. Hence 1:0:0 is a node.

Similarly substitute $U_0 = w_0$, $U_1 = 1$ and $U_2 = w_2$ to get the quadratic form $bw_0^2 + w_0w_2 + bw_2^2$. Conclude that 0:1:0 is a node. Finally substitute $U_0 = w_0$, $U_1 = w_1$ and $U_2 = 1$ to get $sw_0^2 + w_0w_1 + bw_1^2$. Again 0:0:1 is a node. ∎

Lemma 7.3: *The curve* Γ' *has only finitely many inflection points and only finitely many double tangents.*

Proof: Use (6) to compute an equation for the tangent to Γ' at a generic point (u,v) in affine coordinates U, V:

$$v(v+1)U + u(u+1)V + uv(u+v) = 0. \tag{7}$$

Eliminate U from (7) and substitute $U_0 = 1$, $U_1 = U$ and $U_2 = V$ in $f(U_0, U_1, U_2)$ and multiply by $v^2(v+1)^2$ to get a polynomial of degree 4 in V:

$$bu^2(u+1)^2V^4 + u(u+1)(u+v)(u+v+1)V^3 + w_2V^2 + w_1V + w_0. \tag{8}$$

Since (5) is the minimal equation for (u, v) over K the coefficient of V^3 in (8) is nonzero. Therefore (8) has in addition to the double zero v two more distinct zeros. By Bezout's theorem this means that $T_{\Gamma',(u,v)}$ intersects Γ' with multiplicity 2 at (u, v) and with multiplicity 1 at two additional points. Hence (u, v) is a noninflection point and $T_{\Gamma',(u,v)}$ is not a double tangent. By Corollary 3.2 and by Lemma 4.2, Γ' has only finitely many inflection points and only finitely many double tangents. ∎

To conclude the discussion of Case B2 note that the partial derivatives in (6) are linearly independent over $\widetilde{K}$. By Lemma 5.1, Γ' has no strange point.

We summarize the results obtained so far in this section:

Proposition 7.4: *Let F be a function field of genus 0 over a field K. Then F/K has a projective plane node model with only finitely many inflection points, only finitely many double tangents and without strange point.*

Next we add Proposition 7.4 to the results of the previous sections to construct an interim plane model for our function field.

Proposition 7.5: *Let F be a conservative function field of one variable over an infinite field K. Then F/K has a projective plane model with only finitely many inflection points, only finitely many double tangents and without strange points.*

Proof: By Proposition 7.4, it suffices to consider the case where the genus of F/K is positive.

Let x be a separating transcendence element for F/K. Choose a primitive element y for the extension $F/K(x)$, integral over $K[x]$, and let $g(x, Y) = \mathrm{irr}(y, K(x))$. The equation $g(X, Y) = 0$ defines a model Γ_0 for F/K. If $\mathrm{char}(K) \neq 2$ choose $u \in K$ such that $g(X, Y + uX^2) = 0$ defines a model Γ with only finitely many inflection points (Lemma 3.3). By Proposition 4.5, Γ has only finitely many double tangents. If char$(K)=2$ use Lemma 6.1 to choose u such that in addition Γ has only finitely many double tangents. In both cases Lemma 5.4 asserts that Γ has no strange point. It is therefore the desired model for F/K. ∎

REMARK 7.6: (a) *The conservation assumption is dispensable.* Indeed we use this assumption (via Lemma 5.4) only in characteristic 2. In this case however, it is possible to use the transformation $x'=x$ and $y'=y+ux^2+x^m+x^q$ with $u\in K,\ m>2$ odd, and q even and sufficiently large to eliminate the strange point from Γ.

(b) *Proposition 7.5 holds also for finite K.* The only case in which the proof of Proposition 7.5 may break is when the genus is positive and char$(K)=2$, since then there may be no transformation (1) of Section 6 that satisfies (2) of Section 6. However, we may replace (1) of Section 6 by the transformation: $X'=X,\ Y'=Y+h(X^2)$ and prove, as in Lemma 3.3 and Lemma 6.1, that for all but finitely many $h\in K[X]$, the transformed curve Γ' has only finitely many inflection points and only finitely many double tangents. ∎

(c) *The case of genus 0.* If the genus of F is zero, then F/K has a separating transcendence element x such that $[F:K(x)]\le 2$. Hence, $F/K(x)$ is Galois and therefore x is a stabilizing element for F/K. Thus, the models constructed for the proof of Proposition 7.4 are actually dispensable for the proof of Theorem F. Nevertheless, taken into account (a) and (b), these models complete the proof of Proposition 7.5 with no restriction on F nor on K. ∎

8. A smooth model in a higher dimensional projective space.

Let F be a conservative function field of one variable over an infinite field K. We lift the plane model of F/K that Proposition 7.5 gives to a smooth model in a projective space $\mathbb{P}^n$ with only finitely many inflection points and only finitely many double tangents.

Consider the projective space $\mathbb{P}^n$ of dimension n. Present each point $\mathbf{y}$ of $\mathbb{P}^n$ by homogeneous coordinates $y_0{:}y_1{:}\cdots{:}y_n$. If $\mathbf{y}$ is a generic point of a curve Δ defined over a field K with function field F, we always choose the coordinates y_i to lie in F. In particular the higher derivatives of y_i with respect to each separating transcendental element t for $\widetilde{K}F/\widetilde{K}$ are well defined. As for plane curves denote the tangent to Δ at a simple point $\mathbf{a}$ by $T_{\Delta,\mathbf{a}}$.

Lemma 8.1: *Let P be a prime divisor of $\widetilde{K}F/\widetilde{K}$ whose center is a simple point $\mathbf{a}$ of Δ. Then* $\mathrm{rank}(\mathbf{y}(P)\ \ \mathbf{y}'(P)) = 2$ *and* $T_{\Delta,\mathbf{a}} = \{u_0\mathbf{y}(P) + u_1\mathbf{y}'(P)|\ u_0{:}u_1 \in \mathbb{P}^1\}$.

Proof: Let t be a local parameter at $\mathbf{a}$. Then t generates the maximal ideal M of the local ring $O_{\Delta,\mathbf{a}}$ of Δ at $\mathbf{a}$. Without loss assume that $y_0 = 1$. Then also $y_i - a_i$, $i = 1,\ldots,n$, generate M. As $y_i - a_i \cong y_i'(P)t \bmod t^2$ (by (1g) of Section 2), there exists i between 1 and n such that $y_i'(P) \neq 0$. Since $y_0'(P) = 0$ the $(n+1)$-tuples $\mathbf{y}(P)$ and $\mathbf{y}'(P)$ are not proportional. In projective terms, this means that the points $\mathbf{y}(P)$ and $\mathbf{y}'(P)$ are distinct.

Let $f_i(\mathbf{Y})$, $i = 1,\ldots,m$ be homogeneous generators of the ideal of $\widetilde{K}[Y_0,\ldots,Y_n]$ consisting of all polynomials that vanish on Δ. Then the system of linear equations $\sum_{j=0}^n \frac{\partial f_i}{\partial a_j} Y_j = 0$, $i = 1,\ldots,m$ defines $T_{\Delta,\mathbf{a}}$. Take the derivative of $f_i(y_0,\ldots,y_n) = 0$ with respect to t at P to conclude that $\sum_{i=0}^n \frac{\partial f_i}{\partial a_j} y_j'(P) = 0$, $i = 1,\ldots,m$ and that $\mathbf{y}'(P)$ lies on $T_{\Delta,\mathbf{a}}$. Since $\mathbf{y}(P)$ and $\mathbf{y}'(P)$ are distinct $T_{\Delta,\mathbf{a}}$ is the line that goes through them. ∎

Corollary 8.2: *Let P and Q be prime divisors of $\widetilde{K}F/\widetilde{K}$ with simple centers $\mathbf{a}$ and $\mathbf{b}$ of Δ. Then*

$$\begin{aligned} T_{\Delta,\mathbf{a}} = T_{\Delta,\mathbf{b}} \quad &\Longleftrightarrow \quad \mathrm{rank}(\mathbf{y}(P)\ \ \mathbf{y}'(P)\ \ \mathbf{y}(Q)\ \ \mathbf{y}'(Q)) = 2 \\ T_{\Delta,\mathbf{a}} \cap T_{\Delta,\mathbf{b}} \neq \emptyset \quad &\Longleftrightarrow \quad \mathrm{rank}(\mathbf{y}(P)\ \ \mathbf{y}'(P)\ \ \mathbf{y}(Q)\ \ \mathbf{y}'(Q)) \leq 3 \\ T_{\Delta,\mathbf{a}} \cap T_{\Delta,\mathbf{b}} = \emptyset \quad &\Longleftrightarrow \quad \mathrm{rank}(\mathbf{y}(P)\ \ \mathbf{y}'(P)\ \ \mathbf{y}(Q)\ \ \mathbf{y}'(Q)) = 4. \end{aligned}$$

Let P be a prime divisor of $\widetilde{K}F/\widetilde{K}$ whose center $\mathbf{a}$ at Δ is a simple point. We say that $\mathbf{a}$ is an **inflection point** if

$$\text{rank}(\mathbf{y}(P)\ \ \mathbf{y}'(P)\ \ \mathbf{y}^{[2]}(P)) = 2. \tag{1}$$

By Lemma 2.1, this concept is well defined. By Lemma 3.1, this definition agrees in the case $n = 2$ with the old one made in Section 1.

