Reduction of Abelian Varieties and Curves

In Erinnerung an Wulf-Dieter Geyer (1939-2019)

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Abstract

Consider a noetherian domain R_0 with quotient field K_0. Let K be a finitely generated regular transcendental field extension of K_0. We construct a noetherian domain R with $\mathrm{Quot}(R) = K$ that contains R_0 and embed $\mathrm{Spec}(R_0)$ into $\mathrm{Spec}(R)$. Then, we prove that key properties of abelian varieties and smooth geometrically integral projective curves over K are preserved under reduction modulo $\mathfrak{p}$ for "almost all" $\mathfrak{p} \in \mathrm{Spec}(R_0)$ (Remark 1.5).

Notation

- $\tilde{K}$ is the algebraic closure of a field K. Occasionally, we write K_{alg} for $\tilde{K}$.
- K_{sep} is the separable closure of K in $\tilde{K}$.
- K_{ins} is the maximal purely inseparable extension of K in $\tilde{K}$.
- $\mathrm{Gal}(K) := \mathrm{Gal}(K_{\mathrm{sep}}/K)$ is the absolute Galois group of K.
- $\mathbf{o}$ denotes the zero point of a given additive abelian variety A.
- $\mathbf{0} = \{\mathbf{o}\}$ with $\mathbf{o}$ as in the preceding notation.

Introduction

The theory of reduction of algebro-geometric objects has a long history that we won't try to recapitulate here. We only mention Ehud Hrushovski's work [Hru98] in which he proves several "good reduction theorems" modulo prime numbers for algebro-geometric objects over finitely generated transcendental extensions of $\mathbb{Q}$.

We consider an integrally closed noetherian domain R_0 such that for every non-zero $c \in R_0$ there exist infinitely many prime ideals of R_0 that do not contain

c. Then we construct an integrally closed noetherian domain R which is finitely generated as a ring over R_0, and a finitely generated regular transcendental extension K/K_0 of fields such that $K_0 = \mathrm{Quot}(R_0)$ and $K = \mathrm{Quot}(R)$. We embed $\mathrm{Spec}(R_0)$ into $\mathrm{Spec}(R)$, consider each $\mathfrak{p} \in \mathrm{Spec}(R_0)$ as a prime ideal of R (Convention 1.3), and let $\bar{K}_\mathfrak{p}$ be the quotient field of $R/\mathfrak{p}$.

Then, following Hrushovski, we prove in a few cases, that algebro-geometric objects over K retain their properties under reduction modulo $\mathfrak{p}$, for **almost all** $\mathfrak{p} \in \mathrm{Spec}(R_0)$, i.e. for all $\mathfrak{p} \in \mathrm{Spec}(R_0)$ that lie in a non-empty Zariski-open subset of $\mathrm{Spec}(R_0)$ (see Remark 1.5).

Theorem A (Theorem 3.11): *Let A be an abelian variety over K such that $A(K_{0,\mathrm{sep}}K)$ is finitely generated. Then, the following statements hold:*

(a) *For almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$, we have that $\bar{A}_\mathfrak{p}$ is an abelian variety over $\bar{K}_\mathfrak{p}$ with $\dim(\bar{A}_\mathfrak{p}) = \dim(A)$.*

(b) *For almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$, the reduction map $\rho_\mathfrak{p}\colon A(K) \to \bar{A}_\mathfrak{p}(\bar{K}_\mathfrak{p})$ is injective on $A_{\mathrm{tor}}(K)$.*

(c) *If l is a prime number such that $l \neq \mathrm{char}(K_0)$ and $A_l(K_{0,\mathrm{sep}}K) = \mathbf{0}$, then $\bar{A}_{\mathfrak{p},l}(\bar{K}_\mathfrak{p}) = \mathbf{0}$ for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$.*

(d) *For every large prime number l and for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$, the map $\rho_\mathfrak{p}$ induces an injection*

$$\bar{\rho}_{\mathfrak{p},l}\colon A(K)/lA(K) \to \bar{A}_\mathfrak{p}(\bar{K}_\mathfrak{p})/l\bar{A}_\mathfrak{p}(\bar{K}_\mathfrak{p}).$$

(e) *$\rho_\mathfrak{p}\colon A(K) \to \bar{A}_\mathfrak{p}(\bar{K}_\mathfrak{p})$ is an injection for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$.*

In addition to basic properties of abelian varieties and a simple criterion for the injectivity of a homomorphism of abelian groups (Lemma 3.1), the proof of Theorem A applies model theoretic tools, especially ultra-products (Lemma 3.8).

Theorem B (Theorem 4.13): *Let A be an abelian variety over K such that no simple abelian subvariety of $A_{\tilde{K}}$ is defined over $\tilde{K}_0$.*

Then, for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$, no simple abelian subvariety of the abelian variety $\bar{A}_\mathfrak{p}$ over $\bar{K}_\mathfrak{p}$ is defined over $\bar{K}_{0,\mathfrak{p},\mathrm{alg}}$.

This is a generalization to arbitrary characteristic of a result of Hrushovski in characteristic 0. The proof follows that of Hrushovski, adding the necessary adjustments to the general case.

Theorem C (Theorem 5.5): *Let C be a smooth geometrically integral curve over K of genus $g \geq 1$. Suppose that C has a K-rational point, C is conservative (Remark 2.1), and $C_{\tilde{K}}$ is not birationally equivalent to a curve which is defined over $\tilde{K}_0$.*

Then, for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$ the reduced curve $\bar{C}_\mathfrak{p}$ is geometrically integral over $\bar{K}_\mathfrak{p}$, smooth, conservative of genus g, $\bar{C}_\mathfrak{p}(\bar{K}_\mathfrak{p}) \neq \emptyset$, but $\bar{C}_{\mathfrak{p},\bar{K}_{\mathfrak{p},\mathrm{alg}}}$ is not birationally equivalent to a curve which is defined over $\bar{K}_{0,\mathfrak{p},\mathrm{alg}}$.

The proof of Theorem C applies Theorem B for $g = 1$ and the basic tool of the coarse moduli space for curves of a fixed genus g up to isomorphism for $g \geq 2$.

The first four sections of this work follow Hrushovski's style in [Hru98] and mainly use "elementary statements" about algebraically closed fields in order to prove Theorems A and B. In Section 5 we switch to the language of schemes. [1]

Remark D: It turns out that not every algebro-geometric statement defined over K and holds over $\tilde{K}$, where $K = K_0$, is true over $\bar{K}_{0,\mathfrak{p},\mathrm{alg}}$ for almost all prime ideals $\mathfrak{p} \in \mathrm{Spec}(R_0)$.

For example, there are abelian varieties A of dimension 2 defined over a number field K such that $A_{\tilde{\mathbb{Q}}}$ is simple but $\bar{A}_{\mathfrak{p}}$ is not simple for almost all prime ideals $\mathfrak{p}$ of the ring of integers of K [EEHK09, p. 146, Rem. 16]. ∎

1 Reduction modulo almost all $\mathfrak{p}$

{Rmaa}

We fix for the whole work an extension R/R_0 of integral noetherian domains such that $K := \mathrm{Quot}(R)$ is a finitely generated regular transcendental extension of $K_0 := \mathrm{Quot}(R_0)$[2]. Let $r = \mathrm{trans.deg}(K/K_0)$. In Setup 1.1 below we embed $\mathrm{Spec}(R_0)$ into $\mathrm{Spec}(R)$ and observe that for "almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$" the residue field $\bar{K}_{\mathfrak{p}} := \mathrm{Quot}(R/\mathfrak{p})$ is a finitely generated regular extension of $\bar{K}_{0,\mathfrak{p}} := \mathrm{Quot}(R_0/R_0 \cap \mathfrak{p})$ of transcendence degree r. The main result of Section 2 says that if C is a conservative geometrically integral curve of genus g over K, then for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$, the reduced curve $\bar{C}_{\mathfrak{p}}$ is a conservative geometrically integral curve of genus g over $\bar{K}_{\mathfrak{p}}$.

{FGNe}

Setup 1.1 (Finitely generated extension)**.** Our starting point is an integrally closed noetherian domain R_0 with quotient field K_0. We assume that

> for every non-zero $c \in R_0$ there exist infinitely many prime ideals of R_0 that do not contain c. (1) {infpi}

For example, we may take R_0 to be a Dedekind domain with infinitely many maximal ideals. The ring $\mathbb{Z}$ or rings $F[t]$ of polynomials of one variable over an arbitrary field are Dedekind rings with infinitely many prime ideals. Moreover, if R_0 is a Dedekind ring, then its integral closure in any finitely generated extension of $\mathrm{Quot}(R_0)$ is also a Dedekind ring [ZaS75, p. 281, Thm. 19].

We follow [Liu06, p. 55, Def. 3.47] and define an **affine variety over** K_0 to be an affine scheme associated to a finitely generated algebra over K_0 [Liu06, p. 43, Def. 3.2]. Then, an **algebraic variety over** K_0 is a K_0-scheme X which is covered by finitely many affine open subvarieties over K_0. However, in contrast to [Liu06], we assume all of the algebraic varieties in this work to be separated.

Accordingly, a **curve** over K_0 in this work is just an algebraic variety over K_0 whose irreducible components [Liu06, p. 61, first two paragraphs of Section 4.2] are of dimension 1 [Liu06, p. 73, Sec. 5.3].

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[1] The authors are indebted to Gerhard Frey for his contribution to Section 5.

[2] All rings appearing in this work are supposed to be commutative with a unit.

We are especially interested in geometrically integral affine varieties V over K_0 [Liu06, p. 90, Def. 2.8]. In the language of classical algebraic geometry these objects are just called **varieties** defined over K_0. See [Wei62], [Lan58], or [FrJ08, Sections 10.1 and 10.2]. See also Example 1.8.

For example, let K be a finitely generated regular extension of K_0 of transcendence degree $r \geq 1$. Choose a separating transcendence base $u_1, \ldots, u_r$ for K/K_0 and set $\mathbf{u} = (u_1, \ldots, u_r)$. Then, the integral closure R of $R_0[\mathbf{u}]$ in K is a finitely generated $R_0[\mathbf{u}]$-module [Eis95, p. 298, Prop. 13.14], so $R = R_0[\mathbf{x}]$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and $K = \mathrm{Quot}(R)$. In particular, R is a noetherian domain [ZaS75, p. 265, Cor. 1]. By [FrJ08, p. 175, Cor. 10.2.2], the affine variety $V := \mathrm{Spec}(K_0[\mathbf{x}])$ over K_0 is geometrically integral and $\mathbf{x}$ is a **generic point** of V.

Let $w \in K_0[\mathbf{x}]$ be a basic minor of the Jacobian matrix of V with respect to polynomials in $K_0[\mathbf{x}]$ that define V. Adding w^{-1} to $\{x_1, \ldots, x_n\}$, we may assume that V is also smooth [Mum88, p. 233, Cor. 1]. ∎

{ETNs}

Remark 1.2. Let K_0' be a finite separable extension of K_0 and R_0' the integral closure of R_0 in K_0'. Consider a non-zero $c' \in R_0'$. Then, the norm c of c' from K_0' to K_0 lies in R_0 [Lan93, p. 337, Cor. 1.6]. Therefore, if $\mathfrak{p}$ is a prime ideal of R_0 that does not contain c, then each prime ideal of R_0' over $\mathfrak{p}$ does not contain c'. By Condition (1) on R_0, there are infinitely many such prime ideals of R_0. Hence, there are infinitely many prime ideals of R_0' that do not contain c'. Thus, Condition (1) is also satisfied for R_0' replacing R_0. ∎

The most important examples for algebraic varieties over K_0 which are not affine are **projective varieties** defined by homogeneous polynomials [Liu06, p. 55, Def. 3.47]. In particular, **abelian varieties** over K_0 can be represented as projective varieties [Mil85, p. 113, Thm. 7.1].

{ZARo}

Convention 1.3. Let R_0 and R be the integral domains introduced in Setup 1.1. We embed $\mathrm{Spec}(R_0)$ into $\mathrm{Spec}(R)$ and fix this embedding for the whole work in the following way:

For each $\mathfrak{p} \in \mathrm{Spec}(R_0)$ we choose algebraically independent elements $\bar{u}_{\mathfrak{p},1}, \ldots, \bar{u}_{\mathfrak{p},r}$ over $\bar{K}_{0,\mathfrak{p}}$, set $\bar{\mathbf{u}}_{\mathfrak{p}} = (\bar{u}_{\mathfrak{p},1}, \ldots, \bar{u}_{\mathfrak{p},r})$, and let $\mathfrak{p}'$ be the kernel of the map $R_0[\mathbf{u}] \to \bar{K}_{0,\mathfrak{p}}[\bar{\mathbf{u}}_{\mathfrak{p}}]$ that extends the map $R_0 \to \bar{K}_{0,\mathfrak{p}}$ and maps $\mathbf{u}$ onto $\bar{\mathbf{u}}_{\mathfrak{p}}$. Note that $\mathfrak{p}'$ is the smallest prime ideal of $R_0[\mathbf{u}]$ that contains $\mathfrak{p}$.

Then we apply the going up theorem [AtM69, pp. 61, 62, Cor. 5.9, Thm. 5.10] to choose a prime ideal ideal $\mathfrak{p}''$ of R that lies over $\mathfrak{p}'$ and note that $\mathfrak{p}''$ is a minimal prime ideal of R over $\mathfrak{p}'$. Thus, $\mathfrak{p}''$ is also a minimal prime ideal of R over $\mathfrak{p}$.

Finally, we fix $\mathfrak{p}''$ and redenote it by $\mathfrak{p}$.

Claim: *For each non-zero $c \in R$ there exists a non-zero $c_0 \in R_0$ such that if $\mathfrak{p} \in \mathrm{Spec}(R_0)$ and $c_0 \notin \mathfrak{p}$, then $c \notin \mathfrak{p}$.*

Indeed, assume first that $c \in R_0[\mathbf{u}]$. Then, $c = f(\mathbf{u})$ for some non-zero polynomial f with coefficients in R_0. At least one of those coefficients, say c_0, is non-zero. Hence, if $\mathfrak{p} \in \mathrm{Spec}(R_0)$ and $c_0 \notin \mathfrak{p}$, then $\bar{c}_\mathfrak{p} = \bar{f}_\mathfrak{p}(\bar{\mathbf{u}}_\mathfrak{p}) \neq 0$, which means that $c \notin \mathfrak{p}$.

In the general case, R is integral over $R_0[\mathbf{u}]$ (Setup 1.1). Hence, there exist $d_0, \ldots, d_{k-1} \in R_0[\mathbf{u}]$ such that

$$c^k + d_{k-1}c^{k-1} + \cdots + d_1 c + d_0 = 0 \text{ with } d_0 \neq 0. \quad (2)$$ {ukcd}

By the preceding paragraph, there exists a non-zero $c_0 \in R_0$ such that if $\mathfrak{p} \in \mathrm{Spec}(R_0)$ and $c_0 \notin \mathfrak{p}$, then $d_0 \notin \mathfrak{p}$. Hence, by (2), $c \notin \mathfrak{p}$, as claimed.

Having proved the claim, recall that if w is a non-zero element of R, as in the last paragraph of Setup 1.1, then one can identify $\mathrm{Spec}(R[w^{-1}])$ with $\{\mathfrak{p} \in \mathrm{Spec}(R) \mid w \notin \mathfrak{p}\}$. If we now wish to replace R by $R[w^{-1}]$, we may use the claim to choose a non-zero $w_0 \in R_0$ such that if $\mathfrak{p} \in \mathrm{Spec}(R_0)$ and $w_0 \notin \mathfrak{p}$, then $w \notin \mathfrak{p}$. Then, we may replace R_0 by $R_0[w_0^{-1}]$.

