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Embedding Problems with Bounded Ramification over Global Fields of Positive Characteristic

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Abstract: Let K/K_0 be a finite Galois extension of global fields of positive characteristic p. We prove that every finite embedding problem with solvable kernel H over K/K_0 is properly solvable if it is weakly locally solvable and the number of the roots of unity in K is relatively prime to $|H|$.

Moreover, the solution can be chosen to coincide with finitely many (given in advance) weak local solutions. Finally, and this is the main point of this work, the number of primes of K_0 that ramify in the solution field is bounded by the number of primes of K_0 that ramify in K plus the number of prime divisors of $|H|$, counted with multiplicity.

This result completes the main theorem of [JaR18] that demands that p does not divide $|H|$.

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Introduction

Solving finite embedding problems with solvable kernels over a global field K_0 was initiated by Arnold Scholz [Sch37] and Hans Reichardt [Rei37], followed by I. R. Shafarevich [Sha54] and Jürgen Neukirch [Neu79]. The works [GeJ98] and [MaU11] consider the related problem of realizing an l-group over K_0 for a prime $l \neq \mathrm{char}(K_0)$ with the additional constraint of bounding the ramification. A stronger result appears in [JaR18]:

THEOREM A: *Let K/K_0 be a finite Galois extension of global fields, set $\Gamma = \mathrm{Gal}(K/K_0)$, and consider a finite embedding problem*

(1)
$$\begin{tikzcd} & & & \mathrm{Gal}(K_0) \arrow[d, "\rho"] & \\ 1 \arrow[r] & H \arrow[r] & G \arrow[r, "\alpha"] & \Gamma \arrow[r] & 1, \end{tikzcd}$$

with solvable kernel H. Suppose that
(a1) $\mathrm{char}(K_0) \nmid |H|$, $\gcd(|H|, |\mu(K)|) = 1$, *and*
(a2) *for each $\mathfrak{p} \in \mathbb{P}(K_0)$ there exists a homomorphism $\psi_\mathfrak{p}\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}) \to G$ such that $\alpha \circ \psi_\mathfrak{p} = \rho|_{\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})}$ (we call $\psi_\mathfrak{p}$ a* **weak local solution***).*
Let T be a finite subset of $\mathbb{P}(K_0)$ that contains $\mathrm{Ram}(K/K_0)$ and for each $\mathfrak{p} \in T$ let $\varphi_\mathfrak{p}$ be a weak local solution.

Then, there exists an epimorphism $\psi\colon \mathrm{Gal}(K_0) \to G$ such that $\alpha \circ \psi = \rho$ (we call ψ a **proper solution of embedding problem (1)***), and there exists a set $R \subseteq \mathbb{P}(K_0) \smallsetminus T$ with $|R| = \Omega(|H|)$ that satisfies the following conditions:*
(b1) *For each $\mathfrak{p} \in T$ there exists $a \in H$ such that $\psi(\sigma) = a^{-1}\varphi_\mathfrak{p}(\sigma)a$ for all $\sigma \in \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$ (we say that $\psi_\mathfrak{p} := \psi|_{\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})}$ and $\varphi_\mathfrak{p}$ are* **H-equivalent***).*
(b2) *The fixed field N in $K_{0,\mathrm{sep}}$ of $\mathrm{Ker}(\psi)$ satisfies $\mathrm{Ram}(N/K_0) \subseteq T \cup R$, hence $|\mathrm{Ram}(N/K_0)| \leq |T| + \Omega(|H|)$.*

In this theorem we fix a separable algebraic closure $K_{0,\mathrm{sep}}$ of K_0 and let $\mathrm{Gal}(K_0) = \mathrm{Gal}(K_{0,\mathrm{sep}}/K_0)$ be the absolute Galois group of K_0. We denote the set of primes of K_0 by $\mathbb{P}(K_0)$ and for each $\mathfrak{p} \in \mathbb{P}(K_0)$ we choose a completion $\hat{K}_{0,\mathfrak{p}}$ of K_0 at $\mathfrak{p}$. Then, $\mathrm{Ram}(K/K_0)$ denotes the set of all $\mathfrak{p} \in \mathbb{P}(K_0)$ that ramify in K. Finally, $\Omega(|H|)$ is the number of prime divisors of $|H|$, counted with multiplicity.