PROPOSITION 8.3: *Each conservative function field of one variable F/K has a projective smooth model Δ with only finitely many inflection points and with only finitely many double tangents.*

Proof: Let Γ be a projective plane model for F/K with only finitely many inflection points, only finitely many double tangents and without strange point (Proposition 7.5 and Remark 7.6(b)). Let $\mathbf{x} = x_0{:}x_1{:}x_2$ be a generic point of Γ with coordinates in F. For $i = 0, 1, 2$ let $y_i = x_i$ and choose $y_3, \ldots, y_n$ in F such that $\mathbf{y} = y_0{:}y_1{:}\cdots{:}y_n$ generates a smooth curve Δ in $\mathbb{P}^n$.

To choose $y_3, \ldots, y_n$ let g be the genus of F/K and choose a positive divisor of F/K such that $(y_i) + A \geq 0$ for $i = 0, 1, 2$ and $\deg(A) \geq 2g + 1$ (e.g., $A = m(y_0)_\infty + m(y_1)_\infty + m(y_2)_\infty$ for m sufficiently large; here (y_i) is the divisor of y_i and $(y_i)_\infty$ is the divisor of poles of y_i in F.) As Γ is not a line y_0, y_1, y_2 are linearly independent over K. Extend y_0, y_1, y_2 to a basis $y_0, \ldots, y_n$ of the linear space $\mathcal{L}_K(A) = \{z \in F \mid (z) + A \geq 0\}$. As F/K is conservative $y_0, \ldots, y_n$ is also a basis of $\mathcal{L}_{\widetilde{K}}(A)$ [D, p. 144]. Hence A is a very ample divisor [H, p. 308] and the point $\mathbf{y}$ generates a smooth projective curve Δ in $\mathbb{P}^n$ which is obviously defined over K.

By Lemma 3.1, $\text{rank}(\mathbf{x}\ \mathbf{x}'\ \mathbf{x}^{[2]}) = 3$. Hence $\text{rank}(\mathbf{y}\ \mathbf{y}'\ \mathbf{y}^{[2]}) = 3$. Therefore $\text{rank}(\mathbf{y}(P)\ \ \mathbf{y}'(P)\ \ \mathbf{y}^{[2]}(P)) = 3$ for all but finitely many prime divisors of $\widetilde{K}F/\widetilde{K}$. This means that Δ has only finitely many inflection points.

Let $\varphi{:}\ \Delta \to \Gamma$ be the birational morphism defined by $\varphi(\mathbf{y}) = \mathbf{x}$. Each $\mathbf{c} \in \Delta$ determines a unique prime divisor P of $\widetilde{K}F/\widetilde{K}$ such that $\mathbf{y}(P) = \mathbf{c}$. Then $\mathbf{a} = \varphi(\mathbf{c}) = \mathbf{x}(P)$. If $c_0 \neq 0$ or $c_1 \neq 0$ or $c_2 \neq 0$ then $\mathbf{a} = c_0{:}c_1{:}c_1$. Let Δ_0 be the set of all points $\mathbf{c}$ of Δ with $\mathbf{c} = 0{:}0{:}0{:}c_3{:}\cdots{:}c_n$. As $\mathbf{y} \notin \Delta_0$ this is a finite set. Take a cofinite subset Δ_1 of $\Delta - \Delta_0$ on which φ is biregular.

At this point assume without loss that $n \geq 3$; otherwise take $\Delta = \Gamma$. By a theorem of Samuel Δ has no strange point [H, p. 312]. In particular Δ has only finitely many tangents that go through a point of $\Delta - \Delta_1$. Remove each $\mathbf{c}$ from Δ_1 such that $T_{\Delta,\mathbf{c}}$ goes through a point of $\Delta - \Delta_1$. Denote the resulting cofinite subset of Δ_1 by Δ_2.

Consider $\Gamma_2 = \varphi(\Delta_2)$. Let Γ_3 be the set of all points $\mathbf{a} \in \Gamma_2$ such that $T_{\Gamma,\mathbf{a}}$ is neither a double tangent nor does it go through a singular point of Γ or through a point of $\Gamma - \Gamma_2$. Since Γ has no strange point Γ_3 is a confinite subset of Γ_2. Let $\Delta_3 = \varphi^{-1}(\Gamma_3)$. To conclude the proof we show that no tangent to Δ at a point of Δ_3 is double.

Indeed let $\mathbf{c} \in \Delta_3$ and let P and $\mathbf{a}$ be as above. Since φ is biregular on Δ_3 the point $\mathbf{a}$ of Γ is simple. Assume that Δ has another point $\mathbf{d}$ such that $T_{\Delta,\mathbf{c}} = T_{\Delta,\mathbf{d}}$. Let Q be the prime divisor of $F\widetilde{K}/\widetilde{K}$ with center $\mathbf{d}$. By Corollary 8.2,

$$\text{rank}(\mathbf{y}(P)\ \ \mathbf{y}'(P)\ \ \mathbf{y}(Q)\ \ \mathbf{y}'(Q)) = 2. \tag{2}$$

As $\mathbf{d} \in T_{\Delta,\mathbf{c}}$ there exist $\alpha, \beta \in \widetilde{K}$ such that $\mathbf{d} = \alpha\mathbf{y}(P) + \beta\mathbf{y}'(P)$ (Lemma 8.1). Also $\mathbf{d} \in \Delta_1$. In particular $d_0{:}d_1{:}d_2 \neq 0{:}0{:}0$. Hence, $\mathbf{b} = \varphi(\mathbf{d}) = d_0{:}d_1{:}d_2$. Therefore $\mathbf{b} = \alpha\mathbf{x}(P) + \beta\mathbf{x}'(P)$; that is $\mathbf{b} \in T_{\Gamma,\mathbf{a}}$. As $\mathbf{a} \in \Gamma_3$ the tangent $T_{\Gamma,\mathbf{a}}$ is not double, $\mathbf{b}$ is simple on Γ and belongs to Γ_2. On the other hand $\mathbf{a} \neq \mathbf{b}$ (since $\mathbf{c} \in \Delta_1$ and $\mathbf{c} \neq \mathbf{d}$) and by (2), $\text{rank}(\mathbf{x}(P)\ \ \mathbf{x}'(P)\ \ \mathbf{x}(Q)\ \ \mathbf{x}'(Q)) = 2$. Hence $T_{\Gamma,\mathbf{a}} = T_{\Gamma,\mathbf{b}}$; that is, $T_{\Gamma,\mathbf{a}}$ is a double tangent.

This contradiction proves that $T_{\Delta,\mathbf{c}}$ is not a double tangent. ∎

9. Projection from a point.

Each point $\mathbf{o}$ of $\mathbb{P}^n$ defines a morphism π from $\mathbb{P}^n - \mathbf{o}$ into $\mathbb{P}^{n-1}$. To write π explicitly present $\mathbf{o}$ as the intersection of n hyperplanes:

$$(1) \qquad \sum_{j=0}^{n} \alpha_{ij} Y_i = 0, \qquad i = 0, \ldots, n-1.$$

with $\mathrm{rank}(\alpha_{ij}) = n$. If $\mathbf{y} \neq \mathbf{0}$, then the coordinates of $\mathbf{x} = \pi(\mathbf{y})$ are

$$(2) \qquad x_i = \sum_{j=0}^{n} \alpha_{ij} y_j, \qquad i = 0, \ldots, n-1.$$

Other presentation of $\mathbf{o}$ as an intersection of n hyperplanes results in multiplying the matrix (α_{ij}) by a nonsingular $n \times n$ matrix. So π is determined by $\mathbf{o}$ only up to a projective transformation of $\mathbb{P}^{n-1}$. As all the geometrical concepts we consider are invariant under such transformations, this presents no essential difficulty in the sequel. If $\mathbf{o}$ is a K-rational point, choose α_{ij} in K. Thus π is defined over K.