Recall that every non-empty Zariski-open subset S_0 of $\mathrm{Spec}(R)$ (hence, also of $\mathrm{Spec}(R[w^{-1}])$ contains a set of the form $\{\mathfrak{p} \in \mathrm{Spec}(R) \mid c \notin \mathfrak{p}\}$ for some non-zero $c \in R$. Hence, by the claim, S_0 contains a set of the form $\{\mathfrak{p} \in \mathrm{Spec}(R_0) \mid c_0 \notin \mathfrak{p}\}$ with a non-zero $c_0 \in R_0$. Therefore, by our assumption in Setup 1.1 on R_0, S_0 is infinite. ∎

{rrrr}

Remark 1.4. We have used the letter r in Setup 1.1 for the transcendence degree of K/K_0. It is reused with this meaning also in Convention 1.3, but latter on it may get another meaning.

{SPEc}

Remark 1.5 (Reduction modulo almost all $\mathfrak{p}$)**.** Let R be the integral domain introduced in Setup 1.1. For each $\mathfrak{p} \in \mathrm{Spec}(R)$ let $\varphi_\mathfrak{p}\colon R \to R/\mathfrak{p}$ be the residue map. We say that a "mathematical statement θ about $\tilde{K}$" holds **for almost all** $\mathfrak{p} \in \mathrm{Spec}(R)$ if there exists a non-zero $c \in R$ such that θ holds modulo $\mathfrak{p}$ in $\bar{K}_{\mathfrak{p},\mathrm{alg}}$ whenever $\bar{c}_\mathfrak{p} := \varphi_\mathfrak{p}(c) \neq 0$. Thus, θ holds along a non-empty Zariski-open subset of $\mathrm{Spec}(R)$. It follows from Convention 1.3 that θ holds modulo $\mathfrak{p}$ also for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$.

If R_0 is a Dedekind domain, then "for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$" means "for all but finitely many $\mathfrak{p} \in \mathrm{Spec}(R_0)$". In this case, which is our main concern, each $\mathfrak{p} \in \mathrm{Spec}(R_0)$ induces a discrete valuation on $\mathrm{Quot}(R_0)$ and our extension of $\mathfrak{p}$ to $R_0[\mathbf{u}]$ yields a discrete valuation on $\mathrm{Quot}(R_0(\mathbf{u}))$, known as the "Gauss' valuation". Our next extension of $\mathfrak{p}$ (in Convention 1.3) to a prime ideal of R yields a discrete valuation on K but it is not unique. Nevertheless, the "almost all" claim mentioned in the preceding paragraph holds for each choice of the extensions of the $\mathfrak{p}$'s to R. ∎

{ELSt}

Remark 1.6 (Elementary statements)**.** One type of statements about $\tilde{K}$ that we consider are the **elementary statements**, that is, those that are equivalent to sentences in the first order language $\mathcal{L}(\mathrm{ring}, R)$ of rings with a constant symbol b for each element b of R [FrJ08, p. 135, Example 7.3.1 and p. 136, Example 7.3.2]. By [FrJ08, p. 167, Cor. 9.2.2], if a statement θ of this type holds over $\tilde{K}$, then there exists a non-zero $c \in R$ such that θ holds in $\tilde{F}$ for each algebraically closed field $\tilde{F}$ which contains a homomorphic image $\bar{R}$ of R in which the image $\bar{c}$ of c is non-zero. In particular, θ holds in $\bar{K}_{\mathfrak{p},\mathrm{alg}}$ for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$. By Remark 1.5, θ holds in $\bar{K}_{\mathfrak{p},\mathrm{alg}}$ also for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$.

The simplest example for such a θ is "$a \neq b$", where a, b are distinct elements of R. In case $c = a - b$, this statement holds for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $c \notin \mathfrak{p}$.

Note that the proof of Corollary 9.2.2 of [FrJ08] is solely based on the Euclid algorithm for dividing polynomials with residue. This makes it immediately available for all algebro geometric statements that involve finitely many polynomials with bounded degrees.

We consider also statements about algebro-geometric objects defined over $\tilde{K}$ (hence, by elements of R) for which reduction modulo $\mathfrak{p}$ is defined, at least for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$. For many of these statements one may prove that they are elementary. However, a direct proof that a certain mathematical statement θ is elementary could be tedious. In such cases, one may first use algebro geometric tools in order to prove that θ is equivalent to an elementary statement θ'. This has to be done in such a way that the proof of the equivalence $\theta \leftrightarrow \theta'$ itself is formal in the sense of [FrJ08, p. 150] (see also Remark 1.7 below). Then, one may apply the preceding paragraph to θ' and to the proof of $\theta \leftrightarrow \theta'$ to conclude that θ holds for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$. ∎

{frml}

Remark 1.7 (Formal proofs)**.** Following [FrJ08, p. 135, Example 7.3.1], let $\mathcal{L} := \mathcal{L}(\mathrm{ring}, R)$ be the first order language for the theory of fields which contain a homomorphic image of R. Let $\Pi(R)$ be the usual axioms of the theory of fields enhanced by all of the equalities $a_1 + b_1 = c_1$ and $a_2 b_2 = c_2$ with $a_i, b_i, c_i \in R$ that hold in R (i.e. **the positive diagram** of R).

A **formal proof** of a **sentence** φ of $\mathcal{L}$ ([FrJ08, p. 149, Sec. 8.1]) is a finite sequence $(\varphi_1, \dots, \varphi_n)$ of sentences of $\mathcal{L}$ with $\varphi_n = \varphi$ such that each sentence φ_m with $m \le n$ is either a **logical axiom** given by (3a), (3b), or (3c) on pages 150, 151 of [FrJ08], or an axiom in $\Pi(R)$, or φ_m is a consequence of $\{\varphi_1, \dots, \varphi_{m-1}\}$ by one of the **inference rules** (2a) and (2b) on page 150 of [FrJ08]. ∎

{frey}

Example 1.8. (a) Let W be a geometrically integral affine variety over K in $\mathbb{A}_K^{n'}$ of dimension r' with generic point $\mathbf{y} := (y_1, \dots, y_{n'})$ and function field $F := K(\mathbf{y})$. For almost all $\mathfrak{p} \in \mathrm{Spec}(R)$ the variety W is defined by polynomial equations with coefficients in the localization $R_\mathfrak{p}$ of R at $\mathfrak{p}$. For those $\mathfrak{p}$ let $\bar{W}_\mathfrak{p}$ be the Zariski-closed subset of $\mathbb{A}_{\bar{K}_\mathfrak{p}}^{n'}$ defined by the equations that define W reduced modulo $\mathfrak{p}R_\mathfrak{p}$. Thus, one considers the closure of W in $A_R^{n'}$ and passes to the fiber induced by the combined homomorphism $R \to R_\mathfrak{p} \to R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = \bar{K}_\mathfrak{p}$ Then, the Bertini-Noether theorem says that for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$, {good}

(3) $\bar{W}_\mathfrak{p}$ is a geometrically integral affine variety in $\mathbb{A}_{\bar{K}_\mathfrak{p}}^{n'}$ with $\dim(\bar{W}_\mathfrak{p}) = \dim(W)$.

The proof given in [FrJ08, p. 179, Prop. 10.4.2] is not direct. It uses the birational equivalence between W and a hypersurface and applies the absolute irreducibility modulo almost all $\mathfrak{p}$ of the polynomial that defines that hypersurface.

(b) Moreover, in the notation of Remark 1.5, for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$ we may extend the residue map $R \to R/\mathfrak{p}$ to a place $\tilde{K}(\mathbf{y}) \to \bar{K}_{\mathfrak{p},\mathrm{alg}}(\bar{\mathbf{y}}_{\mathfrak{p}})$ that maps $\mathbf{y}$ onto an n'-tuple $\bar{\mathbf{y}}_{\mathfrak{p}} := (\bar{y}_{1,\mathfrak{p}}, \ldots, \bar{y}_{n',\mathfrak{p}})$ which is a generic point of $\bar{W}_{\mathfrak{p}}$. By [FrJ08, p. 175, Cor. 10.2.2(a)], $\bar{F}_{\mathfrak{p}} := \bar{K}_{\mathfrak{p}}(\bar{\mathbf{y}}_{\mathfrak{p}})$ is a regular extension of $\bar{K}_{\mathfrak{p}}$. By (3),

$$\mathrm{trans.deg}(\bar{F}_{\mathfrak{p}}/\bar{K}_{\mathfrak{p}}) = \dim(\bar{W}_{\mathfrak{p}}) = \dim(W) = \mathrm{trans.deg}(F/K) = r'. \quad (4)$$ {trrd}

(c) If $f_1, \ldots, f_m \in K[X_1, \ldots, X_{n'}]$ generate the ideal of polynomials that vanish on W, then by the Jacobian matrix criterion, a point $\mathbf{a} \in W(\tilde{K})$ is simple on W if and only if

$$\mathrm{rank}\Big(\frac{\partial f_i}{\partial X_j}(\mathbf{a})\Big) = n' - r' \quad (5)$$ {jacb}

[Mum88, p. 233, Cor. 1].

Since by (4), $r' = \dim(\bar{W}_{\mathfrak{p}})$ for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$, (5) implies that $\bar{\mathbf{a}}_{\mathfrak{p}} \in \bar{W}_{\mathfrak{p}}(\bar{K}_{\mathfrak{p},\mathrm{alg}})$ is simple on $\bar{W}_{\mathfrak{p}}$, again for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$. Therefore, if W is smooth, then $\bar{W}_{\mathfrak{p}}$ is a smooth affine geometrically integral algebraic variety over $\bar{K}_{\mathfrak{p}}$ for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$.

(d) Following [Liu06, p. 90, Def. 2.8], a **geometrically integral algebraic variety** W **over** K is an algebraic variety over K (see Setup 1.1) such that $W_{\tilde{K}}$ is integral. By [GoW10, p. 70, Prop. 3.10], W can be consider as a union of a finite sets $\{W_i\}_{i \in I}$ of geometrically integral affine open subschemes such that for all $i, j \in I$ there exist a non-empty open subset W_{ij} and an isomorphism $\varphi_{ji}\colon W_{ij} \to W_j$ of schemes such that

$$W_{ii} = W_i, \text{ and } \varphi_{kj} \circ \varphi_{ji} = \varphi_{ki} \text{ on } W_{ij} \cap W_{ik} \text{ for } i, j, k \in I. \quad (6)$$ {abst}

Indeed, W is uniquely determined by the **gluing datum** $\{W_i, W_{ij}, \varphi_{ji}\}_{i,j \in I}$. In particular, $\dim(W) := \dim(W_i)$ is independent of i.

The corresponding object in the classical algebraic geometry is called an **abstract variety**. See [Lan58, Sec. IV6] or [FrJ08, p. 187], where the φ_{ji} in the preceding paragraph are replaced by birational functions that satisfy a modification of Condition (6).

It follows that the mathematical statement "W is a geometrically integral algebraic variety over K of dimension d" is elementary and therefore it remains true under reduction modulo $\mathfrak{p}$ for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$.

Similarly, the analogue statements (b) and (c) about W hold also in the case where W is an abstract variety. ∎

{DFNd}

Notation 1.9. Given morphisms of schemes, $X \to S$ and $T \to S$, we write X_T for the fiber product $X \times_S T$. If $S = \mathrm{Spec}(D)$ for a ring D and $T = \mathrm{Spec}(D')$ for some homomorphism $D \to D'$ of rings, then we often abbreviate $X_{\mathrm{Spec}(D')}$ by $X_{D'}$. If in particular, $D' = D_{\mathfrak{p}}/\mathfrak{p}D_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$ of D and $D \to D'$ is the combined homomorphism $D \to D_{\mathfrak{p}} \to D_{\mathfrak{p}}/\mathfrak{p}D_{\mathfrak{p}}$, then $X_{\mathfrak{p}} := X_{\mathrm{Spec}(D_{\mathfrak{p}}/\mathfrak{p}D_{\mathfrak{p}})}$ is the **fiber of** X **at** $\mathfrak{p}$ [Liu06, p. 83, Def. 1.13 and p. 46, Example 3.18].

Finally, given a homomorphism $D \to D'$ of rings, the canonical isomorphism $D' \otimes_D D_\mathfrak{p} \otimes_{D_\mathfrak{p}} D_\mathfrak{p}/\mathfrak{p}D_\mathfrak{p} \cong D'_\mathfrak{p}/\mathfrak{p}D'_\mathfrak{p}$ allows us to identify the fiber $X_\mathfrak{p}$ with the reduction $\bar{X}_\mathfrak{p}$ of $X := \mathrm{Spec}(D')$ at $\mathfrak{p}$.

However, in Section 5, we use the convention of the theory of schemes and consider the prime ideals of the ring R introduced in Setup 1.1 as points of the scheme $S = \mathrm{Spec}(R)$ for which we use the letter s. Still, the expression "for almost all $s \in S$" will mean "for all $s \in S$ that do not contain a fixed non-zero element c of R", equivalently "for all s in the open subscheme $\mathrm{Spec}(R_c)$ of S", where R_c is the localization of R at c. Also, we drop the bar over the reduced varieties and write for example W_s rather than $\bar{W}_s$ if W is an algebraic variety over K. ∎

2 The genus of a curve

{CRV}

We prove that a conservative geometrically integral curve over K preserves its genus under almost all reductions modulo $\mathfrak{p} \in \mathrm{Spec}(R_0)$.

{RSN1}

Remark 2.1. Let C be a geometrically integral curve over K with function field F. Then, F is a finitely generated regular extension of K [FrJ08, p. 175, Cor. 10.2.2(a)]. Riemann-Roch's theorem supplies a unique non-negative integer $g := \mathrm{genus}(F/K)$, called the **genus of** F/K, such that $\dim(\mathfrak{a}) = \deg(\mathfrak{a}) + 1 - g + \dim(\mathfrak{w} - \mathfrak{a})$ for every divisor $\mathfrak{a}$ and every canonical divisor $\mathfrak{w}$ of F/K [FrJ08, p. 55, Thm. 3.2.1]. One also calls g the **genus of** C and denote it by $\mathrm{genus}(C)$.

Being a regular extension of K, the field F is linearly disjoint from $\tilde{K}$ over K. By [Deu73, p. 132, Thm. 1], $\mathrm{genus}(FL/L) \le \mathrm{genus}(F/K)$ for each algebraic extension L of K. Thus, there exists a finite extension L of K such that the $\mathrm{genus}(FL/L)$ does not drop any more under algebraic extensions of the base field. This means that $\mathrm{genus}(C_L) = \mathrm{genus}(C_{\tilde{K}})$. We say that C_L is **conservative**. Hence, replacing K by L makes C conservative.

If C is conservative, then C is birationally equivalent over K to a smooth projective curve [GeJ89, Prop. 8.3]. Conversely, if C is smooth and projective, then C is conservative [Ros52, Thm. 12].

However, since removing the finitely many singular points from an arbitrary curve C makes it smooth, smoothness by itself does not make C conservative.