The goal of the present work is to improve Theorem A by removing the assumption $\mathrm{char}(K_0) \nmid |H|$ from condition (a1) above. Moreover, in case $\mathrm{char}(K_0)$ divides $|H|$, we replace the function $\Omega(|H|)$ in (b2) by a more economic function $\Omega_p(H, G)$ that however depends on the structure of H as a subgroup of G (Definition 5.2).

THEOREM B (Theorem 5.4): *Let K/K_0 be a finite Galois extension of global fields of positive characteristic p and consider the finite embedding problem (1) with solvable* *kernel H, where $\Gamma = \mathrm{Gal}(K/K_0)$ and $\rho = \mathrm{res}_{K_{0,\mathrm{sep}}/K}$. Let T be a finite set of primes of K_0 that contains $\mathrm{Ram}(K/K_0)$ and $S_0(K) \subseteq T_K$, where $S_0(K)$ is a "basic set of K" introduced before Lemma 3.2 and T_K is the set of all primes of K that lie over T. Let $|H| = np^s$ with $p \nmid n$ and $\gcd(n, |\mu(K)|) = 1$. Suppose that (1) is weakly locally solvable at $\mathfrak{p}$ for each $\mathfrak{p} \in \mathbb{P}(K_0)$. For each $\mathfrak{p} \in T$ let $\varphi_\mathfrak{p}$ be a weak local solution of (1) at $\mathfrak{p}$.*

Then, (1) has a proper solution ψ and there exists a set $R \subseteq \mathbb{P}(K_0) \smallsetminus T$ with $|R| = \Omega_p(H, G)$ such that

(a) *For each $\mathfrak{p} \in T$ there exists $a \in H$ such that $\psi(\sigma) = a^{-1}\varphi_\mathfrak{p}(\sigma)a$ for all $\sigma \in \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$.*

(b) *The fixed field N in $K_{0,\mathrm{sep}}$ of $\mathrm{Ker}(\psi)$ satisfies $\mathrm{Ram}(N/K_0) \subseteq T \cup R$, hence $|\mathrm{Ram}(N/K_0)| \le |T| + \Omega_p(H, G)$ (we call N the* **solution field of (1)***).*

We note that Theorem 9.5.5 on page 563 of [NSW15] implies the proper solvability part of Theorem B, nevertheless, without any information about the ramification of the solution field.

An induction on the structure of G with respect to H, carried out in the proof of Theorem 5.4 using Lemma 5.1, reduces Theorem B to the following result:

Proposition C (Proposition 4.5): *Let K/K_0 be a finite Galois extension of global fields of positive characteristic p and consider an embedding problem*

$$(\rho\colon \mathrm{Gal}(K_0) \to \Gamma,\ \bar{\alpha}\colon \bar{G} \to \Gamma), \tag{2}$$

where $\Gamma = \mathrm{Gal}(K/K_0)$, $\bar{G}$ is a finite group, $\bar{\alpha}$ is an epimorphism, and $\rho = \mathrm{res}_{K_{0,\mathrm{sep}}/K}$. Suppose that $A = \mathrm{Ker}(\bar{\alpha})$ is isomorphic to C_l^r for some positive integer r and a prime number l with $\zeta_l \notin K$ and the action of $\mathrm{Gal}(K_0)$ on A via ρ and via conjugation of $\bar{G}$ on A makes A a simple $\mathrm{Gal}(K_0)$-module.

In addition, let $\lambda\colon G \to \bar{G}$ be an epimorphism of finite groups. Write $|\mathrm{Ker}(\lambda)| = ep^s$ and let n be a positive integer such that $p \nmid en$. Moreover, we assume that $el|n$ if $l \neq p$. Let T be a finite set of primes of K_0 that contains $\mathrm{Ram}(K/K_0)$ and $S_0(K) \subseteq T_K$. Suppose that $\gcd(n, |\mu(K)|) = 1$ and each of the local embedding problems attached to (2) is weakly solvable. In addition, for each $\mathfrak{p} \in T$ let $\varphi_\mathfrak{p}$ be a weak local solution of (2).