Suppose that $\mathbf{y}$ generates a curve Δ over K in $\mathbb{P}^n$. If $\mathbf{o}$ does not lie on Δ, then the restriction of π to Δ is a morphism which maps Δ onto a curve Γ in $\mathbb{P}^{n-1}$ defined over K with generic point $\mathbf{x}$.

LEMMA 9.1: *In the above notation let $P_1, \ldots, P_r$ be prime divisors of $\widetilde{K}F/\widetilde{K}$ and let $k_1, \ldots k_r$ be nonnegative integers that satisfy condition (7) in Lemma 2.1. Let*

$$(3) \qquad d = \mathrm{rank}(x_i^{[k_\rho]}(P_\rho))_{i,\rho} \qquad \text{and} \qquad e = \mathrm{rank}(y_j^{[k_\rho]}(P_\rho))_{j,\rho}.$$

If $\mathbf{o}$ belongs to the linear space L spanned in $\mathbb{P}^n$ by the points $\mathbf{y}^{[k_\rho]}(P_\rho)$, $\rho = 1, \ldots, r$, then $d = e - 1$, otherwise $d = e$.

Proof: Lemma 2.1 asserts that the ranks in (3) are well defined. By (2)

$$(4) \qquad x_i^{[k_\rho]}(P_\rho) = \sum_{j=0}^{n} \alpha_{ij} y_j^{[k_\rho]}(P_\rho) \qquad i = 0, \ldots, n-1; \quad \rho = 1, \ldots, r.$$

Rewrite (4) in terms of matrices:

$$(5) \qquad (x_i^{[k_\rho]}(P_\rho))_{i,\rho} = (\alpha_{ij})_{i,j} (y_j^{[k_\rho]}(P_\rho))_{j,\rho}\,.$$

Let W (resp. V) be the linear space in $\mathring{A}^{n+1}$ (resp. $\mathring{A}^n$) generated by the columns $\mathbf{y}^{[k_\rho]}(P_\rho)$ (resp. $\mathbf{x}^{[k_\rho]}(P_\rho)$), $\rho = 1, \ldots, r$. Consider the matrix (α_{ij}) as a linear map $\alpha\colon \widetilde{K}^{n+1} \to \widetilde{K}^n$ that operates on columns of length $n+1$ by multiplication from the left. Then α maps W onto V.

If $\mathbf{o} \notin L$, then α is injective on W. Hence $e = \dim(W) = \dim(V) = d$. Otherwise the kernel of the restriction of α to W has dimension 1 and the rank decreases by 1. ∎

The following lemma is a geometric reinterpretation of (1).

LEMMA 9.2: *Let* $\mathbf{b}$ *and* $\mathbf{c}$ *be two points of* $\mathbb{P}^n$ *different from* $\mathbf{o}$*. Then* $\pi(\mathbf{b}) = \pi(\mathbf{c})$ *if and only if* $\mathbf{o}$, $\mathbf{b}$ *and* $\mathbf{c}$ *lie on the same line.*

LEMMA 9.3: *Let D be a finitely generated integral domain over a field K. Let P be a prime ideal of D. Then the integral closure of D_P is a finitely generated D_P-module.*

Proof: Let D' be the integral closure of D. By [L1, p. 120] $D' = Dz_1 + \cdots + Dz_m$ for some $z_1, \ldots, z_m \in D'$. The local ring D'_P of D' with respect to the multiplicative set $D - P$ is the integral closure of D_P [L2, p. 8]. It satisfies $D'_P = D_P z_1 + \cdots + D_P z_m$. ∎

The following lemma establishes well known conditions under which π maps a given simple point of Δ onto a simple point of Γ.

LEMMA 9.4: *Let* $\mathbf{b}$ *be a simple point of* Δ *such that the line through* $\mathbf{o}$ *and* $\mathbf{b}$ *neither tangents* Δ *nor bisects it. If* $\mathbf{o} \notin \Delta$*, then the restriction of* π *to* Δ *is a birational morphism which is biregular at* $\mathbf{b}$*, the point* $\mathbf{a} = \pi(\mathbf{b})$ *is simple on* Γ *and* $\pi(T_{\Delta,\mathbf{b}}) = T_{\Gamma,\mathbf{a}}$*. If in addition* $\mathbf{o} \in \mathbb{P}^n(K)$*, then* Γ *is a model of* F/K*.*

Proof: Let S (resp., R) be the local ring of Δ (resp., Γ) at $\mathbf{b}$ (resp., $\mathbf{a}$). Then S is a discrete valuation ring, R is a local ring and $R \subseteq S$. Denote the normalized valuation of $\widetilde{K}F/\widetilde{K}$ that corresponds to S by w. Denote the maximal ideal of S (resp., R) by N (resp., M). Let t be a generator of N.

By Lemma 9.2, $\mathbf{b}$ is the only point of Δ that lies over $\mathbf{a}$. Hence S is the only valuation ring of $\widetilde{K}F$ that contains R. Therefore S is the integral closure of R in $\widetilde{K}F$

[L1, p. 14]. By Lemma 9.3, S is a finitely generated R-module.

Let P the prime divisor of $\widetilde{K}F/\widetilde{K}$ with center $\mathbf{b}$. By (2), $x_i'(P) = \sum_{j=0}^{n} \alpha_{ij} y_j'(P)$, $i = 0, \ldots, n-1$. If $x_i'(P) = 0$ for all i, then $\mathbf{y}'(P) = \mathbf{o}$. By Lemma 8.1, $\mathbf{o}$ would lie on $T_{\Delta,\mathbf{b}}$, a contradiction. Thus $x_i'(P) \neq 0$ for some i between 0 and $n-1$. Deduce from the expansion $x_i = a_i + x_i'(P)t + \cdots$ ((1g) of Section 2) that $w(x_i - a_i) = 1$.

It follows that w is the unique valuation of $\widetilde{K}F/\widetilde{K}$ that lies over its restriction to $\widetilde{K}(\Gamma)$ and that w is unramified. Conclude from the formula $\sum e_i f_i = n$ that $\widetilde{K}(\Gamma) = \widetilde{K}F$. Thus the restriction of π to Δ is a birational morphism.

Also, $N = MS$ and $S/N = \widetilde{K} = R/M$. Hence $S = R + MS$. By Nakayama's Lemma, $S = R$. Hence R is a discrete valuation ring, $\mathbf{a}$ is a simple point of Γ and the restriction of π to Δ is biregular at $\mathbf{b}$. Finally, if $\mathbf{o} \in \mathbb{P}^n(K)$, then $K(\Gamma) \subseteq F$. As F is linearly disjoint from $\widetilde{K}$ over K it is linearly disjoint from $\widetilde{K}(\Gamma)$ over $K(\Gamma)$. Hence $[F : K(\Gamma)] = [\widetilde{K}F : \widetilde{K}(\Gamma)] = 1$. Conclude that $K(\Gamma) = F$ and Γ is a model of F/K. ∎

10. A plane model.

Let F be a conservative function field of one variable over an infinite field K. We start in this section with the smooth model Δ for F/K in $\mathbb{P}^n$ of Proposition 8.3 and project it from a suitable K-rational point onto a model Γ in $\mathbb{P}^{n-1}$ having the same properties as Δ except for $n = 3$ where Γ is allowed to have nodes as singularities. It is essential for the induction process to keep track of one simple point with extra properties. Lemma 10.1 proves the existence of this point on Δ.

Lemma 10.1: *Let Δ be a smooth curve defined over K in $\mathbb{P}^n$ with only finitely many inflection points and only finitely many double tangents. Suppose that $n \geq 3$ and that Δ is contained in no plane. Then $\Delta(\widetilde{K})$ has a noninflection point $\mathbf{a}$ such that $T_{\Delta,\mathbf{a}}$ is not a double tangent and $T_{\Delta,\mathbf{a}}$ intersects only finitely many tangents to Δ.*

Proof: Denote the set of all noninflection points $\mathbf{a}$ of Δ such that $T_{\Delta,\mathbf{a}}$ is not a double tangent by Δ_0. By assumption Δ_0 is a cofinite subset of Δ. We first prove that Δ_0 contains points $\mathbf{a}$ and $\mathbf{b}$ such that $T_{\Delta,\mathbf{a}} \cap T_{\Delta,\mathbf{b}} = \emptyset$.