Finally we note that if C is smooth and projective (hence conservative), then in the language of schemes, $\mathrm{genus}(C_{\tilde{K}}) = \dim_{\tilde{K}} H^1(C_{\tilde{K}}, \mathcal{O}_{C_{\tilde{K}}})$ [Har77, p. 294, Prop. 1.1 and p. 295, Thm. 1.3]. ∎

{ABSg}

Lemma 2.2. *Let C be a conservative geometrically integral curve of genus g over K. Then, for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$, the curve $\bar{C}_\mathfrak{p}$ is a conservative geometrically integral curve of genus g over $\bar{K}_\mathfrak{p}$ and the same statement holds for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$.*

Proof. As in (3), $\bar{C}_\mathfrak{p}$ is a geometrically integral curve over $\bar{K}_\mathfrak{p}$, for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$. By assumption, $\mathrm{genus}(C_{\tilde{K}}) = g$. By [GrR21, Thm. 23],

$\mathrm{genus}(\bar{C}_{\mathfrak{p},\bar{K}_{\mathfrak{p},\mathrm{alg}}}) = g$ for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$. By [GrR21, Cor. 25], $\mathrm{genus}(\bar{C}_\mathfrak{p}) = g$ for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$. Hence, for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$, the curve $\bar{C}_\mathfrak{p}$ is a conservative geometrically integral curve of genus g. By Remark 1.6, this statement holds for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$. $\square$

{ABSh}

Remark 2.3. We supply an alternative proof to Lemma 2.2 which is more elaborate but has the advantage of presenting the genus in terms of the curve.

Since C is conservative, it is birationally equivalent over K to a smooth projective curve C' (Remark 2.1). The birational equivalence of C and C' is an elementary statement on the coefficients of the polynomials that define C and C'. Hence, by Example 1.8, for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$ the curve $\overline{C'}_\mathfrak{p}$ is smooth and projective, and birationally equivalent to $\bar{C}_\mathfrak{p}$ over $\bar{K}_\mathfrak{p}$. It follows from Remark 2.1 that $\bar{C}_\mathfrak{p}$ is conservative for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$. Thus,

$$\mathrm{genus}(\bar{C}_\mathfrak{p}) = \mathrm{genus}(\bar{C}_{\mathfrak{p},\mathrm{alg}}) \text{ for almost all } \mathfrak{p} \in \mathrm{Spec}(R). \qquad (7)$$ {rsnl}

By [GeJ89, Thm. 10.5], $C_{\tilde{K}}$ is birationally equivalent to a **projective plane node model** Γ. Since C is conservative,

$$g = \mathrm{genus}(C) = \mathrm{genus}(C_{\tilde{K}}) = \mathrm{genus}(\Gamma). \qquad (8)$$ {cons}

Let $\mathbf{p}_1, \ldots, \mathbf{p}_d$ be the singular points of Γ. For every $i \in \{1, \ldots, d\}$, Γ is defined, after translating $\mathbf{p}_i$ to the origin (1:0:0), by a homogeneous equation $f_i(X_0, X_1, X_2) = 0$, where

$$f_i(1, X_1, X_2) = (a_{i1}X_2 - a_{i2}X_1)(b_{i1}X_2 - b_{i2}X_1) + \sum_{j=3}^{m_i} g_{ij}(X_1, X_2), \qquad (9)$$ {fgij}

$a_{i1}, a_{i2}, b_{i1}, b_{i2} \in \tilde{K}$, $a_{i1}b_{i2} \neq a_{i2}b_{i1}$, and $g_{ij} \in \tilde{K}[X_1, X_2]$ is a homogeneous polynomial of degree j.

By [Ful89, p. 199, Prop. 5],

$$\mathrm{genus}(\Gamma) = \frac{(\deg(\Gamma) - 1)(\deg(\Gamma) - 2)}{2} - d, \qquad (10)$$ {fulp}

where actually the second term on the right hand side in that proposition is $-\sum_{i=1}^{d} \frac{r_{\mathbf{p}_i}(r_{\mathbf{p}_i}-1)}{2}$, with $r_{\mathbf{p}_i}$ being the smallest degree of the homogeneous terms on the right hand side of equation (9), namely 2.

For almost all $\mathfrak{p} \in \mathrm{Spec}(R)$ the curve $\bar{C}_{\mathfrak{p},\mathrm{alg}}$ is birationally equivalent to $\bar{\Gamma}_\mathfrak{p}$, and by the Jacobian criterion, $\bar{\mathbf{p}}_{1,\mathfrak{p}}, \ldots, \bar{\mathbf{p}}_{d,\mathfrak{p}}$ are the singular points of $\bar{\Gamma}_\mathfrak{p}$. Finally, the presentation (9) for the polynomial defining Γ in the neighborhood of $\bar{\mathbf{p}}_{i,\mathfrak{p}}$ (after translation) has the analogous form also modulo $\mathfrak{p}$. Hence, (10) remains valid modulo $\mathfrak{p}$, so

$$\mathrm{genus}(\bar{C}_\mathfrak{p}) \overset{(7)}{=} \mathrm{genus}(\bar{C}_{\mathfrak{p},\mathrm{alg}}) = \mathrm{genus}(\bar{\Gamma}_\mathfrak{p}) = \mathrm{genus}(\Gamma) \overset{(8)}{=} g,$$

as claimed.

As above, all of this holds also for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$. $\blacksquare$

3 Reduction of Abelian Varieties

{RAB}

Ehud Hrushovski proves in [Hru98, Lemma 4] that if K is a finitely generated extension of $\mathbb{Q}$ and A is an abelian variety over K such that $A(K_{0,\mathrm{sep}}K)$ is finitely generated (with $K_0 = K \cap \tilde{\mathbb{Q}}$), then "almost all" reductions $A \to \bar{A}$ map $A(K)$ injectively into $\bar{A}(\bar{K})$.

We adjust Hrushovski's proof to the field extension K/K_0, introduced in Setup 1.1. To this end, given an abelian additive group C and a positive integer n, we write $C_n = \{c \in C \mid nc = 0\}$, $C_{l^\infty} = \bigcup_{i=1}^\infty C_{l^i}$ for each prime number l, and $C_{\mathrm{tor}} = \bigcup_{n=1}^\infty C_n$. Recall that if C is finitely generated, then $C = C_0 \times C_{\mathrm{tor}}$, where C_0 is a finitely generated free abelian group and C_{tor} is a finite abelian group [Lan93, p. 46, Thm. 8.5]. In particular, C_{l^∞} is a finite group for every prime number l.

The proof relies on a basic lemma about abelian groups.

{INJc}

Lemma 3.1 ([Hru98], p. 198, Lemma 1)**.** *Let $\rho\colon B \to C$ be a homomorphism of abelian groups and let n be a positive integer. Suppose that $\bigcap_{i=1}^\infty n^i B = \mathbf{0}$, $C_n = \mathbf{0}$, and ρ induces an injective map $\bar{\rho}\colon B/nB \to C/nC$. Then, ρ is injective.*

Proof. Let $b \in B$ with $b \neq 0$. Since $\bigcap_{i=1}^\infty n^i B = \mathbf{0}$, there exists a smallest positive integer i such that $b \notin n^i B$. Thus, $b = n^{i-1}b'$ with $i \geq 1$ and $b' \in B \smallsetminus nB$. Since $\bar{\rho}$ is injective, $\rho(b') + nC = \bar{\rho}(b' + nB) \neq 0$, hence $\rho(b') \notin nC$. In particular, $\rho(b') \neq 0$.

Starting from $C_n = \mathbf{0}$, induction implies that $C_{n^j} = \mathbf{0}$ for each $j \geq 1$.

If $i = 1$, then $\rho(b) = \rho(b') \neq 0$. Otherwise, $i \geq 2$ and, by the preceding paragraphs, $\rho(b) = n^{i-1}\rho(b') \neq 0$, as asserted. $\square$

{ABLv}

Remark 3.2 (Abelian variety over K)**.** Recall that a **group variety** over a field K is a geometrically integral algebraic variety A over K equipped with two morphisms $A \times A \to A$ (the **multiplication**) and $A \to A$ (the **inverse operation**), and a distinguished K-rational point $\mathbf{e}$ (the **identity element**) that satisfy the group axioms, thereby make $A(\tilde{K})$ a group (not necessarily commutative). In particular, A is nonsingular [Mil85, p. 104, §1].

The group variety A is an **abelian variety** if A is in addition **complete** [Mil17, p. 155, Def. 7.1]. In particular, A is commutative, and by the preceding paragraph A is nonsingular. See [Mil85, p. 105, Cor. 2.4] or [Mum74, p. 41, (ii)]. In this case we view the group operation as addition and the identity element as the **zero element o**. Moreover, A is projective [Mil85, p. 113, Thm. 7.1]. We fix an embedding of A into $\mathbb{P}_K^m$ for some positive integer m.

Conversely, if a group variety A is a projective algebraic group over a field K, then A is also complete [Mil17, p. 158, Thm. 7.22], hence is an abelian variety.

Recall that a group scheme $\pi\colon \mathcal{A} \to S$ over S is an **abelian scheme** if π is proper [Liu06, p. 103, Def. 3.14] and smooth and the geometric fibers of π are connected [Mil85, p. 145, Sec. 20]. In particular, the fibers of π are abelian varieties. Thus, an abelian scheme S can be thought of as a continuous family of abelian varieties parametrized by S. When $S = \mathrm{Spec}(K)$ is the spectrum of a field K, this is the standard definition of an abelian variety over K.

The polynomials involved in the homogeneous equations that define the abelian variety A as well as those involved in the group operations of A have finitely many non-zero coefficients. Each of these coefficients belongs to K, so we adjoin them and their inverses to the integral domain R introduced in Setup 1.1, if necessary, to assume that A extends to an abelian scheme $\mathcal{A}$ over R, that is $A = \mathcal{A} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K)$ [Mil85, p. 148, Remark 20.9]. Note that the abelian scheme $\mathcal{A}$ depends on the embedding of A into $\mathbb{P}^m_K$. However, the statements "for almost all $\mathfrak{p}$ in $\mathrm{Spec}(R)$" that will follow, do not depend on this choice. Moreover, every point in $A(K)$ has a representation by an $(m+1)$-tuple $(a_0, a_1, \dots, a_m)$ with entries in R (see also the paragraph that follows Lemma 3.3 for the notation $\mathcal{A}(R)$).

However, in order for the latter point to belong to $\mathcal{A}(R)$, the elements $a_0, \dots, a_m$ must generate the unit ideal of the principal ideal domain $R_{\mathfrak{p}}$ for all height 1 prime ideals $\mathfrak{p}$ of the integrally closed noetherian domain R (since, by [Mts94, p. 81, Thm. 11.5(ii)], R is the intersection of all localizations at height 1 prime ideals) [Poo17, p. 42, Example 2.3.17], so at this point we only know that $\mathcal{A}(R) \subseteq A(K)$ [Poo17, p. 43, Cor. 2.3.22]. ∎

We prove that the later inclusion is actually an equality. The starting point is the following result that goes back to André Weil.

Lemma 3.3 ([BLR90, p. 109, Sec. 4.4, Thm. 1])**.** *Let S be a normal noetherian base scheme and let u: $Z \dashrightarrow G$ be an S-rational map from a smooth S-scheme Z to a smooth separated S-group scheme G. Suppose that u is defined in codimension ≤ 1, meaning that the domain of definition of u contains all points of Z of codimenaion ≤ 1. Then, u is defined everywhere.* {BLR}

Let S be a scheme and let X and T be S-schemes. Then, the **set of T-points on** X is $X(T) := \mathrm{Hom}_S(T, X)$ [Poo17, p. 38, Def. 2.3.1]. In the case where $S = \mathrm{Spec}(K)$ and $T = \mathrm{Spec}(L)$ for a field extension L of K, an element of $X(L)$ is called an **L-rational point** or simply an **L-point**. See also [Poo17, p. 41, Example 2.3.5, p. 42, Rem. 2.3.16, Example 2.3.17, and Rem. 2.3.18] for scheme-valued points on projective space.

Proposition 3.4. *Let R be an integrally closed noetherian domain with quotient field K. Let A be an abelian variety over K and assume that A* **extends** *to an abelian scheme $\mathcal{A}$ over* $\mathrm{Spec}(R)$*, i.e. $A = \mathcal{A} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K)$ is the generic fiber of $\mathcal{A}$. Then, the map $\mathcal{A}(R) \to \mathcal{A}(K) = A(K)$ is bijective.* {EXD}

Proof. We follow the proof of [Poo17, p. 65, Thm. 3.2.13(ii)] which proves that if R is a Dedekind domain and X is a proper R-scheme, then the map $X(R) \to X(K)$ is bijective.

Since $\mathcal{A}$ is a projective scheme over R, $\mathcal{A}$ is proper over $\mathrm{Spec}(R)$ [Liu06, p. 108, Thm. 3.30]. In particular, $\mathcal{A}$ is of finite type and separated over $\mathrm{Spec}(R)$ [Liu06, p. 103, Def. 3.14]. Since R is a noetherian ring, this implies that $\mathcal{A}$ is of finite presentation over $\mathrm{Spec}(R)$ [Poo17, p. 59, Def. 3.1.12 and Rem. 3.1.13]. The same holds for K replacing R and A replacing $\mathcal{A}$.

Let $f \in A(K) = \mathcal{A}(K)$. We need to extend $f\colon \operatorname{Spec}(K) \to \mathcal{A}$ to an R-morphism $\operatorname{Spec}(R) \to \mathcal{A}$. To this end we apply [Poo17, p. 60, Thm. 3.2.1(iii)] to find a dense open subscheme U of $\operatorname{Spec}(R)$ such that f extends to a U-morphism $f_U\colon U \to \mathcal{A}_U := \mathcal{A} \times_{\operatorname{Spec}(R)} U$, or equivalently, an R-morphism $f_U\colon U \to \mathcal{A}$.

The rest of the proof breaks up into three parts.

Minimal prime ideals: Since U is a non-empty open subset of $\operatorname{Spec}(R)$, $Z = \operatorname{Spec}(R) \smallsetminus U$ is a proper closed subset of $\operatorname{Spec}(R)$. Endow Z with the structure of a reduced closed subscheme [Liu06, p. 60, Prop. 4.2(e)]. By [Liu06, p. 47, Prop. 3.20], there exists a non-zero ideal $\mathfrak{a}$ of R such that $Z = \operatorname{Spec}(R/\mathfrak{a})$.

Since R is a noetherian ring, so is $R/\mathfrak{a}$ [Mts94, p. 14]. Thus, $\operatorname{Spec}(R/\mathfrak{a})$ is a noetherian scheme [Har77, p. 83, Definition]. By [Liu06, p. 63, Prop. 4.9], $\operatorname{Spec}(R/\mathfrak{a})$ has only finitely many components. Hence, by [Liu06, p. 62, Prop. 4.7(b)], R has only finitely many prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n'}$ that are minimal above $\mathfrak{a}$, each of the schemes $V(\mathfrak{p}_i/\mathfrak{a}) := \{\mathfrak{p}/\mathfrak{a} \mid \mathfrak{p} \in \operatorname{Spec}(R) \text{ and } \mathfrak{p}_i \subseteq \mathfrak{p}\} \cong \operatorname{Spec}(R/\mathfrak{p}_i)$ is an irreducible component of $\operatorname{Spec}(R/\mathfrak{a})$ and $\operatorname{Spec}(R/\mathfrak{a}) = \bigcup_{i=1}^{n'} V(\mathfrak{p}_i/\mathfrak{a})$. If $\mathfrak{p} \in \operatorname{Spec}(R) \smallsetminus U$ is of height 1 (equivalently, of codimension 1 in $\operatorname{Spec}(R)$), then $\mathfrak{p}$ is a minimal prime ideal of R that contains $\mathfrak{a}$, so $\mathfrak{p} = \mathfrak{p}_i$ for some i between 1 and n'. In particular, there are only finitely many $\mathfrak{p} \in \operatorname{Spec}(R) \smallsetminus U$ of height 1, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$.