Then, there exists a set $R \subseteq \mathbb{P}(K_0) \smallsetminus T$ with $|R| = r$ if $l \neq p$ and $|R| = 1$ if $l = p$ such that (2) has a proper solution $\bar{\psi}$ with the following properties:

(a) *$\bar{\psi}_\mathfrak{p} = \bar{\psi}|_{\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})}$ is A-equivalent to $\varphi_\mathfrak{p}$ for each $\mathfrak{p} \in T$,*

(b) *$\bar{\psi}$ is unramified at each $\mathfrak{p} \in \mathbb{P}(K_0) \smallsetminus (T \cup R)$, so if $\bar{N}$ is the solution field of $\bar{\psi}$, then $\mathrm{Ram}(\bar{N}/K_0) \subseteq T \cup R$,*

(c) *the local embedding problem $(\bar{\psi}_\mathfrak{p}\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}) \to \bar{G},\ \lambda\colon G \to \bar{G})$ is weakly solvable for each $\mathfrak{p} \in \mathbb{P}(K_0) \smallsetminus T$, and*

(d) $\gcd(n, |\mu(\bar{N})|) = 1$.

Conditions (c) and (d) in Proposition C are needed in the next stage of the induction.

Proposition 12.3 of [JaR18] considers the case where $l \neq p$ and proves the existence of $\bar{\psi}$ as in Proposition C such that, away from $\mathrm{Ram}(K/K_0) \cup T$, ψ is ramified in at most r primes of K_0.

The proof of [JaR18, Prop. 12.3] depends on [JaR18, Lemma 2.3]. The latter lemma establishes the existence of a homomorphism h from the idele class group C_K of K into C_l with given local behavior and with bounded ramification. Then, the proof applies the reciprocity law and duality theorems of class field theory.

These methods fail if $l = p$. So, we take another route for the proof of Proposition C in this case that turns out to be much simpler than the proof of Proposition C in the case $l \neq p$.

This route goes back to the article [Wit36] of Ernst Witt. In that article Witt uses Artin-Schreier extensions and pre-cohomological methods in order to prove for arbitrary field F of positive characteristic p with $(F^\times : (F^\times)^p) = \infty$ that every finite embedding problem $G \to \mathrm{Gal}(F'/F)$ for $\mathrm{Gal}(F)$, with $\mathrm{Gal}(F'/F)$ a finite p-group and with a kernel which is a finite p-group, is properly solvable. In terms of cohomology, Witt's result implies that $\mathrm{cd}_p(\mathrm{Gal}(F^{(p)}/F)) = 1$, where $F^{(p)}$ is the maximal pro-p extension of F [Ser79, p. 21, Prop. 16].

In the notation of Proposition C, we know by [NSW15, p. 540, Cor. 9.2.6] that embedding problem (2) has a weak solution ψ_0 that is ramified at most at T. If we wish that ψ_0 coincides with $\varphi_\mathfrak{p}$ for each $\mathfrak{p} \in T$, we have to allow ψ_0 ramify at additional prime.

In its full strength, the proof of Proposition C uses [NSW15, p. 539, Thm. 9.2.5] and Lemma 4.3 that guarantees the surjectivity of weak solutions of our embedding problems and on a local-global principle for weak solutions of our embedding problems (Lemma 4.4).

In addition, the proof relies on the following result:

LEMMA D (LEMMA 2.6): *Let K_0 be a global field of positive characteristic p, K a finite Galois extension of K_0, and L a finite Galois extension of K_0 that contains K such that L/K is an abelian p-extension. Let r be a positive integer and $A = C_p^r$ a simple $\mathrm{Gal}(K/K_0)$-module. Let n be a positive integer such that $p \nmid n$. Let T be a finite subset of $\mathbb{P}(K_0)$ that contains $\mathrm{Ram}(K/K_0)$. For each $\mathfrak{p} \in T$, let $y_\mathfrak{p} \in H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$.*