Let $\mathbf{x}$ be a generic point of Δ over $\widetilde{K}$. Denote the set of all pairs $(\mathbf{a}, \mathbf{b}) \in \Delta \times \Delta$ such that $T_{\Delta,\mathbf{a}} \cap T_{\Delta,\mathbf{b}} \neq \emptyset$ by Λ. By Corollary 8.2, Λ is also defined by the condition

$$\text{(1)} \qquad \operatorname{rank}(\mathbf{x}(\mathbf{P}) \;\; \mathbf{x}'(\mathbf{P}) \;\; \mathbf{x}(\mathbf{Q}) \;\; \mathbf{x}'(\mathbf{Q})) \leq 3.$$

where P (resp., Q) is the prime divisor of $\widetilde{K}F/\widetilde{K}$ with center $\mathbf{a}$ (resp., $\mathbf{b}$).

Assertion A: *Λ is a closed subset of $\Delta \times \Delta$.* It suffices to find for given $\mathbf{a}, \mathbf{b} \in \Delta$ an open neighborhood $U = U_{(\mathbf{a},\mathbf{b})}$ of $(\mathbf{a}, \mathbf{b})$ and a closed subset $C = C_{(\mathbf{a},\mathbf{b})}$ of $\Delta \times \Delta$ such that $\Lambda \cap U = C \cap U$. Then $\Delta \times \Delta - \Lambda = \bigcup_{(\mathbf{a},\mathbf{b}) \in \Delta \times \Delta} (U_{(\mathbf{a},\mathbf{b})} - C_{(\mathbf{a},\mathbf{b})})$ is open and therefore Λ is closed.

Indeed let P and Q be as before. Take a homogeneous coordinate system $x_0{:}\cdots{:}x_n$ for $\mathbf{x}$ such that $x_0(P){:}\cdots{:}x_n(P)$ is a homogeneous coordinate system for $\mathbf{a}$. Then $x_i'(P)$ is finite, i.e., $x_i' = f_i(\mathbf{x})/g(\mathbf{x})$ where f_i, g are homogeneous polynomials and $g(\mathbf{a}) \neq 0$. Similarly there exists homogeneous coordinate system $ux_0{:}\cdots{:}ux_n$ for $\mathbf{x}$, with $u \in \widetilde{K}F$ such that $(ux_0)(Q){:}\cdots{:}(ux_n)(Q)$ is a homogeneous coordinate system for $\mathbf{b}$, and there exist homogeneous polynomials $p_0, \ldots, p_n, q$ such that $(ux_i)' = p_i(\mathbf{x})/q(\mathbf{x})$ and $q(\mathbf{b}) \neq 0$. Condition (1) becomes

$$\text{(2)} \qquad \operatorname{rank}(\mathbf{a} \;\; \mathbf{f}(\mathbf{a})/g(\mathbf{a}) \;\; \mathbf{b} \;\; \mathbf{p}(\mathbf{a})/q(\mathbf{a})) \leq 3,$$

which means that all 4×4 subdeterminants of the $(n+1) \times 4$ matrix

$$(\mathbf{a} \;\; \mathbf{f}(\mathbf{a})/g(\mathbf{a}) \;\; \mathbf{b} \;\; \mathbf{p}(\mathbf{a})/q(\mathbf{a}))$$

vanish. Take U to be the open subset of $\Delta \times \Delta$ determined by the condition $g(\mathbf{a}) \neq 0$ and $q(\mathbf{b}) \neq 0$. Take C as the closed subset of $\Delta \times \Delta$ determined by the vanishing of all the above 4×4 determinants. Then $\Lambda \cap U = C \cap U$, as desired.

Remark: If we hold $\mathbf{a} \in \Delta$ fixed, the same argument shows that the set of all $\mathbf{b} \in \Delta$ such that $T_{\Delta,\mathbf{a}} \cap T_{\Delta,\mathbf{b}} \neq \emptyset$ is closed.

Assertion B: *Λ is a proper subset of Δ.* Otherwise choose $\mathbf{a}, \mathbf{b} \in \Delta$ such that $T_{\Delta,\mathbf{a}} \neq T_{\Delta,\mathbf{b}}$. Let $\mathbf{s}$ be the intersection point of $T_{\Delta,\mathbf{a}}$ and $T_{\Delta,\mathbf{b}}$ and let E be the plane spanned by them. Choose a point $\mathbf{c} \in \Delta - E$. Denote the intersection points of $T_{\Delta,\mathbf{c}}$ with $T_{\Delta,\mathbf{a}}$ and $T_{\Delta,\mathbf{b}}$, respectively, by $\mathbf{s}_1$ and $\mathbf{s}_2$. Since $T_{\Delta,\mathbf{c}}$ is not contained in E the points $\mathbf{s}_1$ and $\mathbf{s}_2$ coincide with $\mathbf{s}$. This means that infinitely many tangents to Δ go through $\mathbf{s}$. In other words, $\mathbf{s}$ is a strange point of Δ. This contradiction to Samuel's theorem [H, p. 312] proves our assertion.

Since $\Delta \times \Delta$ is an irreducible variety $\dim(\Lambda) \le 1$. In particular there exists $(\mathbf{a}, \mathbf{b}) \in \Delta_0 \times \Delta - \Lambda$. Thus $\mathbf{a} \times \Delta \not\subseteq \Lambda$. Since $\mathbf{a} \times \Delta$ is an irreducible curve (in $\mathbb{P}^n \times \mathbb{P}^n$) the set $(\mathbf{a} \times \Delta) \cap \Lambda$ is finite. Conclude that $T_{\Delta,\mathbf{a}}$ intersects only finitely many tangents to Δ. ∎

Corollary 10.2: *Let Δ be a smooth curve as in Lemma 10.1. Denote the union of all lines that intersect Δ at two distinct points whose tangents intersect by W. Then W is contained in a K-closed set of dimension at most 2.*

Proof: It suffices to prove that $D = \{(\mathbf{a}, \mathbf{b}) \in \Delta \times \Delta |\ T_{\Delta,\mathbf{a}} \cap T_{\Delta,\mathbf{b}} \ne \emptyset\}$ is a K-closed set of dimension at most 1. Indeed, let $f_1, \dots, f_m \in K[X_0, \dots, X_n]$ be homogeneous polynomials that generate the ideal of all polynomials which vanish on Δ. For $(\mathbf{a}, \mathbf{b}) \in \Delta \times \Delta$ consider the $2m \times (n+1)$ matrix M whose ith row is $(\partial f_i/\partial a_0 \ \cdots \ \partial f_i/\partial a_n)$ for $i = 1, \dots, m$ and $(\partial f_i/\partial b_0 \ \cdots \ \partial f_i/\partial b_n)$ for $i = m+1, \dots, 2m$. Then $(\mathbf{a}, \mathbf{b}) \in D$ if and only if $\mathrm{rank}(M) \le n$, i.e., all $(n+1) \times (n+1)$ subdeterminants of M vanish. Conclude that D is K-closed.

Let $\mathbf{a}$ be a point of Δ such that $T_{\Delta,\mathbf{a}}$ intersects only finitely many tangents of Δ (Lemma 10.1). In particular there exists $\mathbf{b} \in \Delta$ such that $T_{\Delta,\mathbf{a}} \cap T_{\Delta,\mathbf{b}} = \emptyset$. Conclude that D is a proper subset of $\Delta \times \Delta$ and therefore $\dim(D) \le 1$. ∎

Lemma 10.3: *Let Δ be a smooth curve in $\mathbb{P}^3$ defined over K. Suppose that Δ is not contained in a plane. Then, for each $\mathbf{a} \in \Delta(\widetilde{K})$ the union of all secants of Δ which intersect $T_{\Delta,\mathbf{a}}$ is contained in a closed set of dimension at most 2.*

Proof: Let $\mathbf{b}$ be a point of $T_{\Delta,\mathbf{a}}$ different from $\mathbf{a}$. Choose $\mathbf{c} \in \Delta(\widetilde{K}) - T_{\Delta,\mathbf{a}}$. Then choose a point $\mathbf{d} \in \Delta(\widetilde{K})$ which does not belong to the plane spanned by $T_{\Delta,\mathbf{a}}$ and $\mathbf{c}$. In other words, $\det(\mathbf{a}\ \mathbf{b}\ \mathbf{c}\ \mathbf{d}) \ne 0$. Conclude that $\dim\{(\mathbf{c}, \mathbf{d}) \in \Delta \times \Delta |\ \det(\mathbf{a}\ \mathbf{b}\ \mathbf{c}\ \mathbf{d}) = 0\} \le 1$.