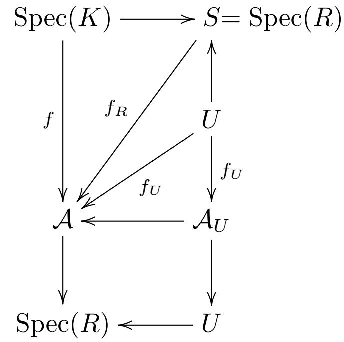
Claim: We can extend f_U to a morphism from an open neighborhood of $U \cup \{\mathfrak{p}_1, \ldots \mathfrak{p}_n\}$ into $\mathcal{A}$.

Indeed, it suffices to extend f_U to a morphism from an open neighborhood of $U \cup \{\mathfrak{p}\}$ into $\mathcal{A}$ for each $\mathfrak{p} \in \operatorname{Spec}(R) \smallsetminus U$ of height 1, since then we can repeat the extension argument for each missing point.

Note that $R_\mathfrak{p}$ is a discrete valuation ring [Mts94, p. 82, Corollary] with quotient field K. Hence, since $\mathcal{A}$ is proper over $\operatorname{Spec}(R)$, it follows from the valuative criterion for properness [Poo17, p. 65, Thm. 3.2.12] that we can extend $f\colon \operatorname{Spec}(K) \to \mathcal{A}$ to a morphism $\operatorname{Spec}(R_\mathfrak{p}) \to \mathcal{A}$. Next, apply [Poo17, p. 61, Remark 3.2.2] to spread out this morphism to an R-morphism $f_V\colon V \to \mathcal{A}_V \subseteq \mathcal{A}$ for some dense open $V \subseteq \operatorname{Spec}(R)$. Suppose that $\bigcup_{j=1}^{k} \operatorname{Spec}(R_j)$ is an affine cover of $U \cap V$. By [Poo17, p. 65, Thm. 3.2.13(i)], $\mathcal{A}(R_j) \subseteq \mathcal{A}(K)$, so $(f_U)|_{\operatorname{Spec}(R_j)}$ and $(f_V)|_{\operatorname{Spec}(R_j)}$ define the same point f of $\mathcal{A}(K)$, $j = 1, \ldots, k$. Hence, the restrictions of f_U and f_V to $U \cap V$ must agree. Thus, we can glue to obtain an extension of f to $U \cup V$, which contains both U and $\mathfrak{p}$. This proves the claim.

End of the proof: By Lemma 3.3, applied to $S = \operatorname{Spec}(R)$, $Z = S$ and $G = \mathcal{A}$, the R-morphism f_U, which as an R-rational map $\operatorname{Spec}(R) \dashrightarrow \mathcal{A}$ is defined in codimension ≤ 1 by the claim above, extends to an R-morphism $f_R\colon \operatorname{Spec}(R) \to$

$\mathcal{A}$,



as desired. □

Remark 3.5. (On the elementary nature of Abelian varieties.) We observe that the statement about the group operations of A satisfying the group axioms is equivalent to an elementary statement about $A(\tilde{K})$ with parameters in R. Hence, by the elimination of quantifiers of the theory of algebraically closed fields (Remark 1.6) and as in Example 1.8, for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$ the reduced variety $\bar{A}_\mathfrak{p}$ is a group variety over $\bar{K}_\mathfrak{p}$, $\bar{A}_\mathfrak{p}$ is projective, hence complete, and $\dim(\bar{A}_\mathfrak{p}) = \dim(A)$ (by (3)). It follows that $\bar{A}_\mathfrak{p}$ is an abelian variety. By Remark 1.6, those statements hold also for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$. {ELN}

If $f\colon A \to B$ is a morphism (resp. homomorphism, epimorphism) of abelian varieties over K, then so is the reduction map $f_\mathfrak{p}\colon \bar{A}_\mathfrak{p} \to \bar{B}_\mathfrak{p}$, again for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$, so also for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$.

By Proposition 3.4, the ring homomorphism $R \to \bar{K}_\mathfrak{p}$ induces a group homomorphism $\rho_\mathfrak{p}\colon A(K) = \mathcal{A}(R) \to \bar{A}_\mathfrak{p}(\bar{K}_\mathfrak{p})$. Let L be a finite separable extension of K, let R_L be the integral closure of R in L, and extend $\mathfrak{p}$ to a prime ideal of R_L. Then, $\rho_\mathfrak{p}$ extends to a group homomorphism $\rho_\mathfrak{p}\colon A(L) \to \bar{A}_\mathfrak{p}(\bar{L}_\mathfrak{p})$. Indeed, as in Setup 1.1, R_L is noetherian [ZaS75, p. 265, Cor. 1]. Thus, by Proposition 3.4, $A(L) = \mathcal{A}(R_L)$.

Finally, we note that [Shi98, p. 95, Prop. 25] proves that $\bar{A}_\mathfrak{p}$ is an abelian variety for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$ in the case where R is a Dedekind domain. ∎

The following result is well-known.

{SEPp}

Lemma 3.6. *Let A be an abelian variety over K, consider $\mathbf{a} \in A(K)$, and let n be a positive integer with $\mathrm{char}(K) \nmid n$. Then, every point $\mathbf{b} \in A$ with $n\mathbf{b} = \mathbf{a}$ lies in $A(K_{\mathrm{sep}})$. In particular, $A_n(\tilde{K}) \subseteq A(K_{\mathrm{sep}})$.*

Proof. By [Mil85, p. 115, Thm. 8.2], the map $n_A\colon A \to A$, defined by $n_A(\mathbf{b}) = n\mathbf{b}$ is étale. By [Mum88, p. 245, Cor. 1], $n_A^{-1}(\mathbf{a}) \subseteq A(K_{\mathrm{sep}})$, as claimed.

In particular, $A_n(\tilde{K}) = n_A^{-1}(\mathbf{o}) \subseteq A(K_{\mathrm{sep}})$. □

{ULTp}

Setup 3.7. By Convention 1.3, last paragraph, the intersection of finitely many non-empty Zariski-open subsets of $\mathrm{Spec}(R_0)$ is infinite. Hence, [FrJ08, p. 139, Lemma 7.5.4] yields an ultrafilter $\mathcal{D}$ on $\mathrm{Spec}(R_0)$ that contains every non-empty Zariski-open subset of $\mathrm{Spec}(R_0)$. We call an ultrafilter $\mathcal{D}$ on $\mathrm{Spec}(R_0)$ that satisfies this condition a **Zariski-ultrafilter** on $\mathrm{Spec}(R_0)$. In particular, a Zariski-ultrafilter on $\mathrm{Spec}(R_0)$ is **non-principal**, i.e. $\mathcal{D}$ contains no finite subset of $\mathrm{Spec}(R_0)$ [FrJ08, p. 139, Example 7.5.1(b)].

Let $K^* = \prod \bar{K}_{\mathfrak{p}}/\mathcal{D}$, where $\mathfrak{p}$ ranges over $\mathrm{Spec}(R_0)$, be the corresponding ultraproduct [FrJ08, Sections 7.5 and 7.7]. As in Convention 1.3, we consider $\mathrm{Spec}(R_0)$ as a subset of $\mathrm{Spec}(R)$. Taking the ultraproduct of the residue maps $\rho_{\mathfrak{p}}\colon R \to \bar{K}_{\mathfrak{p}}$, we obtain a homomorphism $\rho^*\colon R \to K^*$. Moreover, by that convention, for every non-zero $c \in R$ there exists a non-zero $c_0 \in R_0$ such that

$$\{\mathfrak{p} \in \mathrm{Spec}(R_0) \mid c_0 \notin \mathfrak{p}\} \subseteq \{\mathfrak{p} \in \mathrm{Spec}(R_0) \mid c \notin \mathfrak{p}\}. \qquad (11)$$ {ulfr}

Since the left hand side of (11) belongs to $\mathcal{D}$, so is the right hand side and therefore $\{\mathfrak{p} \in \mathrm{Spec}(R_0) \mid c \in \mathfrak{p}\} \notin \mathcal{D}$ (by the definition of ultrafilter [FrJ08, p. 138, Sec. 7.5]). Hence, the map ρ^* is injective. It follows that ρ extends to an embedding $\rho^*\colon K \to K^*$. We identify K as a subfield of K^* under ρ^* and consider the following diagram of fields:

$$\begin{array}{ccccc} & & K_{\mathrm{sep}} & & \\ & & | & & \\ K_{0,\mathrm{sep}} & — & K_{0,\mathrm{sep}}K & — & K_{0,\mathrm{sep}}K^* \\ | & & | & & | \\ K_0 & ——— & K & ——— & K^*. \end{array} \qquad \blacksquare$$

The following result is a generalization of [Hru98, p. 199, Lemma 3].

Lemma 3.8. *K_{sep} is linearly disjoint from $K_{0,\mathrm{sep}}K^*$ over $K_{0,\mathrm{sep}}K$.* {LNDj}

Proof. By Setup 1.1, K/K_0 is a finitely generated regular extension, $K = K_0(\mathbf{x})$, and $V = \mathrm{Spec}(K_0[\mathbf{x}])$ is the geometrically integral affine variety with generic point $\mathbf{x} = (x_1, \ldots, x_n)$.

Part A: *We prove that if K' is a finite separable extension of K which is regular over K_0, then K' is linearly disjoint from K^* over K.*

To this end, we set $d = [K' : K]$. Then, $[K'K_{0,\mathrm{sep}} : KK_{0,\mathrm{sep}}] = d$. Also, there exists a geometrically integral affine variety V' over K_0 such that $K' = K_0(V')$. Replacing V and V' by appropriate non-empty Zariski-open subsets, we may assume that there exists a finite separable morphism $f\colon V' \to V$ such that

$$|f^{-1}(\mathbf{a})| = d \text{ for each } \mathbf{a} \in V(\tilde{K}). \qquad (12)$$ {degr}

Since (12) is an elementary statement on $\tilde{K}_0$, it holds over $F := \prod \bar{K}_{\mathfrak{p},\mathrm{alg}}/\mathcal{D}$. Hence, $[K'F : KF] = [F(V') : F(V)] \geq d$.

Note that $K^* \subseteq F$ and observe the following diagram of fields.

$$\begin{array}{ccccc}
K' & \text{———} & K'K^* & \text{———} & K'F = F(V') \\
{\scriptstyle d}\Big| & & \Big| & & \Big|{\scriptstyle \geq d} \\
K & \text{———} & K^* & \text{———} & KF = F(V) \\
\Big| & & \Big| & & \Big| \\
K_0 & = & K_0 & \text{———} & F = \prod \tilde{K}_{\mathfrak{p},\mathrm{alg}}/\mathcal{D}.
\end{array}$$

Then,

$$d = [K' : K] \geq [K'K^* : KK^*] = [K'K^* : K^*] \geq [K'F : KF] \geq d, \qquad (13)$$ {inql}

so all of the terms appearing in (13) are equal to d. In particular, $[K'K^* : K^*] = d = [K' : K]$. This implies that K' is linearly disjoint from K^* over K, as claimed.

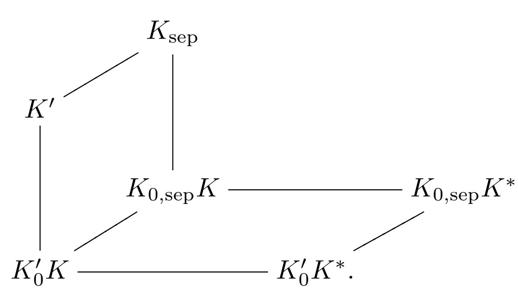
Part B: For an arbitrary finite separable extension K' of K we set $K_0' = K' \cap \tilde{K}_0$. Since the extension K/K_0 is regular, so is KK_0'/K_0' [FrJ08, p. 35, Lemma 2.5.3]. In particular, KK_0'/K_0' is separable. Since K'/K is a finite separable extension, K'/KK_0' is also separable. Therefore, K'/K_0' is separable [FrJ08, p. 39, Cor. 2.6.2]. By definition, K_0' is algebraically closed in K'. Hence, K'/K_0' is regular [FrJ08, p. 39, Lemma 2.6.4].

Note that since K'/K and K/K_0 are separable extensions, so is K'/K_0 [FrJ08, p. 39, Cor. 2.6.2(a)]. Hence, K_0' is also a separable extension of K_0. Since K_0'/K_0 is algebraic, $K_0' \subseteq K_{0,\mathrm{sep}}$. It follows that $K_0' = K' \cap K_{0,\mathrm{sep}}$.

By Part A, applied to K', KK_0', and K_0' rather than to K', K, and K_0, we have that K' is linearly disjoint from K^*K_0' over KK_0'.

Conclusion of the proof: Assume by contradiction that K_{sep} is not linearly disjoint from $K_{0,\mathrm{sep}}K^*$ over $K_{0,\mathrm{sep}}K$. Then, there exist $z_1, \ldots, z_m \in K_{\mathrm{sep}}$ that are linearly independent over $K_{0,\mathrm{sep}}K$ but linearly dependent over $K_{0,\mathrm{sep}}K^*$. Thus, there exist $v_1, \ldots, v_m \in K_{0,\mathrm{sep}}K^*$, not all zero, such that $\sum_{i=1}^m v_i z_i = 0$.

Without loss we may assume that $v_i = \sum_{j=1}^{r_i} a_{ij} u_{ij}$, with $a_{ij} \in K_{0,\mathrm{sep}}$ and $u_{ij} \in K^*$ for all i and j. Then, we choose a finite separable extension K' of K such that $z_1, \ldots, z_m \in K'$ and $a_{ij} \in K_0'$ for all i, j.



Thus, $v_i \in K_0'K^*$ for $i = 1, \ldots, m$.

Since $z_1, \ldots, z_m$ are linearly independent over $K_{0,\mathrm{sep}}K$, they are linearly independent also over $K_0'K$. Hence, by Part B, $z_1, \ldots, z_m$ are linearly independent over $K_0'K^*$. But this contradicts the relation $\sum_{i=1}^m v_i z_i = 0$ established above.

We conclude from this contradiction that K_{sep} is linearly disjoint from $K_{0,\mathrm{sep}}K^*$ over $K_{0,\mathrm{sep}}K$, as claimed. □

Next we prove an analog of [Hru98, p. 199, Lemma 4] that for itself partially strengthen [Lan62, p. 161, Cor.]. As in Convention 1.3, we consider $\mathrm{Spec}(R_0)$ as a subset of $\mathrm{Spec}(R)$.

The proof of Part (d) of Theorem 3.11 uses the following lemma.