Then, there exist a prime $\mathfrak{q} \in \mathbb{P}(K_0) \smallsetminus T$ and an element $x \in H^1(\mathrm{Gal}(K_0), A)$ such that

(a) *for each $\mathfrak{p} \in T$ we have $\mathrm{res}_\mathfrak{p}(x) = y_\mathfrak{p}$,*

(b) *for each $\mathfrak{p} \in \mathbb{P}(K_0) \smallsetminus (T \cup \{\mathfrak{q}\})$ the element $\mathrm{res}_\mathfrak{p}(x)$ of $H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$ is unramified (Definition 2.4), and*

(c) *$\mathfrak{q}$ totally splits in $L(\zeta_n)$ and $\mathrm{res}_\mathfrak{q}(x)\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{q}}) \to A$ is a homomorphism whose image is contained in a subgroup of A which is isomorphic to C_p.*

(d) *Moreover, let G and $\bar{G}$ be finite groups such that $A \le \bar{G}$ and let $\lambda\colon G \to \bar{G}$ be an epimorphism. Then, there exists a homomorphism $x'_{\mathfrak{q}}\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{q}}) \to G$ such that $\lambda \circ x'_{\mathfrak{q}} = \mathrm{res}_{\mathfrak{q}}(x)$.*

Here, $\mathrm{res}_{\mathfrak{p}}\colon H^1(\mathrm{Gal}(K_0), A) \to H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$ is the usual restriction map of cohomology groups.

Part (b) in Lemma D follows from [NSW15, p. 539, Thm. 9.2.5] whose proof is much simpler than the corresponding result for $A = C_l^r$ with $l \ne p$ that uses the Poitou-Tate duality theorem.

Finally, Lemma D depends on Lemma 2.5 and on the following analog of [JaR18, Lemma 2.3]:

PROPOSITION E (Proposition 1.2): *Let K be a global field of positive characteristic p, let r be a positive integer, let $A = C_p^r$, and let S be a finite set of primes of K. For each $\mathfrak{P} \in S$ let $h_{\mathfrak{P}}\colon \mathrm{Gal}(\hat{K}_{\mathfrak{P}}) \to A$ be a homomorphism, where $\hat{K}_{\mathfrak{P}}$ is a completion of K at $\mathfrak{P}$ and $\mathrm{Gal}(\hat{K}_{\mathfrak{P}})$ is embedded in $\mathrm{Gal}(K)$. Finally, consider $\mathfrak{P}_0 \in \mathbb{P}(K) \smallsetminus S$.*

Then, there exists a homomorphism $h\colon \mathrm{Gal}(K) \to A$ such that:

(a) $h|_{\mathrm{Gal}(\hat{K}_{\mathfrak{P}})} = h_{\mathfrak{P}}$ *for each* $\mathfrak{P} \in \{\mathfrak{P}_0\} \cup S$.

(b) *For each prime $\mathfrak{P}$ of K away from $\{\mathfrak{P}_0\} \cup S$ the restriction of h to the inertia subgroup of $\mathrm{Gal}(\hat{K}_{\mathfrak{P}})$ is trivial.*

The proof of the latter result uses the fact that each Galois extension L of K of degree p is generated by a root x of an irreducible Artin-Schreier polynomial $X^p - X - a$ with $a \in K$. The latter is a specialization of the polynomial $X^p - X - t$, with t transcendental, and with Galois group C_p over $K(t)$ as well as over $K_{\mathrm{sep}}(t)$. Thus, instead of class field theory, we use Hilbert's irreducibility theorem for our function field K_0 intensified by the strong approximation theorem [FrJ08, p. 241, Thm. 13.3.5]. In contrast to the case $A = C_l^r$ with $l \ne p$, only one prime $\mathfrak{q} \in \mathbb{P}(K_0) \smallsetminus \mathrm{Ram}(K/K_0)$ may need to ramify in the solution field of (1).

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1. Artin-Schreier Extensions

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Let K be a global field of positive characteristic p. We use Artin-Schreier extensions to prove a restricted version of [JaR18, Lemma 2.3] that constructs a homomorphism $h\colon \mathrm{Gal}(K) \to C_p^r$ with a given local behavior.