Hence

$$\dim\{u_0\mathbf{c}+u_1\mathbf{d}|\ \mathbf{c},\mathbf{d}\in\Delta,\quad \det(\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}\ \ \mathbf{d})=0\quad\text{and}\quad u_0{:}u_1\in\mathbb{P}^1\}\le 2,$$

which proves the Lemma. ∎

We are now ready to project a smooth n-dimensional projective curve one dimension lower.

LEMMA 10.4: *Let Δ be a smooth curve in $\mathbb{P}^n$, $n\ge 3$, defined over K with a function field F. Suppose that $\mathbf{a}\in\Delta(\widetilde{K})$ is a noninflection point such that*

(a1) *$T_{\Delta,\mathbf{a}}$ is not a double tangent, and*

(a2) *$T_{\Delta,\mathbf{a}}$ intersects only finitely many tangents of Δ.*

Then $\mathbb{P}^n$ has a nonempty open set U such that for each $\mathbf{o}\in U(K)$ the projection $\pi{:}\ \mathbb{P}^n\to\mathbb{P}^{n-1}$ from $\mathbf{o}$ maps Δ onto a curve $\bar{\Delta}$ such that

(b1) *$\bar{\Delta}$ is birationally equivalent to Δ over K and $\pi(\mathbf{a})$ is a simple point of $\bar{\Delta}$,*

(b2) *$\pi(\mathbf{a})$ is a noninflection point,*

(b3) *$T_{\bar{\Delta},\pi(\mathbf{a})}$ is not a double tangent,*

(b4) *if $n\ge 4$, then $\bar{\Delta}$ is a smooth curve; if $n=3$, then $\bar{\Delta}$ is a node curve,*

(b5) *$T_{\bar{\Delta},\pi(\mathbf{a})}$ goes through no singular point of $\bar{\Delta}$, and*

(b6) *if $n\ge 4$, then $T_{\bar{\Delta},\pi(\mathbf{a})}$ intersects only finitely many tangents of $\bar{\Delta}$.*

Proof: Let $\mathbf{y}=y_0{:}y_1{:}\cdots{:}y_n$ be a generic point of Δ over K and let $F=K(\mathbf{y})$ be its function field. Let P be the prime divisor of $\widetilde{K}F/\widetilde{K}$ with $\mathbf{y}(P)=\mathbf{a}$. After having chosen $\mathbf{o}$ we let $\mathbf{x}=\pi(\mathbf{y})$. Then $\bar{\mathbf{a}}=\pi(\mathbf{a})=\mathbf{x}(P)$.

For each i between 1 and 6 we construct a proper closed subset C_i of $\mathbb{P}^n$ such that if $\mathbf{o}\in\mathbb{P}^n(K)-C_i$, then condition (b$i$) is fulfilled. Assume without loss that Δ is contained in no plane.

Proof of (b1): Denote the set of all secants of Δ that go through $\mathbf{a}$ by $\mathrm{Sec}(\Delta,\mathbf{a})$. Let $C_1=\Delta\cup T_{\Delta,\mathbf{a}}\cup\mathrm{Sec}(\Delta,\mathbf{a})$. Then C_1 is a proper closed subset of $\mathbb{P}^n$ of dimension 2. If $\mathbf{o}\in\mathbb{P}^n(K)-C_1$, then π is a birational morphism over K and $\bar{\mathbf{a}}$ is a simple point of $\bar{\Delta}$ (Lemma 9.4).

Proof of (b2): By (1) of Section 8, $\mathrm{rank}(\mathbf{y}(P)\ \mathbf{y}'(P)\ \mathbf{y}^{[2]}(P)) = 3$. This means that the points $\mathbf{y}(P)$, $\mathbf{y}'(P)$ and $\mathbf{y}^{[2]}(P)$ span a plane E in $\mathbb{P}^n$. If $\mathbf{o} \notin C_2 = C_1 \cup E$, then $\mathrm{rank}(\mathbf{x}(P)\ \mathbf{x}'(P)\ \mathbf{x}^{[2]}(P)) = 3$ (Lemma 9.1). Hence, $\bar{\mathbf{a}}$ is a noninflection point of $\bar{\Delta}$.

Proof of (b3): Let $\mathbf{b}_1, \ldots, \mathbf{b}_m$ the only points of Δ with tangents that intersect $T_{\Delta,\mathbf{a}}$. Let Q_i be the prime divisor of $\widetilde{K}F/\widetilde{K}$ such that $\mathbf{y}(Q_i) = \mathbf{b}_i,\ i = 1, \ldots, m$. By Corollary 8.2

$$\mathrm{rank}(\mathbf{y}(P)\ \mathbf{y}'(P)\ \mathbf{y}(Q_i)\ \mathbf{y}'(Q_i)) \leq 3, \qquad i = 1, \ldots, m. \tag{3}$$

Thus, for each i between 1 and m the points $\mathbf{y}(P)$, $\mathbf{y}'(P)$, $\mathbf{y}(Q_i)$ and $\mathbf{y}'(Q_i)$ span a linear variety E_i of dimension at most 2. Therefore the dimension of $C_3 = C_2 \cup E_1 \cup \cdots \cup E_m$ is 2.

Suppose that $\mathbf{o} \notin C_3$. Assume that $\bar{\Delta}$ has a simple point $\bar{\mathbf{b}}$ different from $\bar{\mathbf{a}}$ such that $T_{\bar{\Delta},\bar{\mathbf{a}}} = T_{\bar{\Delta},\bar{\mathbf{b}}}$. Let Q be the prime divisor of $\widetilde{K}F/\widetilde{K}$ such that $\mathbf{x}(Q) = \bar{\mathbf{b}}$. By Corollary 8.2,

$$\mathrm{rank}(\mathbf{x}(P)\ \mathbf{x}'(P)\ \mathbf{x}(Q)\ \mathbf{x}'(Q)) = 2. \tag{4}$$

Let $\mathbf{b}$ be the unique point of Δ such that $\pi(\mathbf{b}) = \bar{\mathbf{b}}$. By Lemma 9.1

$$\mathrm{rank}(\mathbf{y}(P)\ \mathbf{y}'(P)\ \mathbf{y}(Q)\ \mathbf{y}'(Q)) \leq 3. \tag{5}$$

As $\mathbf{a} \neq \mathbf{b}$ condition (a1) asserts that $T_{\Delta,\mathbf{a}} \neq T_{\Delta,\mathbf{b}}$. Hence Corollary 8.2 strengthens (5) to

$$\mathrm{rank}(\mathbf{y}(P)\ \mathbf{y}'(P)\ \mathbf{y}(Q)\ \mathbf{y}'(Q)) = 3. \tag{6}$$

By the choice of $\mathbf{b}_1, \ldots, \mathbf{b}_m$ and by Corollary 8.2, $Q = Q_i$ for some i between 1 and m. Since $\mathbf{o} \notin E_i$, Lemma 9.1 implies that the left hand side of (4) equals 3. This contradiction proves that $T_{\bar{\Delta},\bar{\mathbf{a}}}$ is not a double tangent.

Proof of (b4): The union $\mathrm{Tan}(\Delta)$ of all tangents to Δ has dimension 2. The union $\mathrm{Sec}(\Delta)$ of all secants of Δ is contained in a closed set of dimension 3. The union

$\mathrm{Tri}(\Delta)$ of all multisecants of Δ (i.e., lines that intersect Δ in at least three distinct points) is contained in a closed set of dimension at most 2 [H, p. 314]. The union W of lines that intersect Δ at two distinct points such that the tangents at these points intersect is contained in a closed set of dimension at most 2 (Lemma 10.2). If $n \geq 4$ let $C_4 = C_3 \cup \mathrm{Tan}(\Delta) \cup \mathrm{Sec}(\Delta)$. If $n = 3$ let C_4 be the closure of $C_3 \cup \mathrm{Tan}(\Delta) \cup \mathrm{Tri}(\Delta) \cup W$. Its dimension is at most 2.