{ML1K}

Lemma 3.9. *Let $\Gamma \le \Delta$ be abelian groups such that $(\Delta : \Gamma) < \infty$. Let l be a prime number with $l \nmid (\Delta : \Gamma)$. Then, $l\Delta \cap \Gamma = l\Gamma$.*

Proof. Consider $\delta \in \Delta$ and $\gamma \in \Gamma$ such that $l\delta = \gamma$. Since $l \nmid (\Delta : \Gamma)$, there are $k, m \in \mathbb{Z}$ such that $ml = 1 + k(\Delta : \Gamma)$. Hence,

$$m\gamma = ml\delta = \delta + k(\Delta : \Gamma)\delta. \qquad (14)$$ {mglk}

Since $(\Delta : \Gamma)\delta, m\gamma \in \Gamma$, we have by (14) that $\delta \in \Gamma$, so $\gamma \in l\Gamma$, as claimed. □

{INJj}

Remark 3.10. The assumption "$A(K_{0,\mathrm{sep}}K)$ is finitely generated" that enters in the next result, holds by Corollary 4.9, if $A_{\tilde{K}}$ has no simple quotient which is defined over $\tilde{K}_0$. ∎

{INJk}

Theorem 3.11. *Let A be an abelian variety over K such that $A(K_{0,\mathrm{sep}}K)$ is finitely generated. Then, the following statements hold:*

(a) *For almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$, we have that $\bar{A}_\mathfrak{p}$ is an abelian variety over $\bar{K}_\mathfrak{p}$ with $\dim(\bar{A}_\mathfrak{p}) = \dim(A)$.*

(b) *For almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$, the reduction map $\rho_\mathfrak{p}\colon A(K) \to \bar{A}_\mathfrak{p}(\bar{K}_\mathfrak{p})$ is injective on $A_{\mathrm{tor}}(K)$.*

(c) *If l is a prime number such that $l \ne \mathrm{char}(K_0)$ and $A_l(K_{0,\mathrm{sep}}K) = \mathbf{0}$, then $\bar{A}_{\mathfrak{p},l}(\bar{K}_\mathfrak{p}) = \mathbf{0}$ for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$.*

(d) *For every large prime number l and for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$, the map $\rho_\mathfrak{p}$ induces an injection*

$$\bar{\rho}_{\mathfrak{p},l}\colon A(K)/lA(K) \to \bar{A}_\mathfrak{p}(\bar{K}_\mathfrak{p})/l\bar{A}_\mathfrak{p}(\bar{K}_\mathfrak{p}).$$

(e) *$\rho_\mathfrak{p}\colon A(K) \to \bar{A}_\mathfrak{p}(\bar{K}_\mathfrak{p})$ is an injection for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$.*

In both (c) and (d), the exceptional sets of $\mathfrak{p}$'s depend on l.

Proof of (a). See Remark 3.5.

Proof of (b). Since $A(K)$ is a finitely generated abelian group, $A_{\mathrm{tor}}(K)$ is finite. For a point of $A(K)$, being different from $\mathbf{o}$ is an elementary property. Hence, for each non-zero $\mathbf{a} \in A_{\mathrm{tor}}(K)$, and for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$, the element $\rho_{\mathfrak{p}}(\mathbf{a})$ is non-zero. By Convention 1.3, the same statement holds for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$. Hence, for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$, the map $\rho_{\mathfrak{p}}$ is injective on $A_{\mathrm{tor}}(K)$.

Proof of (c). Assume by contradiction that for all $\mathfrak{p}$ in an infinite subset S_l of $\mathrm{Spec}(R_0)$ there exists a non-zero point $\mathbf{a}_{\mathfrak{p}} \in \bar{A}_{\mathfrak{p},l}(\bar{K}_{\mathfrak{p}})$. We choose a non-principal ultrafilter $\mathcal{D}$ on $\mathrm{Spec}(R_0)$ that contains S_l as an element [FrJ08, p. 139, Lemma 7.5.4]. As in Setup 3.7, let $K^* = \prod \bar{K}_{\mathfrak{p}}/\mathcal{D}$. Then, the points $\mathbf{a}_{\mathfrak{p}}$ with $\mathfrak{p} \in S_l$ yield a non-zero point $\mathbf{a}$ in $A_l(K^*)$ [FrJ08, p. 142, Cor. 7.7.2], hence also in $A_l(K_{0,\mathrm{sep}}K^*)$.

In addition, since A is defined over K and since $l \neq \mathrm{char}(K)$, the point $\mathbf{a}$ belongs to $A(K_{\mathrm{sep}})$ (by Lemma 3.6). But, by Lemma 3.8, K_{sep} is linearly disjoint from $K_{0,\mathrm{sep}}K^*$ over $K_{0,\mathrm{sep}}K$. Hence, $\mathbf{a} \in A(K_{0,\mathrm{sep}}K)$. Therefore, $\mathbf{a} \in A_l(K_{0,\mathrm{sep}}K)$. This contradicts the assumption we have made in (c).

Proof of (d). Since $A(K_{0,\mathrm{sep}}K)$ is a finitely generated abelian group, there exists a finite separable extension K_0' of K_0 such that $A(K_0'K)$ contains all of the generators of that group. Let R_0' be the integral closure of R_0 in K_0'. For each $\mathfrak{p} \in \mathrm{Spec}(R_0)$ extend $\mathfrak{p}$ to a prime ideal of the integral closure of R_0' and then to the integral closure $R_{KK_0'}$ of RR_0' in KK_0'. Note that by Remark 3.5, $A(KK_0') = \mathcal{A}(R_{KK_0'})$. Then consider the following commutative diagram,

$$\begin{array}{ccc} A(KK_0')/lA(KK_0') & \longrightarrow & \bar{A}_{\mathfrak{p}}((\overline{KK_0'})_{\mathfrak{p}})/l\bar{A}_{\mathfrak{p}}((\overline{KK_0'})_{\mathfrak{p}}) \\ \uparrow & & \uparrow \\ A(K)/lA(K) & \longrightarrow & \bar{A}_{\mathfrak{p}}(\bar{K}_{\mathfrak{p}})/l\bar{A}_{\mathfrak{p}}(\bar{K}_{\mathfrak{p}}), \end{array}$$

where the vertical arrows are the natural homomorphisms and the horizontal arrows are the corresponding reduction modulo $\mathfrak{p}$. By Lemma 3.9, the left vertical map is injective if l does not divide the finite index $(A(KK_0') : A(K))$. Therefore, if the upper horizontal map is injective, then so is the lower horizontal map.

By [ZaS75, p. 265, Cor. 1], R_0' is a noetherian domain. By Remark 1.2 R_0', replacing R_0, satisfies Condition 1. Thus, replacing R_0 by R_0', K_0 by K_0', and K by $K_0'K$, we may assume that

$$A(K) = A(K_{0,\mathrm{sep}}K). \tag{15}$$ {finx}

As in the proof of (c), assume by contradiction that the map $\bar{\rho}_{\mathfrak{p},l}$ is non-injective for all $\mathfrak{p}$ in an infinite subset S_l of $\mathrm{Spec}(R_0)$. Again, let $\mathcal{D}$ be a non-principal ultrafilter on $\mathrm{Spec}(R_0)$ that contains S_l as an element and let $K^* = \prod \bar{K}_{\mathfrak{p}}/\mathcal{D}$. Since the non-injectivity of $\bar{\rho}_{\mathfrak{p},l}$ is an elementary statement on $A(K)$, Łoš' theorem [FrJ08, p. 142, Prop. 7.7.1], implies that the map

$$\bar{\rho}_l^* := \prod \bar{\rho}_{\mathfrak{p},l}/\mathcal{D}\colon A(K^{\mathrm{Spec}(R_0)}/\mathcal{D})/lA(K^{\mathrm{Spec}(R_0)}/\mathcal{D}) \to A(K^*)/lA(K^*) \tag{16}$$ {noni}

is non-injective.

On the other hand, consider $\mathbf{a} \in A(K)$ for which there exists $\mathbf{b} \in A(K^*)$ with $l\mathbf{b} = \mathbf{a}$. By Lemma 3.6, $\mathbf{b} \in A(K_{\mathrm{sep}})$. By Lemma 3.8, K_{sep} is linearly disjoint from $K_{0,\mathrm{sep}}K^*$ over $K_{0,\mathrm{sep}}K$. Hence,

$$\mathbf{b} \in A(K_{0,\mathrm{sep}}K) \overset{(15)}{=} A(K).$$

It follows that the map

$$\varphi_l \colon A(K)/lA(K) \to A(K^*)/lA(K^*) \tag{17}$$ {injk}

induced by the $\bar{\rho}_{\mathfrak{p},l}$'s is injective.

By assumption, $A(K)$ is a finitely generated abelian group. Hence, the quotient $A(K)/lA(K)$ is a finite abelian group. Therefore, again by Łoš' theorem, both groups $A(K)/lA(K)$ and $A(K^{\mathrm{Spec}(R_0)}/\mathcal{D})/lA(K^{\mathrm{Spec}(R_0)}/\mathcal{D})$ have the same number of elements and the map

$$\psi_l \colon A(K)/lA(K) \to A(K^{\mathrm{Spec}(R_0)}/\mathcal{D})/lA(K^{\mathrm{Spec}(R_0)}/\mathcal{D})$$

is injective [FrJ08, last paragraph of p. 143]. It follows that ψ_l is even bijective. Moreover, $\bar{\rho}_l^* \circ \psi_l = \varphi_l$. Comparing (16) and (17), we get a contradiction.

Proof of (e). By assumption, $A(K_{0,\mathrm{sep}}K)$ is a finitely generated abelian group. Hence, for each large l, we have $A_l(K_{0,\mathrm{sep}}K) = \mathbf{0}$.

As in the proof of (d), we may replace K_0 by a suitable finite separable extension K_0' to assume that $A(K) = A(K_{0,\mathrm{sep}}K)$ is finitely generated. Note that if the reduction map $A(KK_0') \to \bar{A}_{\mathfrak{p}}(\overline{(KK_0')}_{\mathfrak{p}})$ is injective, then so is the reduction map $A(K) \to \bar{A}_{\mathfrak{p}}(\bar{K}_{\mathfrak{p}})$. Let $l \neq \mathrm{char}(K_0)$ be a large prime number. In particular,

$$A_l(K_{0,\mathrm{sep}}K) = \mathbf{0}. \tag{18}$$ {alk0}

Then, by (d), (18), and (c),

$$\bar{\rho}_{\mathfrak{p},l} \text{ is injective and } \bar{A}_{\mathfrak{p},l}(\bar{K}_{\mathfrak{p}}) = \mathbf{0} \text{ for almost all } \mathfrak{p} \in \mathrm{Spec}(R_0). \tag{19}$$ {injc}

By (b),

$$\rho_{\mathfrak{p}} \text{ is injective on } A_{\mathrm{tor}}(K) \text{ for almost all } \mathfrak{p} \in \mathrm{Spec}(R_0). \tag{20}$$ {ontr}

Since $A(K)$ is a finitely generated abelian group,

$$A(K) = A_{\mathrm{tor}}(K) \oplus B, \text{ where } B \text{ is a finitely generated free abelian group} \tag{21}$$ {fabg}

[Lan93, p. 147, Thm. 7.3]. Hence, $\bigcap_{i=1}^{\infty} l^i B = \mathbf{0}$.

Now consider $\mathfrak{p} \in \mathrm{Spec}(R_0)$ that satisfies (19) and (20). Then, $\bar{A}_{\mathfrak{p},l}(\bar{K}_{\mathfrak{p}}) = \mathbf{0}$. Let $\mathbf{b} \in B$ and suppose that $\bar{\rho}_{\mathfrak{p},l}(\mathbf{b} + lA(K)) \in l\bar{A}_{\mathfrak{p}}(\bar{K}_{\mathfrak{p}})$. By (19), $\mathbf{b} \in lA(K)$, so there exist $\mathbf{a}' \in A_{\mathrm{tor}}(K)$ and $\mathbf{b}' \in B$ such that $\mathbf{b} = l\mathbf{a}' + l\mathbf{b}'$. Hence, by (21), $\mathbf{b} = l\mathbf{b}'$. Thus, $\bar{\rho}_{\mathfrak{p},l}$ is injective on B/lB. Therefore, by the preceding paragraph and by Lemma 3.1, with $C = \bar{A}_{\mathfrak{p}}(\bar{K}_{\mathfrak{p}})$, we have that $\rho_{\mathfrak{p}}$ is injective on B. This means that $\mathrm{Ker}(\rho_{\mathfrak{p}}) \subseteq A_{\mathrm{tor}}(K)$. We conclude from (20) that $\rho_{\mathfrak{p}}$ is injective, as claimed. $\Box$

4 Isotriviality of Abelian Varieties

{ISOt}

We introduce the notion of $\tilde{K}/\tilde{K}_0$-isotriviality of abelian varieties and prove that if an abelian variety has no $\tilde{K}/\tilde{K}_0$-isotrivial quotients, then the same holds for almost all of its reductions. Again, K_0 and K are the fields introduced in Setup 1.1.

{ISOu}

Remark 4.1 (Isogenies of abelian varieties)**.** We say that the abelian variety A over K is **simple** if A is non-zero and has no non-zero proper abelian subvarieties over K.

Every morphism $\alpha\colon A \to B$ of abelian varieties over K that maps the zero point of A onto the zero point of B is a homomorphism [Mil85, p. 107, Cor. 3.6]. Thus, $\alpha(\mathbf{a}+\mathbf{a}') = \alpha(\mathbf{a})+\alpha(\mathbf{a}')$ for all $\mathbf{a}, \mathbf{a}' \in A(\tilde{K})$. If, in addition, α is surjective and $\dim(A) = \dim(B)$, then $\mathrm{Ker}(\alpha)$ is a finite group scheme and α is an **isogeny** [Mil85, p. 114, Prop. 8.1].

In particular, multiplication of A by a positive integer n is an isogeny that we denote by n_A and set $A_n = \mathrm{Ker}(n_A)$. By [Mil85, p. 115, Thm. 8.2], n_A is étale if and only if $\mathrm{char}(K) \nmid n$. In that case

$$|A_n(K_{\mathrm{sep}})| = n^{2\dim(A)} \qquad (22)$$ {dimn}

[Mil85, p. 116, Rem. 8.4].

If $\alpha\colon A \to B$ is an isogeny of abelian varieties over K, then there exists an isogeny $\beta\colon B \to A$ and a positive integer n such that $\beta \circ \alpha = n_A$ [Mum74, p. 169, Rem.].

Every birational map $A \to B$ between abelian varieties over K that maps the zero point of A onto the zero point of B is an isomorphism [Mil85, p. 107, Rem. 3.7]. ∎

{CHEv}

Remark 4.2. Let A be an abelian variety over K and let B be an abelian subvariety of A over K. By a theorem of Poincaré, A has an abelian subvariety B' over K such that $A = B + B'$ and $B \cap B'$ is a finite group (see [Lan59, p. 28, Thm. 6] or [Mil85, p. 122, Prop. 12.1]). This gives a short exact sequence

$$\mathbf{0} \longrightarrow C \longrightarrow B \times B' \xrightarrow{\beta} A \longrightarrow \mathbf{0}$$

with $\beta(\mathbf{b}, \mathbf{b}') = \mathbf{b} + \mathbf{b}'$ and

$$C = \{(\mathbf{b}, \mathbf{b}') \in B \times B' \mid \mathbf{b} + \mathbf{b}' = \mathbf{o}\} = \{(\mathbf{b}, -\mathbf{b}) \in B \times B' \mid b \in B\} \cong B \cap B'$$

is finite. Thus, β is an isogeny.

Using induction on $\dim(A)$, we find a short exact sequence

$$\mathbf{0} \longrightarrow A_0 \longrightarrow A_1 \times \cdots \times A_r \xrightarrow{\alpha} A \longrightarrow \mathbf{0}, \qquad (23)$$ {chvb}

where $A_1, \ldots, A_r$ are simple abelian subvarieties of A, defined over K, such that $A_1 + \cdots + A_r = A$. Thus, A_0 is a finite subgroup of A. In particular, α is an isogeny.

Claim: *Every simple abelian subvariety B of A is isogeneous to A_i for some i between 1 and r.*

Indeed, by Remark 4.1, the short exact sequence (23) yields another short exact sequence

$$\mathbf{0} \longrightarrow A_0' \longrightarrow A \xrightarrow{\alpha'} A_1 \times \cdots \times A_r \longrightarrow \mathbf{0}, \tag{24}$$ {chva}

with A_0' finite.