In this result we consider for each prime $\mathfrak{P}$ of K, a completion $\hat{K}_{\mathfrak{P}}$ of K at $\mathfrak{P}$. Let $\hat{K}_{\mathfrak{P},\mathrm{ur}}$ be the maximal unramified extension of $\hat{K}_{\mathfrak{P}}$ and let $\hat{I}_{\mathfrak{P}} = \mathrm{Gal}(\hat{K}_{\mathfrak{P},\mathrm{ur}})$ be the inertia subgroup of $\mathrm{Gal}(\hat{K}_{\mathfrak{P}})$. We fix an embedding $\lambda_{\mathfrak{P}}\colon K_{\mathrm{sep}} \to \hat{K}_{\mathfrak{P},\mathrm{sep}}$ that maps K onto itself, set $K_{\mathfrak{P}} = \hat{K}_{\mathfrak{P}}^{\lambda_{\mathfrak{P}}^{-1}}$ and observe that $\mathrm{Gal}(K_{\mathfrak{P}})^{\lambda_{\mathfrak{P}}} = \mathrm{Gal}(\hat{K}_{\mathfrak{P}})$.

LEMMA 1.1: *Let K be a global field of positive characteristic p, let L be a finite Galois extension of K, let s be a positive integer, and let $\mathfrak{P}_0, \mathfrak{P}_1, \ldots, \mathfrak{P}_s$ be primes of K such that $\mathfrak{P}_0 \notin \{\mathfrak{P}_1, \ldots, \mathfrak{P}_s\}$ but $\mathfrak{P}_1, \ldots, \mathfrak{P}_s$ are not necessarily distinct. For $i = 0, 1, \ldots, s$ let $\hat{N}_i$ be either $\hat{K}_{\mathfrak{P}_i}$ or an Artin-Schreier extension of $\hat{K}_{\mathfrak{P}_i}$. Then, there exist Galois extensions $N_0, N_1, \ldots, N_s$ of K such that* CHAa input, 33

(a) *$N_i = K$ if $\hat{N}_i = \hat{K}_{\mathfrak{P}_i}$ and N_i is an Artin-Schreier extension of K if $\hat{N}_i$ is an Artin-Schreier extension of $\hat{K}_{\mathfrak{P}_i}$ for $i = 0, 1, \ldots, s$,*

(b) *$\lambda_{\mathfrak{P}_i}(N_i)\hat{K}_{\mathfrak{P}_i} = \hat{N}_i$ and $\lambda_{\mathfrak{P}_i}(N_i) \cap \hat{K}_{\mathfrak{P}_i} = K$ for $i = 0, 1, \ldots, s$,*

(c) *$\mathrm{Ram}(N_0/K) \subseteq \{\mathfrak{P}_0, \mathfrak{P}_1\}$, $\mathrm{Ram}(N_i/K) \subseteq \{\mathfrak{P}_0, \mathfrak{P}_i\}$ for $i = 1, \ldots, s$, and*

(d) *the fields $N_0, N_1, \ldots, N_s, L$ are linearly disjoint over K.*

Proof: In order to simplify our notation, we assume that $\lambda_{\mathfrak{P}_i}\colon K_{\mathrm{sep}} \to \hat{K}_{\mathfrak{P}_i,\mathrm{sep}}$ is the inclusion map, $i = 0, 1, \ldots, s$. We assume by induction that $N_0, N_1, \ldots, N_{s-1}$ are fields that satisfy (b), (c), and (d) for $s-1$ rather than for s.

If $\hat{N}_s = \hat{K}_{\mathfrak{p}_s}$, we set $N_s = K$ and observe that (a), (b), (c), and (d) hold for $i = s$. Thus, we may assume that $\hat{N}_{\mathfrak{P}_s}/\hat{K}_{\mathfrak{P}_s}$ is an Artin-Schreier extension. Hence, $\hat{N}_s = \hat{K}_{\mathfrak{P}_s}(\hat{x}_s)$, where $\hat{x}_s$ is a root of an irreducible polynomial $X^p - X - \hat{a}_s$ in $\hat{K}_{\mathfrak{P}_s}[X]$. Krasner's lemma (e.g. [Jar91, Prop. 12.3]) gives a positive integer m such that if $a \in K$ satisfies $\mathrm{ord}_{\mathfrak{P}_s}(a - \hat{a}_s) > m$, if the polynomial $X^p - X - a$ is irreducible over K, if the element x_s is a root of $X^p - X - a$ in K_{sep}, and we set $N_s = K(x_s)$, then

(1) $N_s\hat{K}_{\mathfrak{P}_s} = \hat{N}_s$.