Suppose that $\mathbf{o} \notin C_4$. If $n \geq 4$, then by Lemma 9.4, each point of $\bar{\Delta}$ is simple. If $n = 3$, then the only singular points of $\bar{\Delta}$ are nodes [H, p. 313].

Indeed, since $\mathbf{o} \notin \mathrm{Tan}(\Delta)$, a point $\bar{\mathbf{c}} \in \bar{\Delta}$ is singular only if $\pi(\mathbf{c}_1) = \pi(\mathbf{c}_2) = \bar{\mathbf{c}}$ for two distinct points $\mathbf{c}_1, \mathbf{c}_2 \in \Delta$ (Lemma 9.4). By Lemma 9.2, the line L through $\mathbf{c}_1$ and $\mathbf{c}_2$ goes through $\mathbf{o}$. Since $\mathbf{o} \notin \mathrm{Tri}(\Delta)$, it goes through no other point of Δ. As $\mathbf{o} \notin W$ the tangents $T_{\Delta,\mathbf{c}_1}$ and $T_{\Delta,\mathbf{c}}$ do not intersect. So $\bar{\Delta}$ has exactly two distinct tangents $\pi(T_{\Delta,\mathbf{c}_1})$ and $\pi(T_{\Delta,\mathbf{c}_2})$ at $\bar{\mathbf{c}}$. This means that $\bar{\mathbf{c}}$ is a node [H, p. 310].

Proof of (b5): As $\bar{\Delta}$ is smooth for $n \geq 4$ we have to prove (b5) only for $n = 3$.

The union of all secants of Δ which intersect $T_{\Delta,\mathbf{a}}$ is contained in a closed set V of dimension at most 2 (Lemma 10.3). Let $C_5 = C_4 \cup V$ and suppose that $\mathbf{o} \notin C_5$. Assume that $T_{\bar{\Delta},\bar{\mathbf{a}}}$ goes through a singular point $\bar{\mathbf{c}}$ of $\bar{\Delta}$. By Lemma 9.4 and since $\mathbf{o} \notin \mathrm{Tan}(\Delta)$ there exist two distinct points $\mathbf{c}_1$ and $\mathbf{c}_2$ of Δ such that $\pi(\mathbf{c}_1) = \pi(\mathbf{c}_2) = \bar{\mathbf{c}}$. In particular $\mathbf{o}$ lies on the line L through $\mathbf{c}_1$ and $\mathbf{c}_2$ (Lemma 9.2). Since $\pi(T_{\Delta,\mathbf{a}}) = T_{\bar{\Delta},\bar{\mathbf{a}}}$ (Lemma 9.4) there exists $\mathbf{c}_3 \in T_{\Delta,\mathbf{a}}$ such that $\pi(\mathbf{c}_3) = \mathbf{c}$. By Lemma 9.2, $\mathbf{c}_3 \in L$. Thus $T_{\Delta,\mathbf{a}}$ intersects the secant L of Δ. It follows that $L \subseteq V$ and therefore $\mathbf{o} \notin L$. This contradiction proves that $T_{\bar{\Delta},\bar{\mathbf{a}}}$ goes through no singular point of $\bar{\Delta}$, as desired.

Proof of (b6): Choose a point $\mathbf{b}_0 \in \Delta(\widetilde{K}) - \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ (we use the notation of the proof of (b3)). Let Q_0 be the prime divisor of $\widetilde{K}F/\widetilde{K}$ such that $\mathbf{y}(Q_0) = \mathbf{b}_0$. By the coice of $\mathbf{b}_1, \ldots, \mathbf{b}_m$ the tangents at $\mathbf{a}$ and $\mathbf{b}_0$ do not intersect:

$$\mathrm{rank}(\mathbf{y}(P)\ \ \mathbf{y}'(P)\ \ \mathbf{y}(Q_0)\ \ \mathbf{y}'(Q_0)) = 4 \tag{7}$$

(Corollary 8.2). Therefore the tangents span a three dimensional space S in $\mathbb{P}^n$. Consider the three dimensional closed subset $C_6 = C_5 \cup S$ of $\mathbb{P}^n$. Suppose that $\mathbf{o} \notin C_6$. By

Lemma 9.1 and by (7)

$$\operatorname{rank}(\mathbf{x}(P)\ \ \mathbf{x}'(P)\ \ \mathbf{x}(Q_0)\ \ \mathbf{x}'(Q_0)) = 4. \tag{8}$$

By the remark in the proof of Lemma 10.1 which proceeds Assertion A, the set $\bar{\Delta}_0$ of all $\bar{\mathbf{b}} \in \bar{\Delta}$ such that $T_{\bar{\Delta},\bar{\mathbf{b}}}$ intersects $T_{\bar{\Delta},\bar{\mathbf{a}}}$ is closed. As $\pi(\mathbf{b}_0)$ does not belong to $\bar{\Delta}_0$ (by (8) and Corollary 8.2) this set is finite. ∎

Everything is now ready for the proof of Theorem F of the introduction.

THEOREM 10.5: *Let F be a conservative function field of one variable over an infinite field K. Then F/K has a projective plane node model Γ with only finitely many inflection points, only finitely many double tangents, and without strange point.*

Proof: By Proposition 7.4 we may assume that the genus of F/K is positive. Let Δ be a smooth model in $\mathbb{P}^n$ for F/K with only finitely many inflection points and only finitely many double tangents (Proposition 8.3). Assume without loss that Δ is contained in no plane. In particular $n \geq 3$. Let $\mathbf{a} \in \Delta(\widetilde{K})$ be a noninflection point such that $T_{\Delta,\mathbf{a}}$ is not a double tangent and it intersects only finitely many tangents to Δ (Lemma 10.1). Choose a point $\mathbf{o} \in \mathbb{P}^n(K)$ that satisfies condition (b) of Lemma 10.4. As in that Lemma, let $\pi\colon \mathbb{P}^n \to \mathbb{P}^{n-1}$ be the projection from $\mathbf{o}$ and let $\bar{\Delta} = \pi(\Delta)$. If $n = 3$, take $\Gamma = \bar{\Delta}$. If $n \geq 4$, then $\bar{\Delta}$ and $\bar{\mathbf{a}}$ satisfy the assumptions of Lemma 10.4. Now use induction on n to conclude that F/K has a projective plane node model Γ with only finitely many inflection points and only finitely many double tangents.

In each case, Lemma 5.4 assures that Γ has no strange point. ∎

We have mentioned in the introduction that Theorem 10.5 implies our main result:

THEOREM 10.6: *Let F be a conservative function field of one variable over an infinite field K. Then F/K is a stable extension.*

Since every function field of one variable over a perfect field is conservative [D, p. 132] Theorem 10.6 can be reformulated for perfect fields:

THEOREM 10.7: *Every infinite perfect field K is stable in dimension 1.*

11. Function fields of higher transcendence degree.

Proposition 8.3 implies that each conservative function field F/K of one variable has a model normal over $\widetilde{K}$. Conversely, if a function field F/K has a normal model, then it is conservative [R, Thm. 12].

In accordance with this equivalence we say that a finitely generated regular extension F/K is **conservative** if it has a normal model, i.e., there exists a projective variety V defined over K whose function field is F such that each algebraic point of V is normal (over $\widetilde{K}$). As with curves, we use the term **variety** for V only when it is absolutely irreducible. With this interpretation of conservation theorem 10.6 can be generalized to function fields of several variables.

In the following Proposition we denote the hyperplane $u_0X_0+u_1X_1+\cdots+u_nX_n = 0$ by $H_{\mathbf{u}}$. A theorem of Bertini says that if V is a nonsingular variety, then for each $\mathbf{u}$ in a certain nonempty open subset of $\mathbb{P}^n$, $V \cap H_{\mathbf{u}}$ is also a nonsingular variety ([H, p. 179] or [L1, p. 217]). To prove the analogous result for normal varieties let us first recall some concepts and results from commutative algebra.