Now note that $\mathrm{Ker}(\alpha'|_B)$ as a subgroup of $\mathrm{Ker}(\alpha')$ is finite. Hence, $\alpha'|_B \colon B \to \alpha'(B)$ is an isogeny and therefore $\alpha'(B)$ is a simple abelian subvariety of $A_1 \times \cdots \times A_r$, in particular $\alpha'(B) \neq \mathbf{0}$. Therefore, there exists i between 1 and r such that the projection $\pi_i \colon A_1 \times \cdots \times A_r \to A_i$ is non-zero on $\alpha'(B)$. Since A_i and $\alpha'(B)$ are simple, $\pi_i|_{\alpha'(B)} \colon \alpha'(B) \to A_i$ is an isogeny. Thus, B is isogeneous to A_i, as claimed.

Following the claim we call $A_1, \dots, A_r$ the **simple quotients** of A. The existence and the uniqueness (up to isogenies) of the simple quotients is **Poincaré's complete reducibility theorem** (see [Lan59, p. 30, Cor.] or [Mil85, p. 122, Prop. 12.1]).

By our construction, every simple quotient of A is isomorphic to a simple abelian subvariety of A. Conversely, by the Claim, every simple abelian subvariety of A is also a simple quotient of A.

Finally, we note that if K is separably closed and in particular if K is algebraically closed, then the decomposition of A into a direct product of simple abelian varieties does not change, up to isogeny, under extensions of K [Con06, Cor. 3.21]. ∎

As usual, we say that a geometrically integral algebraic variety V over K is **defined over a subfield** K_0 if there exists a geometrically integral variety V_0 over K_0 such that $V_{0,K} := V_0 \times_{\mathrm{Spec}(K_0)} \mathrm{Spec}(K) \cong V$.

Analogous definition applies to the notion "a morphism $f \colon V \to W$ between geometrically integral varieties".

{MSPc}

Lemma 4.3. *Let A be an abelian variety over $\tilde{K}_0$ and let B be an abelian variety over $\tilde{K}$. Then:*

(a) $A_{\mathrm{tor}}(\tilde{K}) = A_{\mathrm{tor}}(\tilde{K}_0)$.

(b) $A_{\mathrm{tor}}(\tilde{K}_0)$ *is Zariski-dense in A.*

(c) *If B is already defined over $\tilde{K}_0$, then every abelian subvariety of B and every homomorphism $\alpha \colon A_{\tilde{K}} \to B$ are already defined over $\tilde{K}_0$.*

(d) *Every automorphism of $A_{\tilde{K}}$ is already defined over $\tilde{K}_0$.*

Proof of (a). Let $\mathbf{a} \in A_{\mathrm{tor}}(\tilde{K})$ and let n be the order of $\mathbf{a}$. Then, $\mathbf{a}$ is a $\tilde{K}$-rational point of the finite subgroup scheme A_n of A (Remark 4.1). Since A_n is defined over $\tilde{K}_0$, all of its points are $\tilde{K}_0$-rational, as claimed.

Proof of (b). We follow [Spe14].

The Zariski-closure of $A_{\mathrm{tor}}(\tilde{K}_0)$ is an abelian algebraic subgroup T of A over $\tilde{K}_0$. Hence, $A_{\mathrm{tor}}(\tilde{K}_0) \subseteq T(\tilde{K}_0) \subseteq A(\tilde{K}_0)$, so $A_{\mathrm{tor}}(\tilde{K}_0) \subseteq T_{\mathrm{tor}}(\tilde{K}_0) \subseteq A_{\mathrm{tor}}(\tilde{K}_0)$. Therefore,

$$A_{\mathrm{tor}}(\tilde{K}_0) = T_{\mathrm{tor}}(\tilde{K}_0) \qquad (25)$$ {atbt}

and $\dim(T) \leq \dim(A)$. The connected component C of the zero point of T is a projective group variety, hence an abelian variety (Remark 3.2, third paragraph). Moreover, $T(\tilde{K}_0)/C(\tilde{K}_0)$ is a finite group [Bor91, p. 46, Prop.(b)] which is abelian.

Choose a prime number $l > \max(|T(\tilde{K}_0)/C(\tilde{K}_0)|, \mathrm{char}(K))$. Since $T(\tilde{K}_0)$ is an abelian group, we have $|T_l| = |C_l|$. Hence,

$$l^{2\dim(A)} \overset{(22)}{=} |A_l| \overset{(25)}{=} |T_l| = |C_l| \overset{(22)}{=} l^{2\dim(C)}.$$

Therefore, $\dim(A) = \dim(C)$, hence $A = C \leq T$, so $A = T$, as claimed.

Proof of (c). See [Mil85, p. 146, Cor. 20.4].

Proof of (d). Statement (d) is a special case of Statement (c). □

{DEFi}

Corollary 4.4. *Let A be an abelian variety over $\tilde{K}$.*

(a) *If all of the simple quotients of A are defined over $\tilde{K}_0$, then A is defined over $\tilde{K}_0$.*

(b) *If A is defined over $\tilde{K}_0$ and B is an abelian variety over $\tilde{K}$ which is isogeneous to A, then B is also defined over $\tilde{K}_0$.*

Proof of (a). The abelian varieties $A_1, \ldots, A_r$ that appear in the short exact sequence (23) are the simple quotients of A, so by our assumption, they are defined over $\tilde{K}_0$. Moreover, A_0 is a finite subgroup of A (second paragraph of Remark 4.2). Hence, by Lemma 4.3(a),

$$A_0(\tilde{K}) \subseteq A_{1,\mathrm{tor}}(\tilde{K}) \times \cdots \times A_{r,\mathrm{tor}}(\tilde{K}) \subseteq A_1(\tilde{K}_0) \times \cdots \times A_r(\tilde{K}_0).$$

Hence, $A_0(\tilde{K}) = A_0(\tilde{K}_0)$, so by (23), A is isomorphic over $\tilde{K}$ to the K_0-abelian variety $(A_1 \times \cdots \times A_r)/A_0$. Thus, A is defined over $\tilde{K}_0$.

Proof of (b). The simple quotients of B are isogeneous to the simple quotients of A, so as in the proof of (a), each of them is defined over $\tilde{K}_0$. It follows again by (a) that B is defined over $\tilde{K}_0$. □

{COMp}

Definition 4.5 (Isotriviality)**.** Let A be an abelian variety over K. We say that $A_{\tilde{K}}$ **has a $\tilde{K}/\tilde{K}_0$-isotrivial quotient** if there exist an abelian variety T over $\tilde{K}_0$ and a non-zero homomorphism $\tau\colon T_{\tilde{K}} \to A_{\tilde{K}}$. By Remark 4.2, this is equivalent for $A_{\tilde{K}}$ to have a quotient which is defined over $\tilde{K}_0$. ∎

{TRAv}

Remark 4.6 (The trace of an abelian variety)**.** Let A be an abelian variety over K. Then, there exists an abelian variety $\mathrm{Tr}_{K/K_0}(A)$ over K_0 and a homomorphism

$$\tau_{A,K/K_0}\colon \mathrm{Tr}_{K/K_0}(A)_K \to A \qquad (26)$$ {trac}

(defined over K) satisfying the following universal property:

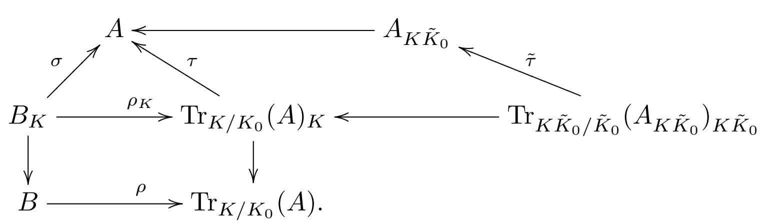
Given an abelian variety B over K_0 and a homomorphism $\sigma\colon B_K \to A$, there exists a unique homomorphism $\rho\colon B \to \mathrm{Tr}_{K/K_0}(A)$ such that $\sigma = \tau_{A,K/K_0} \circ \rho_K$. See [Lan59, p. 213, Thm. 8] or [Con06, Thm. 6.2]. (Note that by Setup 1.1, K/K_0 is a regular extension, in particular K/K_0 is a primary extension, as needed in Conrad's theorem.)

The pair $(\mathrm{Tr}_{K/K_0}(A), \tau_{A,K/K_0})$ is called the **K/K_0-trace of** A.

By [Con06, Thm. 6.8], the base change from K_0 to $\tilde{K}_0$ of (26) yields the trace

$$\tau_{A_{K\tilde{K}_0},\tilde{K}/\tilde{K}_0}\colon \mathrm{Tr}_{K\tilde{K}_0/\tilde{K}_0}(A_{K\tilde{K}_0})_{K\tilde{K}_0} \to A_{K\tilde{K}_0}.$$

With $\tau := \tau_{A,K/K_0}$ and $\tilde{\tau} := \tau_{A_{\tilde{K}_0},\tilde{K}/\tilde{K}_0}$ the above mentioned objects fit into the following commutative diagram:



In addition, the map τ is injective on K-points, so $\mathrm{Tr}_{K/K_0}(A)(K_0)$ is naturally a subgroup of $A(K)$ [Con06, first paragraph of §7]. In particular, if A has no $\tilde{K}/\tilde{K}_0$-isotrivial quotients, alternatively, A has no simple quotient which is defined over $\tilde{K}_0$, then $\mathrm{Tr}_{K\tilde{K}_0/\tilde{K}_0}(A_{K\tilde{K}_0})(\tilde{K}_0) = \mathbf{0}$, so $\mathrm{Tr}_{K\tilde{K}_0/\tilde{K}_0}(A_{K\tilde{K}_0}) = \mathbf{0}$. Hence, $\mathrm{Tr}_{K/K_0}(A) = \mathbf{0}$. ∎

The next result is a relative Mordell-Weil theorem and is due to Lang-Néron [Lan62, Chap. V]. See also [Con06, Thm. 7.1].

Proposition 4.7. *Let A be an abelian variety over K. Then, the quotient group* {MWLn}

$$A(K)/\mathrm{Tr}_{K/K_0}(A)(K_0)$$

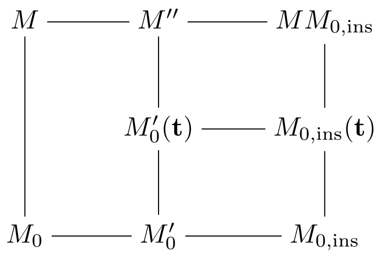
is finitely generated.

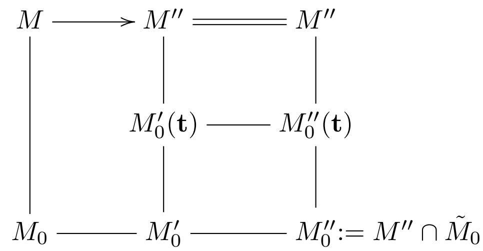
Non-regularity of finitely generated extension of fields can be "corrected" by going over to finite extensions:

Lemma 4.8. *Let M/M_0 be a finitely generated extension of fields. Then, M_0 has a finite extension M_0'' and M has a finite extension M'' such that M''/M_0'' is a finitely generated regular extension.* {rglr}

Proof. The maximal purely inseparable extension $M_{0,\mathrm{ins}}$ of M_0 is perfect. Hence, $MM_{0,\mathrm{ins}}/M_{0,\mathrm{ins}}$ is a finitely generated separable extension. Let $\mathbf{t} := (t_1, \ldots, t_r)$, with $t_1, \ldots, t_r \in M$, be a separating transcendence base for the latter extension. In particular, $MM_{0,\mathrm{ins}}/M_{0,\mathrm{ins}}(\mathbf{t})$ is a finite separable extension.

Let $f \in M_{0,\mathrm{ins}}(\mathbf{t})[X]$ be an irreducible polynomial for a primitive element x of the latter extension and choose a finite extension M_0' of M_0 in $M_{0,\mathrm{ins}}$ that contains the coefficients of the rational functions that appear as coefficients of $f(\mathbf{t}, X)$ as a polynomial in X. Also, suppose that $M = M_0(t_1, \dots, t_r, s_1, \dots, s_m)$ and enlarge M_0' to assume that $s_1, \dots, s_m \in M'' := M_0'(\mathbf{t}, x)$. Then, $M \subseteq M''$ and M'' is a finite separable extension of $M_0'(\mathbf{t})$.





Now observe that M_0'' is algebraically closed in M''. Moreover, since $M''/M_0'(\mathbf{t})$ is a finite separable extension, so is $M''/M_0''(\mathbf{t})$. Since $t_1, \dots, t_r$ are algebraically independent over M_0'', we conclude that M''/M_0'' is finitely generated and separable. Therefore, by [FrJ08, p. 39, Lemma 2.6.4], M''/M_0'' is regular, as desired. □

If in addition to the assumptions of Proposition 4.7, A has no $\tilde{K}/\tilde{K}_0$-isotrivial quotients, then by Remark 4.6, $\mathrm{Tr}_{K/K_0}(A) = \mathbf{0}$. This yields the following result.

{FINg}

Corollary 4.9. *Let M/M_0 be a finitely generated extension of fields and let A be an abelian variety over M. Suppose that $A_{\tilde{M}}$ has no simple quotient which is defined over $\tilde{M}_0$. Then, $A(M)$ is finitely generated.*

Proof. We use Lemma 4.8 to choose finite extensions M_0'' and M'' of M_0 and M, respectively, such that $M_0'' \subseteq M''$ and M''/M_0'' is a finitely generated regular extension. Then, $(A_{M''})_{\tilde{M}} \cong A_{\tilde{M}}$ has no simple quotient which is defined over $\tilde{M}_0$. By Remark 4.6, $\mathrm{Tr}_{M''/M_0''}(A_{M''}) = \mathbf{0}$. Hence, by Proposition 4.7, $A(M'')$ is finitely generated. Since $A(M) \subseteq A(M'')$, also $A(M)$ is finitely generated, as claimed. □

The next result is Corollary 7 on page 201 of [Hru98].

{BETa}

Lemma 4.10. *Let B be an abelian variety over an algebraically closed field F_0. Let F be an extension of F_0, let A be an abelian variety over F, and let $h\colon B_F \to A$ be a homomorphism. Then, F has an extension F' of degree at most β, where $\beta = \beta(\dim(A))$ depends only on $\dim(A)$, such that $h(B_F)_{\mathrm{tor}}(\tilde{F}) \subseteq A(F')$.*

Lemma 4.10 also follows from [Sil92, Thm. 4.2 and Cor. 3.3], with $\beta(\dim(A)) = 2(9\dim(A))^{2\dim(A)}$, and the fact that a surjective homomorphism of abelian varieties over an algebraically closed field induces an epimorphism on the torsion points. See https://mathoverflow.net/questions/266512/a-surjective-morphism-of-abelian-varieties-induces-an-epimorphism-on-the-torsion

{SURj}

Lemma 4.11. *Let A, K, R be as in Remark 3.2 and let n be a positive integer with* $\mathrm{char}(K) \nmid n$. *Then, for almost all* $\mathfrak{p} \in \mathrm{Spec}(R)$, *reduction modulo* $\mathfrak{p}$ *maps* $A_n(\tilde{K})$ *isomorphically onto* $\bar{A}_{\mathfrak{p},n}(\bar{K}_{\mathfrak{p},\mathrm{alg}})$. *Hence, the same holds for almost all* $\mathfrak{p} \in \mathrm{Spec}(R_0)$.

Proof. The case where $R = R_0$ is a Dedekind ring follows from [SeT68, Lemma 2]. Indeed, in this case for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$, $R_\mathfrak{p}$ is a discrete valuation ring with a trivial inertia group.

We prove the general case by model theory as follows.