It follows that

(2) N_s is an Artin-Schreier extension of K and $N_s \cap \hat{K}_{\mathfrak{P}_s} = K$.

Corollary 12.2.3 on page 224 of [FrJ08] gives a separable Hilbert subset H of K such that if $a \in H$, then $X^p - X - a$ is irreducible over $N_0N_1\cdots N_{s-1}L$.

For $s = 0$ we use [FrJ08, p. 241, Thm. 13.3.5] to choose a_0 in H with

(3) $\mathrm{ord}_{\mathfrak{P}_0}(a_0 - \hat{a}_0) > m$ and $\mathrm{ord}_{\mathfrak{P}}(a_0) \geq 0$ for all $\mathfrak{P} \in \mathbb{P}(K) \smallsetminus \{\mathfrak{P}_0, \mathfrak{P}_1\}$.

If $s \geq 1$, we apply [FrJ08, p. 241, Thm. 13.3.5] to choose a_s in H such that

(4) $\mathrm{ord}_{\mathfrak{P}_s}(a_s - \hat{a}_s) > m$ and $\mathrm{ord}_{\mathfrak{P}}(a_s) \geq 0$ for all $\mathfrak{P} \in \mathbb{P}(K) \smallsetminus \{\mathfrak{P}_0, \mathfrak{P}_s\}$.

In each case let x_s be a root of $X^p - X - a_s$ and set $N_s = K(x_s)$. By (1) and (2), $N_s\hat{K}_{\mathfrak{P}_s} = \hat{N}_s$ and $N_s \cap \hat{K}_{\mathfrak{P}_s} = K$. Since $a_s \in H$, the field N_s is linearly disjoint from $N_0N_1\cdots N_{s-1}L$ over K. Since, by our induction hypotheses, $N_0, N_1, \ldots, N_{s-1}, L$ are linearly disjoint over K, we conclude that $N_0, N_1, \ldots, N_s, L$ are linearly disjoint over K.

Finally, [FrJ08, p. 29, Example 2.3.9] and (3) imply that each $\mathfrak{P} \in \mathbb{P}(K) \smallsetminus \{\mathfrak{P}_0, \mathfrak{P}_1\}$ is unramified in N_0, so $\mathrm{Ram}(N_0/K) \subseteq \{\mathfrak{P}_0, \mathfrak{P}_1\}$. For $s \geq 1$, [FrJ08, p. 29, Example 2.3.9] and (4) imply that each $\mathfrak{P} \in \mathbb{P}(K) \smallsetminus \{\mathfrak{P}_0, \mathfrak{P}_s\}$ is unramified in N_s, so

$\mathrm{Ram}(N_s/K) \subseteq \{\mathfrak{P}_0, \mathfrak{P}_s\}$. This concludes the induction. ∎

PROPOSITION 1.2: *Let K be a global field of positive characteristic p, let r be a positive integer, and let $A = C_{p,1} \times \cdots \times C_{p,r}$, where $C_{p,1}, \ldots, C_{p,r}$ are isomorphic copies of C_p. Let S be a non-empty finite set of primes of K and let $\mathfrak{P}_0$ be a prime in $\mathbb{P}(K) \smallsetminus S$. For each $\mathfrak{P} \in \{\mathfrak{P}_0\} \cup S$ let $h_\mathfrak{P}\colon \mathrm{Gal}(\hat{K}_\mathfrak{P}) \to A$ be a homomorphism.* CHAb input, 141