Consider a Noetherian local ring A with a maximal ideal P and let M be a nonzero finitely generated A-module. The (Krull) **dimension** of A is the maximal length of a descending sequence of prime ideals of A. The **dimension** of M is the dimension of the quotient ring $A/\mathrm{Ann}(M)$, where $\mathrm{Ann}(M) = \{a \in A|\ aM = 0\}$. A sequence $a_1, \ldots, a_m$ of elements of P is M**-regular** if a_i is not a zero divisor of $M/(a_1M + \cdots + a_{i-1}M)$, $i = 1, \ldots, m$. The **depth** of M is the length of the maximal regular sequence of elements of P. It is known that $\mathrm{depth}(M) \le \dim(M)$ ([G1, p. 33] or [M, p. 100]). Let therefore $\mathrm{codepth}(M) = \dim(M) - \mathrm{depth}(M)$.

Lemma 11.1: *With the above assumptions, we have:*

(a) *For each M-regular element $a \in P$, we have* $\mathrm{codepth}(M) = \mathrm{codepth}(M/aM)$ *[G1, p. 36].*

(b) *If B is another local Noetherian ring and ρ: $B \to A$ is a local homomorphism (i.e., ρ maps the maximal ideal of B into P) and M is also finitely generated over B (via ρ), then* $\mathrm{codepth}_A(M) = \mathrm{codepth}_B(M)$ *[G1, p. 36].*

(c) *If B is a local Noetherian domain and b is a nonzero element of the maximal ideal of B, then for $A = B/bB$ we have* $\mathrm{codepth}_B(B) = \mathrm{codepth}_A(A)$.

Proof of (c): Just notice that b is a B-regular element and use (a) and (b) to conclude that $\mathrm{codepth}_B(B) = \mathrm{codepth}_B(B/bB) = \mathrm{codepth}_A(A)$. ∎

These notions apply in particular to local rings of varieties. If $\mathbf{x}$ is a point on a variety V defined over K we denote its local ring (over $\widetilde{K}$) by $O_{V,\mathbf{x}}$. The dimension of $O_{V,\mathbf{x}}$ is equal to $\dim(V)$ minus $\dim_K(\mathbf{x})$. The following criterion for V to be normal is due to Krull ([G2, p. 108] or [M, p. 125]):

LEMMA 11.2: *A variety V of dimension r is normal if and only if for each $\mathbf{x} \in V$ we have:*

(a) *V is nonsingular in codimension 1 (i.e., its singular locus, V_{sing}, has dimension at most $r-2$), and*

(b) *if* $\dim(O_{V,\mathbf{x}}) \geq 2$, *then* $\mathrm{depth}(O_{V,\mathbf{x}}) \geq 2$.

Note that (a) replaces the equivalent condition of [G2] that if $\dim(O_{V,\mathbf{x}}) = 1$, then $O_{V,\mathbf{x}}$ is regular.

PROPOSITION 11.3: *Let V be a normal variety in $\mathbb{A}^n$ (resp. $\mathbb{P}^n$) of dimension $r \geq 2$ defined over a field K. Then there exists a nonempty open subset U of $\mathbb{P}^n$ such that for each $\mathbf{u} \in U$ the intersection $V \cap H_{\mathbf{u}}$ is a normal variety in $\mathbb{A}^n$ (resp. $\mathbb{P}^n$) of dimension $r-1$ defined over $K(\mathbf{u})$.*

Proof: It suffices to to prove the Lemma only in the affine case. Let $\mathbf{x}$ be a generic point of V over K. If $u_1, \ldots, u_n$ are algebraically independent elements over K and $u_0 = -\sum_{i=1}^n u_i x_i$, then, by [L1, p. 212], $V \cap H_{\mathbf{u}}$ is a variety of dimension $r-1$ defined over $K(\mathbf{u})$ (Here we put $X_0 = 1$ and consider $H_{\mathbf{u}}$ as a hyperplane in $\mathbb{A}^n$.) Apply Bertini's principle to find a nonempty open subset U_1 of $\mathbb{P}^n$ such that $V \cap H_{\mathbf{u}'}$ is a variety of dimension $r-1$ defined over $K(\mathbf{u}')$ for each $\mathbf{u}' \in U_1$ [FJ3, p.120].

Also, each point $\mathbf{y} \in V \cap H_{\mathbf{u}}$ is simple on V if and only if it is simple on $V \cap H_{\mathbf{u}}$ [L1, p. 217]. Let $f_1, \ldots, f_m$ be a system of generators for the ideal of all polynomials in $K[X_1, \ldots, X_n]$ that vanish on V. Then for $\mathbf{y}$ to be simple on V (resp., on $V \cap H_{\mathbf{u}}$)

means that the rank of the Jacobian matrix $(\partial f_i/\partial \mathbf{y}_j)$ (resp., the same matrix with the extra line $(u_j)_{1\le j\le n}$) is $n-r$ (resp., $n-r+1$). Apply elimination of quantifiers over algebraically closed fields to find an open subset U_2 of U_1 such that for each $\mathbf{u}'\in U_2$ and for each $\mathbf{y}\in V\cap H_{\mathbf{u}'}$ the point $\mathbf{y}$ is simple on V if and only if it is simple on $V\cap H_{\mathbf{u}'}$ [FJ3, p. 103].

If V_{sing} is empty, let $U_3=U_2$, otherwise let $V_{\mathrm{sing},i}$, $i=1,\ldots,m$ be the $\widetilde{K}$-irreducible components of V_{sing}. By Lemma 11.2(a), $\dim(V_{\mathrm{sing},i})\le r-2$. Choose a $\widetilde{K}$-rational point $\mathbf{a}_i$ of $V_{\mathrm{sing},i}$. Let U_3 be the set of all $\mathbf{u}\in U_2$ such that $\mathbf{a}_i\notin H_{\mathbf{u}}$, $i=1,\ldots,m$. Then U_3 is a nonempty open set and for each $\mathbf{u}\in U_3$ the intersection $V_{\mathrm{sing}}\cap H_{\mathbf{u}}$ has dimension at most $r-3$ [L1, p. 36]. Hence $V\cap H_{\mathbf{u}}$ satisfies condition (a) of Lemma 11.2.

To deal with condition (b) of Lemma 11.2 consider for each $s\ge 0$ the closed subset $Z_s(V)=\{\mathbf{y}\in V|\ \mathrm{codepth}(O_{V,\mathbf{y}})>s\}$ of V (This is a special case of Theorem 12.1.1(v) of [G3, p. 174] in which X is the variety V, $Y=\mathrm{Spec}(K)$, f is the constant map and $\mathcal{F}=O_V$ is the structure sheaf of V.) If $s\ge r-1$, then $Z_s(V)$ is empty. Recall that the codimension of a closed subset Z of V is given by $\mathrm{codim}(Z,V)=\dim(V)-\dim(Z)$ [Ha, p. 128]. By Proposition 5.7.4(i) of [G2, p. 104] (or directly from the definitions), condition (b) of Lemma 11.2 is equivalent to the following one:

(1) $\mathrm{codim}(Z_s(V),V)>s+2$ for each $0\le s\le r-2$.

Let U_4 be a nonempty open set consisting of all $\mathbf{u}\in U_3$ such that for each s between 0 and $r-2$, if $Z_s(V)$ is nonempty, then $H_{\mathbf{u}}$ contains no $\widetilde{K}$-irreducible component of $Z_s(V)$.

To conclude the proof of the proposition we prove that if $\mathbf{u}\in U_4$, and $W=V\cap V_u$, then W satisfies condition (1). Indeed, let $s\ge 0$ such that $Z_s(V)$ is nonempty. If $\mathbf{y}\in W$, then $O_{W,y}=O_{V,y}/(u_0+u_1x_1+\cdots+u_nx_n)O_{V,y}$. By lemma 11.1(c), $\mathrm{codepth}(O_{W,y})=\mathrm{codepth}(O_{V,y})$. Hence $Z_s(W)=Z_s(V)\cap H_{\mathbf{u}}$. Conclude that $\mathrm{codim}(Z_s(W),W)=\mathrm{codim}(Z_s(V),V)>s+2$ [L1, p. 36], as desired. ∎

THEOREM 11.4: *Let F be a finitely generated regular conservative extension of an infinite field K. Then F/K is a stable extension.*

Proof: Do induction on $r = \text{trans.deg}(F/K)$. Theorem 10.6 covers the case $r = 1$. So suppose that $r \geq 2$.