For almost all $\mathfrak{p} \in \mathrm{Spec}(R)$ we consider the abelian variety $\bar{A}_\mathfrak{p}$ and the homomorphism $\rho_\mathfrak{p}$ induced by reduction modulo $\mathfrak{p}$ which is introduced in Remark 3.5. In particular, $\rho_\mathfrak{p}$ maps $A_n(\tilde{K})$ into $\bar{A}_{\mathfrak{p},n}(\bar{K}_{\mathfrak{p},\mathrm{alg}})$.

Since the statement "$\mathbf{y}, \mathbf{y}' \in A_n(\tilde{K})$ and $\mathbf{y} \neq \mathbf{y}'$" is elementary, we find that for almost all $\mathfrak{p}$, $\rho_\mathfrak{p}$ maps $A_n(\tilde{K})$ injectively into $\bar{A}_{\mathfrak{p},n}(\bar{K}_{\mathfrak{p},\mathrm{alg}})$.

By Remark 3.5, $\dim(A) = \dim(\bar{A}_\mathfrak{p})$ for almost all $\mathfrak{p}$. Hence,

$$|A_n(\tilde{K})| \overset{(22)}{=} n^{2\dim(A)} = n^{2\dim(\bar{A}_\mathfrak{p})} \overset{(22)}{=} |\bar{A}_{\mathfrak{p},n}(\bar{K}_{\mathfrak{p},\mathrm{alg}})|$$

for almost all $\mathfrak{p}$. It follows from the preceding paragraph that for almost all $\mathfrak{p}$, $\rho_\mathfrak{p}$ maps $A_n(\tilde{K})$ isomorphically onto $\bar{A}_{\mathfrak{p},n}(\bar{K}_{\mathfrak{p},\mathrm{alg}})$, as claimed. □

The following lemma is not optimal, but it is all we need for the proof of Theorem 4.13 below.

{HTOr}

Lemma 4.12. *Let F be an algebraically closed field and $h\colon B \to B'$ a non-zero homomorphism of abelian varieties over F. Let n be a positive integer which is not a multiple of* $\mathrm{char}(F)$. *Then, $h(B(F))$ contains a point of order n.*

proof. By assumption, $B'' := h(B)$ is an abelian subvariety of B' of positive dimension. Since F is algebraically closed, $B''(F) = h(B(F))$. By (22), $B''_n(F) \cong (\mathbb{Z}/n\mathbb{Z})^{2\dim(B'')} \neq \mathbf{0}$, as stated. □

We prove an analog of [Hru98, p. 201, Cor. 8].

{HRSo}

Theorem 4.13. *Let R_0, K_0, R, and K be as in Setup 1.1 and let A be an abelian variety over K such that no simple quotient of $A_{\tilde{K}}$ is defined over $\tilde{K}_0$.*

Then, for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$, *$\bar{A}_\mathfrak{p}$ is an abelian variety over $\bar{K}_\mathfrak{p}$ and no simple quotient of $\bar{A}_{\mathfrak{p},\bar{K}_{\mathfrak{p},\mathrm{alg}}}$ is defined over $\bar{K}_{0,\mathfrak{p},\mathrm{alg}}$.*

Proof. We fix a prime number $l \neq \mathrm{char}(K)$ and let $\beta := \beta(\dim(A))$ be the constant introduced in Lemma 4.10.

Part A: *There exists a positive integer i such that for each $\mathbf{y}$ in $A(\tilde{K})$ of order l^i we have $[K\tilde{K}_0(\mathbf{y}) : K\tilde{K}_0] > \beta$.*

Indeed, we assume by contradiction that for each positive integer i the set

$$S_i = \{\mathbf{y} \in A(\tilde{K}) \mid \mathrm{ord}(\mathbf{y}) = l^i \text{ and } [K\tilde{K}_0(\mathbf{y}) : K\tilde{K}_0] \leq \beta\}$$

is non-empty. Since $S_i \subseteq A_{l^i}(\tilde{K})$, the set S_i is finite (Remark 4.1).

If $\mathbf{y} \in S_{i+1}$, then $l\mathbf{y} \in S_i$. Since the inverse limit of finite non-empty sets is non-empty [FrJ08, p. 3, Cor. 1.1.4], this yields an infinite sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \dots$ of points in $A_{l^\infty}(\tilde{K})$ such that $l\mathbf{y}_{i+1} = \mathbf{y}_i$ for $i = 1, 2, 3, \dots$ and $[K\tilde{K}_0(\mathbf{y}_i) : K\tilde{K}_0] \leq \beta$.

Note that $K\tilde{K}_0(\mathbf{y}_i) \subseteq K\tilde{K}_0(\mathbf{y}_{i+1})$. Hence, by the preceding paragraph, the sequence $K\tilde{K}_0(\mathbf{y}_1) \subseteq K\tilde{K}_0(\mathbf{y}_2) \subseteq K\tilde{K}_0(\mathbf{y}_3) \subseteq \cdots$ becomes stationary at some point. Thus, $K\tilde{K}_0$ has a finite extension M such that $\mathbf{y}_i \in A(M)$ for all i. It follows that $A_{l^\infty}(M)$ is infinite.

On the other hand, $\tilde{M} = \tilde{K}$. Since no simple quotient of $A_{\tilde{M}}$ is defined over $\tilde{K}_0$, the abelian group $A(M)$ is finitely generated (Corollary 4.9). In particular, $A_{l^\infty}(M)$ is finite (see the second paragraph of Section 3). This contradiction to the preceding paragraph proves our claim.

Part B: *Reduction modulo* $\mathfrak{p}$. By Setup 1.1, $K = K_0(\mathbf{x})$ with $\mathbf{x} = (x_1, \dots, x_n)$. Thus, $K\tilde{K}_0 = \tilde{K}_0(\mathbf{x})$, so by Part A

$$[\tilde{K}_0(\mathbf{x}, \mathbf{y}) : \tilde{K}_0(\mathbf{x})] > \beta \text{ for every } \mathbf{y} \in A(\tilde{K}) \text{ of order } l^i. \qquad (27) \quad \{\texttt{kxya}\}$$

We embed A in $\mathbb{P}^m_K$ for some positive integer m (Remark 3.2). Let V be the integral affine variety over $\tilde{K}_0$ with generic point $\mathbf{x}$ and recall that $\mathbf{x}$ has been chosen in Setup 1.1 such that V is smooth. For every $\mathbf{y} \in A(\tilde{K})$ of order l^i we denote the integral subvariety of $\mathbb{A}^n_{\tilde{K}_0} \times \mathbb{P}^m_{\tilde{K}_0}$ with generic point $(\mathbf{x}, \mathbf{y})$ by $W_{\mathbf{y}}$.

Claim: *For almost all* $\mathfrak{p} \in \mathrm{Spec}(R_0)$ *and every* $\mathbf{y} \in A(\tilde{K})$ *of order* l^i, *we have*

$$[\bar{K}_{0,\mathfrak{p},\mathrm{alg}}(\bar{\mathbf{x}}_{\mathfrak{p}}, \bar{\mathbf{y}}_{\mathfrak{p}}) : \bar{K}_{0,\mathfrak{p},\mathrm{alg}}(\bar{\mathbf{x}}_{\mathfrak{p}})] > \beta, \qquad (28) \quad \{\texttt{hrtb}\}$$

where $\bar{\mathbf{x}}_{\mathfrak{p}}$ *is a generic point of* $\bar{V}_{\mathfrak{p}}$ *and such that, as in Example 1.8,* $(\bar{\mathbf{x}}_{\mathfrak{p}}, \bar{\mathbf{y}}_{\mathfrak{p}})$ *is a reduction modulo* $\mathfrak{p}$ *of* $(\mathbf{x}, \mathbf{y})$ *that generates* $\bar{W}_{\mathbf{y},\mathfrak{p}}$.

Indeed, by Remark 4.1, $A(\tilde{K})$ has only finitely many points $\mathbf{y}$ whose order is l^i. Hence, it suffices to consider $\mathbf{y} \in A(\tilde{K})$ of order l^i and to prove (28) for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$.

By Lemma 3.6, $\tilde{K}_0(\mathbf{x}, \mathbf{y})/\tilde{K}_0(\mathbf{x})$ is a finite separable extension. Let $\varphi\colon W_{\mathbf{y}} \to V$ be the rational map defined by $\varphi(\mathbf{x}, \mathbf{y}) = \mathbf{x}$. Since φ is separable and V is normal (because V is smooth), $d := [\tilde{K}_0(\mathbf{x}, \mathbf{y}) : \tilde{K}_0(\mathbf{x})] = \deg(\varphi)$ is the number of points in $\varphi^{-1}(\mathbf{a})$ for every $\mathbf{a}$ in $V_0(\tilde{K}_0)$ for some non-empty open subset V_0 of V [Mil17, p. 182, Thm. 8.40]. Thus, the equality $d = \deg(\varphi)$ is an elementary statement on $\tilde{K}_0$.

It follows by Remarks 3.5 and 1.6 that $\bar{\varphi}_{\mathfrak{p}}\colon \bar{W}_{\mathbf{y},\mathfrak{p}} \to \bar{V}_{\mathfrak{p}}$ is a separable rational map with $\deg(\bar{\varphi}_{\mathfrak{p}}) = d$ for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$. Hence, by the preceding paragraph, for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$ we have

$$\begin{aligned}[\bar{K}_{0,\mathfrak{p},\mathrm{alg}}(\bar{\mathbf{x}}_{\mathfrak{p}}, \bar{\mathbf{y}}_{\mathfrak{p}}) : \bar{K}_{0,\mathfrak{p},\mathrm{alg}}(\bar{\mathbf{x}}_{\mathfrak{p}})] &= \deg(\bar{\varphi}_{\mathfrak{p}}) = d \\ &= \deg(\varphi) = [\tilde{K}_0(\mathbf{x}, \mathbf{y}) : \tilde{K}_0(\mathbf{x})] \overset{(27)}{>} \beta,\end{aligned}$$

as claimed.

Conclusion of the proof: For almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$, $\bar{A}_\mathfrak{p}$ is an abelian variety over $\bar{K}_\mathfrak{p}$ with $\dim(\bar{A}_\mathfrak{p}) = \dim(A)$ (Remark 3.5). By Lemma 4.11, for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$, reduction modulo $\mathfrak{p}$ maps $A_{l^i}(\tilde{K})$ isomorphically onto $\bar{A}_{\mathfrak{p},l^i}(\bar{K}_{\mathfrak{p},\mathrm{alg}})$. Hence, by the claim,

$$\text{for all } \bar{\mathbf{y}} \in \bar{A}_{\mathfrak{p},l^i}(\bar{K}_{\mathfrak{p},\mathrm{alg}}) \text{ of order } l^i \quad \text{we have } [\bar{K}_{0,\mathfrak{p},\mathrm{alg}}(\bar{\mathbf{x}}_\mathfrak{p}, \bar{\mathbf{y}}) : \bar{K}_{0,\mathfrak{p},\mathrm{alg}}(\bar{\mathbf{x}}_\mathfrak{p})] > \beta. \tag{29}$$ {cncm}

Let $\mathfrak{p}$ be a prime ideal of R_0 that satisfies (29). We assume by contradiction that $\bar{A}_{\mathfrak{p},\bar{K}_{\mathfrak{p},\mathrm{alg}}}$ has a non-trivial $\bar{K}_{0,\mathfrak{p},\mathrm{alg}}$-quotient. Thus, by Definition 4.5, there exist an abelian variety B over a finite extension of $\bar{K}_{0,\mathfrak{p}}$ and a non-zero homomorphism $h\colon B_{\bar{K}_{\mathfrak{p},\mathrm{alg}}} \to \bar{A}_{\mathfrak{p},\bar{K}_{\mathfrak{p},\mathrm{alg}}}$. By the preceding paragraph, $\beta(\dim(A)) = \beta(\dim(\bar{A}_\mathfrak{p}))$. By Lemma 4.10 with $\bar{K}_{0,\mathfrak{p},\mathrm{alg}}$ and $\bar{K}_{0,\mathfrak{p},\mathrm{alg}}(\bar{\mathbf{x}}_\mathfrak{p})$ replacing F_0 and F, respectively, all torsion points of $h(B_{\bar{K}_{0,\mathfrak{p},\mathrm{alg}}})$ are rational over a finite extension of $\bar{K}_{0,\mathfrak{p},\mathrm{alg}}(\bar{\mathbf{x}}_\mathfrak{p})$ of degree at most β. But by Lemma 4.12, $h(B(\bar{K}_{0,\mathfrak{p},\mathrm{alg}}))$ contains a point $\bar{\mathbf{y}}$ of order l^i. By what we have just said, the degree of $\bar{\mathbf{y}}$ over $\bar{K}_{0,\mathfrak{p},\mathrm{alg}}(\bar{\mathbf{x}}_\mathfrak{p})$ is at most β. This contradiction to (29) proves that $\bar{A}_{\mathfrak{p},\bar{K}_{\mathfrak{p},\mathrm{alg}}}$ has no $\bar{K}_{0,\mathfrak{p},\mathrm{alg}}$-quotient, as claimed. □

{HRSp}

Corollary 4.14. *Let R_0, K_0, R, and K be as in Setup 1.1 and let C be an elliptic curve over K such that $C_{\tilde{K}}$ is not defined over $\tilde{K}_0$.*

Then, for almost all $\mathfrak{p} \in \mathrm{Spec}(R_0)$, $\bar{C}_\mathfrak{p}$ is an elliptic curve over $\bar{K}_\mathfrak{p}$ and $\bar{C}_{\mathfrak{p},\bar{K}_\mathfrak{p},\mathrm{alg}}$ is not defined over $\bar{K}_{0,\mathfrak{p},\mathrm{alg}}$.

5 A Moduli Space

{MSp}

Let F/F_0 be an extension of fields. We say that a geometrically integral curve C over F is **$\tilde{F}/\tilde{F}_0$-isotrivial** if there exists a geometrically integral curve C_0 over $\tilde{F}_0$ such that $C_{0,\tilde{F}}$ is birationally equivalent to $C_{\tilde{F}}$. Recall that if both C and C_0 are smooth and projective, then the latter condition implies that $C_{0,\tilde{F}}$ is isomorphic to $C_{\tilde{F}}$ [Har77, p. 45, Cor. 6.12].

We prove that "$\tilde{K}/\tilde{K}_0$-non-isotriviality" for curves over K is preserved under almost all reductions with respect to prime ideals of R_0. As in the preceding sections, K/K_0 is the finitely generated field extension introduced in Setup 1.1 and R_0 is a noetherian domain with $\mathrm{Quot}(R_0) = K_0$.

{morpm}

Remark 5.1. Recall that a **quasi-projective** morphism (see [Liu06, p. 109, Def. 3.35] for a definition) is stable under base change. See [Liu06, p. 112, Exer. 3.20(a)] or [GoW10, p. 575, quasi-projective satisfies (BC)]. ∎

{crgn}

Remark 5.2. A **curve of genus g over a scheme S** is a smooth and proper morphism $\pi\colon C \to S$ of schemes whose geometric fibers $C_{\tilde{s}} = C \times_S \mathrm{Spec}(\Omega)$, for each morphism $\tilde{s}\colon \mathrm{Spec}(\Omega) \to S$, where Ω is an algebraically closed field, are irreducible curves of genus g. By [Liu06, p. 104, Prop. 3.16(c) and p. 143, Prop. 3.38], $C_{\tilde{s}}$ is proper and smooth over $\mathrm{Spec}(\Omega)$. Hence, by [Liu06, p. 109, Rem. 3.33], $C_{\tilde{s}}$ is projective over Spec(Ω). Therefore, by Remark 2.1, $C_{\tilde{s}}$ is also conservative. ∎

{rpfn}

Remark 5.3. For a scheme M we denote by h_M the **representable functor** from the category of schemes to the category of sets defined by $h_M(T) = \mathrm{Hom}(T, M)$ for each scheme T, where $\mathrm{Hom}(T, M)$ is the set of morphisms of schemes from T to M [GoW10, p. 93, Section 4.1].