Let V be a normal model for F/K in $\mathbb{P}^n$. Assume without loss that V is contained in no hyperplane. Take a generic point $\mathbf{x} = 1{:}x_1{:}\cdots{:}x_n$ of V over K such that $F = K(\mathbf{x})$. With U as in Proposition 11.3, choose $\mathbf{u} \in U$ such that u_0 is transcendental over K and $u_1, \ldots, u_n \in K$. Then $V \cap H_{\mathbf{u}}$ is a normal variety of dimension $r - 1$ over $K(u_0)$. If $\mathbf{x}'$ is a generic point of $V \cap H_{\mathbf{u}}$ over $K(u_0)$, then $K(u_0, \mathbf{x}') = K(\mathbf{x}')$ and $\text{trans.deg}(K(\mathbf{x}')/K) = r$. So, we may identify $\mathbf{x}'$ with $\mathbf{x}$. In particular we identify $K(\mathbf{u}_0)$ as a subfield of F and find that F is a regular conservative extension of $K(u_0)$.

Now use induction on r to choose a stabilizing basis $t_1, \ldots, t_r$ for $F/K(u_0)$. The elements $u_0, t_1, \ldots, t_{r-1}$ form a stabilizing basis for F/K. Conclude that F/K is stable.

∎

12. A normal PAC extension of a Hilbertian field.

Recall that a field N is PAC if every nonvoid absolutely irreducible variety defined over N has an N-rational point [FJ3, p. 129]. Section 4 of [FJ2] applies the stability of fields of characteristic 0 to construct to each countable Hilbertian field K of characteristic 0 a normal PAC extension N which is itself Hilbertian. This section generalizes the construction to the case where $\text{char}(K) > 0$.

Let $f \in K[T, X]$ be an absolutely irreducible polynomial such that $\partial f/\partial X \neq 0$. Let $F = K(t, x)$, with $f(t, x) = 0$, be the function field of the plane curve $f(T, X) = 0$. We say that f is **stable with respect to** X if $\mathcal{G}(f(t, X), K(t)) = \mathcal{G}(f(t, X), \widetilde{K}(t))$ [FJ3, p. 186]. This is equivalent to the condition that the Galois hull $\widehat{F}$ of the separable extension $F/K(t)$ is a regular extension of K, i.e., that t is a stabilizing element for the extension F/K.

Lemma 12.1: *Let N be an algebraic extension of a perfect infinite field K. A sufficient condition for N to be* PAC *is that for every absolutely irreducible polynomial $f \in K[T, X]$ which is stable with respect to X and for each finite subset A of K there exist $a, b \in N$ such that $f(a, b) = 0$ and $a \notin A$.*

Proof: It suffices to prove that each plane curve Γ defined over K has an N-rational point [FJ3, Thm. 10.4]. So let F be the function field of a plane curve Γ over K. Choose a stabilizing transcendental element t for F/K (Theorem 10.6). Choose a primitive element x for the extension $F/K(t)$ and let $f(T,X)$ be an irreducible polynomial with coefficients in K such that $f(t,x) = 0$. Then f is stable with respect to X. By assumption $f(T,X)$ has infinitely many N-rational zeros. As the curve $f(T,X) = 0$ is K-birationally equivalent to Γ almost each of these points gives an N-rational point of Γ. ∎

Let $f_1, \ldots, f_m \in K[T,X]$ be irreducible polynomials separable with respect to X. Let $g \in K[T]$ be a nonzero polynomial. The set $H(f_1, \ldots, f_m; g)$ of all $a \in K$ such that $g(a) \neq 0$ and $f_1(a,X), \ldots, f_m(a,X)$ are irreducible and separable in $K[X]$ is a **separable Hilbert set**. A field L is **separably Hilbertian** if each separable Hilbert subset of L is nonempty [FJ3, p. 147]. It is not difficult to show that the maximal purely inseparable extension of a Hilbertian field is separably Hilbertian [FJ3, p. 149]. In particular if K is a finitely generated transcendental extension of a field K_0, then the maximal purely inseparable extension of K is separably Hilbertian. Also if K' is a finite separable extension of a separably Hilbertian field K, then each separable Hilbert subset of K' contains a Hilbert subset of K (the proof of [FJ3, Lemma 11.7] is valid with minor modifications also for separable Hilbert sets.)

We denote the absolute Galois group of a perfect field K by $G(K)$. If $\sigma_1, \ldots, \sigma_e \in G(K)$, then $\widetilde{K}(\sigma_1, \ldots, \sigma_e)$ denotes the fixed field of $\sigma_1, \ldots, \sigma_e$ in $\widetilde{K}$.

THEOREM 12.2: *Let K be a countable perfect separably Hilbertian field. Then K has a Galois extension N with the following properties:*

(a) *$\mathcal{G}(N/K)$ is isomorphic to the direct product of infinitely many finite groups,*

(b) *N is a separably Hilbertian field,*

(c) *N contains no field of the form $\widetilde{K}(\sigma_1, \ldots, \sigma_e)$, with $\sigma_1, \ldots, \sigma_e \in G(K)^e$, and*

(d) *N is* PAC.

Proof: Order the countable set of all absolutely irreducible polynomials in $K[T,X]$ which are stable with respect to X in a sequence: $f_1(T,X), f_2(T,X), f_3(T,X), \ldots$ such that each polynomial occurs infinitely many times. Take a sequence

$$g_1(T,X), g_2(T,X), g_3(T,X), \dots$$

of absolutely irreducible polynomials with Galois groups over $K(T)$ isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Let t be transcendental over K. Let $G_i = \mathcal{G}(f_i(t,X), K(t))$, $i = 1,2,3,\dots$. Construct by induction a linearly disjoint sequence $K_1, L_1, K_2, L_2, K_3, L_3, \dots$ of Galois extensions of K such that K_i contains a zero of f_i, $\mathcal{G}(K_i/K) \cong G_i$, and $\mathcal{G}(L_i/K) \cong \mathbb{Z}/2\mathbb{Z}$, $i = 1,2,3,\dots$ [FJ3, Lemma 15.8]. Let N be the compositum of all K_i. Then N is a Galois extension of K and $\mathcal{G}(N/K) \cong \prod_{i=1}^{\infty} G_i$.

To show that N is separably Hilbertian assume without loss that G_1 is a nontrivial group. Then N is a finite proper separable extension of $N_0 = \prod_{i=2}^{\infty} K_i$ and N_0/K is Galois. The desired result is a special case of [FJ3, Cor. 12.15].

Denote the compositum of all L_i by L. It is a Galois extension of K with Galois group isomorphic to the direct product of countably many isomorphic copies of $\mathbb{Z}/2\mathbb{Z}$. In particular its rank is infinite. If there were a positive integer e and $\sigma_1, \dots, \sigma_e \in G(K)$ such that $\widetilde{K}(\sigma_1, \dots, \sigma_e) \subseteq N$, then $\widetilde{K}(\sigma_1, \dots, \sigma_e)$ would be linearly disjoint from L over K. Hence $\mathcal{G}(L/K)$ would be a quotient of the group $G(\widetilde{K}(\sigma_1, \dots, \sigma_e))$ and therefore its rank would be at most e, a contradiction.

Finally Lemma 12.1 implies that N is PAC. ∎

Remark 12.3: If we could strengthen Theorem 10.7 and prove that every infinite field is stable in dimension 1, then we could take K in Theorem 12.2 to be Hilbertian. ∎

Problem 12.4: *Let L be a purely inseparable extension of a field K. Suppose that L is* PAC. *Is K also* PAC*?*

13. Concluding remarks.

The assumption in Theorem 10.5 that F/K is conservative is indispensable. Indeed, it is known that if a function field F/K has a projective smooth model, then F/K is conservative [R, Thm. 12]. Since the node model of F/K we construct is a projection of a smooth space model, the assumption that F/K is conservative is vital to our proof. Moreover, it is vital for the theorem itself. Indeed if F/K has a projective plane node model Γ, then F/K is conservative. To prove this claim (without verifying each detail) let $\mathbf{a}$ be a singular point of Γ. Denote the local ring of Γ at $\mathbf{a}$ by O and let O' be the integral closure of O in F. We have to prove that $\widetilde{K}O'$ is integrally closed. This is so because the integral closure R of $\widetilde{K}O'$ in $\widetilde{K}F$ is the intersection of two distinct valuation rings, R_1 and R_2, corresponding to the two tangents of Γ at $\mathbf{a}$. Since the intersection multiplicity of each of the tangents with Γ at $\mathbf{a}$ is 3 the semilocal ring $\widetilde{K}O'$ contains for each i a prime element of R_i. An application of Nakayama's lemma implies that $\widetilde{K}O' = R$.

The second case which our methods fail to cover is when K is finite. In this case projections cannot avoid bad points.

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