Then, h_M is a contravariant functor from the category of schemes to the category of sets. Thus, for every morphism $f\colon T \to S$ of schemes we have a map $h_M(f)\colon h_M(S) \to h_M(T)$ that attaches to each morphism $\varphi\colon S \to M$ the morphism $\varphi \circ f\colon T \to M$. ∎

{MgS}

Remark 5.4. Suppose that $g \geq 2$ and let S be a noetherian scheme. We denote by $\mathcal{M}_g(S)$ the set of all curves of genus g over S, modulo isomorphism. Then, $\mathcal{M}_g$ is a contravariant functor from the category of noetherian schemes to the category of sets. Thus, for every morphism $f\colon T \to S$ of noetherian schemes we have a map $\mathcal{M}_g(f)\colon \mathcal{M}_g(S) \to \mathcal{M}_g(T)$ that attaches to each curve $\pi\colon C \to S$ of genus g the curve $\pi_T\colon C \times_S T \to T$, which is also of genus g, with π_T being the projection on the second factor.

By [MFK94, p. 143, Cor. 7.14 and p. 99, Def. 5.6], there exists a scheme M_g over $\mathrm{Spec}(\mathbb{Z})$ which satisfies

$$\begin{aligned}&M_g \text{ is quasi-projective over the open subset } \mathrm{Spec}(\mathbb{Z}) \smallsetminus \{p\mathbb{Z}\}\\ &\text{of } \mathrm{Spec}(\mathbb{Z}), \text{ for each prime number } p,\end{aligned} \tag{30}$$ {Eq5_1}

and there exists a morphism Φ_g from the functor $\mathcal{M}_g$ to the functor h_{M_g}, in particular for each noetherian scheme S there is a map $\Phi_g(S)\colon \mathcal{M}_g(S) \to \mathrm{Hom}(S, M_g)$, such that (M_g, Φ_g) is a **coarse moduli scheme**. That is, {CMSa}

(a) for all algebraically closed fields Ω, the map

$$\Phi_g(\mathrm{Spec}(\Omega))\colon \mathcal{M}_g(\mathrm{Spec}(\Omega)) \to h_{M_g}(\mathrm{Spec}(\Omega)) = \mathrm{Hom}(\mathrm{Spec}(\Omega), M_g))$$

is bijective, and {ABSj}

(b) for every scheme N and morphism ψ from $\mathcal{M}_g$ to h_N, there is a unique morphism $\chi\colon h_{M_g} \to h_N$ such that $\psi = \chi \circ \Phi_g$. [3]

In particular, by (30) and Remark 5.1, for every field F, the scheme $M_{g,F}$ is quasi-projective over $\mathrm{Spec}(F)$. Although we don't use it, we mention that $M_{g,F}$ is irreducible [DeM69].

Consider a curve $\pi\colon C \to S$ of genus g and a geometric fiber $C_{\tilde{s}} = C \times_S \mathrm{Spec}(\Omega)$ as in Remark 5.2. Denote by $[\pi]$ the corresponding element in $\mathcal{M}_g(S)$. By definition,

$$[\pi_\Omega] = \mathcal{M}_g(\tilde{s})([\pi]), \tag{31}$$ {piOm}

where $\pi_\Omega := \pi_{\mathrm{Spec}(\Omega)}$ is as in the first paragraph of the present remark. Let

$$\varphi = \Phi_g(S)([\pi]) \in h_{M_g}(S) = \mathrm{Hom}(S, M_g)\,. \tag{32}$$ {Eq5_2}

[3] We don't use condition (b) in the sequel.

Thus, $\varphi\colon S \to M_g$ is a morphism of schemes and, since Φ_g is a morphism between two contravariant functors,

$$\Phi_g(\mathrm{Spec}(\Omega))([\pi_\Omega]) = \varphi \circ \tilde{s} \in \mathrm{Hom}(\mathrm{Spec}(\Omega), M_g), \tag{33}$$ {Eq5_3}

as follows from the following commutative square:

$$\begin{array}{ccccccc}
[\pi] & \in & \mathcal{M}_g(S) & \xrightarrow{\Phi_g(S)} & \mathrm{Hom}(S, M_g) & \ni & \varphi \\
\downarrow & & \downarrow{\scriptstyle \mathcal{M}_g(\tilde{s})} & & \downarrow{\scriptstyle h_{M_g}(\tilde{s})} & & \downarrow \\
[\pi_\Omega] & \in & \mathcal{M}_g(\mathrm{Spec}(\Omega)) & \xrightarrow[\Phi_g(\mathrm{Spec}(\Omega))]{} & \mathrm{Hom}(\mathrm{Spec}(\Omega), M_g) & \ni & \varphi \circ \tilde{s}\,.
\end{array}$$ ∎

{ISoT}

Theorem 5.5. *Let C be a smooth geometrically integral curve over K of genus $g \geq 1$. Suppose that $C(K) \neq \emptyset$, C is conservative, and $C_{\tilde{K}}$ is not birationally equivalent to a curve which is defined over $\tilde{K}_0$.*

Then, for almost all $s \in \mathrm{Spec}(R_0)$ the reduced curve C_s over $\bar{K}_s$ is geometrically integral, smooth, conservative of genus g, and $C_s(\bar{K}_s) \neq \emptyset$.

In addition, $C_{s,\bar{K}_{s,\mathrm{alg}}}$ is not birationally equivalent to a curve which is defined over $\bar{K}_{0,s,\mathrm{alg}}$. In other words, if C is non-$\tilde{K}/\tilde{K}_0$-isotrivial, then $C_{s,\bar{K}_{s,\mathrm{alg}}}$ is non-$\bar{K}_{s,\mathrm{alg}}/\bar{K}_{0,s,\mathrm{alg}}$-isotrivial for almost all $s \in \mathrm{Spec}(R_0)$.

Proof. Replacing C by a birationally equivalent curve, we may assume that C is, in addition to being smooth and geometrically integral, also projective [GeJ89, Prop. 8.3]. By assumption and the first paragraph of this section, $C_{\tilde{K}}$ is not defined over $\tilde{K}_0$.

By Example 1.8(c),(d), and Lemma 2.2, smoothness, being geometrically integral, projective, and being conservative of genus g, are preserved under reduction with respect to almost all $s \in \mathrm{Spec}(R)$ (see also Remark 5.2), hence also with respect to almost all $s \in \mathrm{Spec}(R_0)$ (Remark 1.5). Also, the K-rational point of C yields a $\bar{K}_s$-rational point of C_s for almost all $s \in \mathrm{Spec}(R)$, hence also for almost all $s \in \mathrm{Spec}(R_0)$. It remains to prove:

Claim: *For almost all $s \in \mathrm{Spec}(R_0)$ the curve $\tilde{C}_s := C_{s,\bar{K}_{s,\mathrm{alg}}}$ is not defined over $\bar{K}_{0,s,\mathrm{alg}}$.*

The case $g = 1$ is covered by Corollary 4.14, since then C is an elliptic curve over K.

Assume $g \geq 2$ and let (M_g, Φ_g) be the coarse moduli scheme that corresponds to the functor $\mathcal{M}_g$. Let $\pi\colon \mathcal{C} \to \mathrm{Spec}(R)$ be a curve of genus g whose generic fiber is C. Then, $C_s = \mathcal{C} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\bar{K}_s)$ and $\tilde{C}_s = C_s \times_{\mathrm{Spec}(\bar{K}_s)} \mathrm{Spec}(\bar{K}_{s,\mathrm{alg}})$ for each $s \in \mathrm{Spec}(R)$. Let $[\pi]$ be the corresponding element in $\mathcal{M}_g(\mathrm{Spec}(R))$ (last paragraph of Remark 5.4) and let $\varphi := \Phi_g(\mathrm{Spec}(R))([\pi]) \in \mathrm{Hom}(\mathrm{Spec}(R), M_g)$ be as in (32).

Since $C_{\tilde{K}}$ is not defined over $\tilde{K}_0$,

$$\text{there is no curve } \pi_0\colon C_0 \to \mathrm{Spec}(\tilde{K}_0) \text{ of genus } g \text{ such that } [\pi_{\tilde{K}}] = [\pi_{0,\tilde{K}}]. \tag{34}$$ {Eq5_4}

Let $j\colon \mathrm{Spec}(\tilde{K}) \to \mathrm{Spec}(R)$ (resp. $j_0\colon \mathrm{Spec}(\tilde{K}) \to \mathrm{Spec}(\tilde{K}_0)$) be the morphism induced from the inclusion $R \subset \tilde{K}$ (resp. $\tilde{K}_0 \subset \tilde{K}$). Then, by (33),

$$\Phi_g(\mathrm{Spec}(\tilde{K}))([\pi_{\tilde{K}}]) = \varphi \circ j \in \mathrm{Hom}(\mathrm{Spec}(\tilde{K}), M_g)\,.$$

The morphism $\varphi \circ j\colon \mathrm{Spec}(\tilde{K}) \to M_g$ defines a $\tilde{K}$-rational point $\mathbf{a}$ of M_g.

Subclaim A: *There is no morphism* $\varphi_0\colon \mathrm{Spec}(\tilde{K}_0) \to M_g$ *such that*

$$\varphi \circ j = \varphi_0 \circ j_0\,. \tag{35}$$ {Eq5_5}

Otherwise, since $\varphi_0 \in \mathrm{Hom}(\mathrm{Spec}(\tilde{K}_0), M_g)$, there is by (a), a curve $\pi_0\colon C_0 \to \mathrm{Spec}(\tilde{K}_0)$ of genus g which satisfies $\Phi_g(\mathrm{Spec}(\tilde{K}_0))([\pi_0]) = \varphi_0$. Therefore,

$$\Phi_g(\mathrm{Spec}(\tilde{K}))([\pi_{\tilde{K}}]) \overset{(33)}{=} \varphi \circ j \overset{(35)}{=} \varphi_0 \circ j_0 \overset{(33)}{=} \Phi_g(\mathrm{Spec}(\tilde{K}))([\pi_{0,\tilde{K}}]).$$

Hence, by (a) again, $[\pi_{\tilde{K}}] = [\pi_{0,\tilde{K}}]$, contrary to (34). Thus, the $\tilde{K}$-rational point $\mathbf{a}$ of M_g is not $\tilde{K}_0$-rational, which proves the subclaim.

By (1), we may assume that some prime number is invertible in R_0. Hence, by (30) and Remark 5.1, $M_{g,R} := M_g \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(R)$ is quasi-projective over $\mathrm{Spec}(R)$, say $M_{g,R} \subseteq \mathbb{P}^r_R$ for some positive integer r. Then, by Subclaim A, there exists $\mathbf{a} = (a_0 : a_1 : \cdots : a_r) \in M_{g,R}(\tilde{K})$ and there exist distinct k, l between 0 and r such that $a_l \neq 0$ and $\frac{a_k}{a_l} \notin \tilde{K}_0$. Therefore, for almost all $s \in \mathrm{Spec}(R_0)$, we have that $\bar{\mathbf{a}}_s = (\bar{a}_{0,s} : \bar{a}_{1,s} : \cdots : \bar{a}_{r,s}) \in M_{g,\bar{K}_s}(\bar{K}_{s,alg})$ and $\frac{\bar{a}_{k,s}}{\bar{a}_{l,s}} \notin \bar{K}_{0,s,\mathrm{alg}}$. Thus, for almost all $s \in \mathrm{Spec}(R_0)$, the $\bar{K}_{s,\mathrm{alg}}$-rational point $\bar{\mathbf{a}}_s$ of $M_{g,\bar{K}_s}$ is not $\bar{K}_{0,s,\mathrm{alg}}$-rational.

Consider such $s \in \mathrm{Spec}(R_0)$ and let

$$j_s\colon \mathrm{Spec}(\bar{K}_{s,\mathrm{alg}}) \to \mathrm{Spec}(\bar{K}_s) \to \mathrm{Spec}(R)$$

(resp. $j_{0,s}\colon \mathrm{Spec}(\bar{K}_{s,\mathrm{alg}}) \to \mathrm{Spec}(\bar{K}_{0,s,\mathrm{alg}})$) be the morphism induced by the reduction $R \to \bar{K}_s$ followed by the inclusion $\bar{K}_s \subset \bar{K}_{s,\mathrm{alg}}$ (resp. the inclusion $\bar{K}_{0,s,\mathrm{alg}} \subset \bar{K}_{s,\mathrm{alg}}$). Then, $\bar{\mathbf{a}}_s$ is the $\bar{K}_{s,\mathrm{alg}}$-rational point of M_g corresponding to the morphism $\varphi \circ j_s\colon \mathrm{Spec}(\bar{K}_{s,\mathrm{alg}}) \to M_g$ and, by (33),

$$\Phi_g(\mathrm{Spec}(\bar{K}_{s,\mathrm{alg}}))([\pi_{\bar{K}_{s,\mathrm{alg}}}]) = \varphi \circ j_s \in \mathrm{Hom}(\mathrm{Spec}(\bar{K}_{s,\mathrm{alg}}), M_g)\,.$$

Since the $\bar{K}_{s,\mathrm{alg}}$-rational point $\bar{\mathbf{a}}_s$ of M_g is not $\bar{K}_{0,s,\mathrm{alg}}$-rational,

$$\begin{array}{l}\text{there is no morphism } \varphi_{0,s}\colon \mathrm{Spec}(\bar{K}_{0,s,\mathrm{alg}}) \to M_g \\ \text{such that } \varphi \circ j_s = \varphi_{0,s} \circ j_{0,s}.\end{array} \tag{36}$$ {Eq5_6}

Subclaim B: *There is no curve* $\pi_{0,s}\colon C_{0,s} \to \mathrm{Spec}(\bar{K}_{0,s,\mathrm{alg}})$ *of genus g such that*

$$[\pi_{0,s,\bar{K}_{s,\mathrm{alg}}}] \overset{(31)}{=} \mathcal{M}_g(j_{0,s})([\pi_{0,s}]) = \mathcal{M}_g(j_s)([\pi]) \overset{(31)}{=} [\pi_{\bar{K}_{s,\mathrm{alg}}}]\,. \tag{37}$$ {Eq5_7}

Otherwise, let $\varphi_{0,s} := \Phi_g(\mathrm{Spec}(\bar{K}_{0,s,\mathrm{alg}}))([\pi_{0,s}]) \in \mathrm{Hom}(\mathrm{Spec}(\bar{K}_{0,s,\mathrm{alg}}), M_g)$. Then,

$$\begin{aligned}\varphi_{0,s} \circ j_{0,s} &\overset{(33)}{=} \Phi_g(\mathrm{Spec}(\bar{K}_{s,\mathrm{alg}}))([\pi_{0,s,\bar{K}_{s,\mathrm{alg}}}]) \\ &\overset{(37)}{=} \Phi_g(\mathrm{Spec}(\bar{K}_{s,\mathrm{alg}}))([\pi_{\bar{K}_{s,\mathrm{alg}}}]) \overset{(33)}{=} \varphi \circ j_s \,,\end{aligned}$$

which contradicts (36). This proves the subclaim.

By Subclaim B, the curve $\pi_{\bar{K}_{s,\mathrm{alg}}}\colon \tilde{C}_s \to \mathrm{Spec}(\bar{K}_{s,\mathrm{alg}})$ is not defined over $\bar{K}_{0,s,\mathrm{alg}}$. This proves the claim. □

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