Then, there exists a homomorphism $h\colon \mathrm{Gal}(K) \to A$ such that

(a) *$\mathrm{res}_\mathfrak{P}(h) = h_\mathfrak{P}$ for each $\mathfrak{P} \in \{\mathfrak{P}_0\} \cup S$ and*

(b) *$\mathrm{res}_\mathfrak{P}(h)(\hat{I}_\mathfrak{P}) = \mathbf{1}_A$ for each $\mathfrak{P} \in \mathbb{P}(K) \smallsetminus (\{\mathfrak{P}_0\} \cup S)$.*

Here, $\mathrm{res}_\mathfrak{P}(h)\colon \mathrm{Gal}(\hat{K}_\mathfrak{P}) \to A$ is the homomorphism defined by $\mathrm{res}_\mathfrak{P}(h)(\sigma) = h(\sigma^{\lambda_\mathfrak{P}^{-1}})$ for each $\sigma \in \mathrm{Gal}(\hat{K}_\mathfrak{P})$.

Proof: As in the proof of Lemma 1.1, we assume that the maps $\lambda_\mathfrak{P}\colon K_{\mathrm{sep}} \to \hat{K}_{\mathfrak{P},\mathrm{sep}}$ are inclusions.

Suppose that S consists of s distinct primes $\mathfrak{P}_1, \ldots, \mathfrak{P}_s$ of K. Since S is non-empty, $s \geq 1$. For each $0 \leq j \leq s$ let $A_j = h_{\mathfrak{P}_j}(\mathrm{Gal}(\hat{K}_{\mathfrak{P}_j}))$. As a subgroup of the p-elementary abelian group A, the group A_j is p-elementary abelian. Thus, $A_j = A_{j,1} \times \cdots \times A_{j,r_j}$, where $0 \leq r_j \leq r$ and $A_{jk} \cong C_p$ for $k = 1, \ldots, r_j$. If $r_j = 0$, then A_j is the trivial group. Let $\pi_{jk}\colon A_j \to A_{jk}$ be the projection on the kth factor of A_j. Then, $h_{\mathfrak{P}_j} = (\pi_{j,1} \circ h_{\mathfrak{P}_j}, \ldots, \pi_{j,r_j} \circ h_{\mathfrak{P}_j})$. In particular, if $r_j = 0$, then $A_j = \mathbf{1}$ and $h_{\mathfrak{P}_j}\colon \mathrm{Gal}(\hat{K}_{\mathfrak{P}_j}) \to A$ is the trivial homomorphism. In addition, let $\hat{N}_{jk}$ be the fixed field of $\mathrm{Ker}(\pi_{jk} \circ h_{\mathfrak{P}_j})$ in $\hat{K}_{\mathfrak{P}_j,\mathrm{sep}}$. Then, for each j and k,

(5) $\hat{N}_{jk} = \hat{K}_{\mathfrak{P}_j}$ or $\hat{N}_{jk}$ is an Artin-Schreier extension of $\hat{K}_{\mathfrak{P}_j}$.

It follows that $\hat{N}_j = \hat{K}_{\mathfrak{P}_j} \hat{N}_{j,1} \cdots \hat{N}_{j,r_j}$ is the fixed field of $\mathrm{Ker}(h_{\mathfrak{P}_j})$ in $\hat{K}_{\mathfrak{P}_j,\mathrm{sep}}$. In particular, $\mathrm{Gal}(\hat{N}_j/\hat{K}_{\mathfrak{P}_j}) \cong A_j$. Note that $\hat{N}_j = \hat{K}_{\mathfrak{P}_j}$ if $r_j = 0$.

By Lemma 1.1, applied to the primes $\mathfrak{P}_{0,1}$ and $\mathfrak{P}_{1,1}, \ldots, \mathfrak{P}_{1,r_1}, \ldots, \mathfrak{P}_{s,1}, \ldots, \mathfrak{P}_{s,r_s}$ with $\mathfrak{P}_{j,k} = \mathfrak{P}_j$ for each j and k, K has a Galois extension N_{jk} such that

(6a) $N_{jk}\hat{K}_{\mathfrak{P}_j} = \hat{N}_{jk}$, $N_{jk} \cap \hat{K}_{\mathfrak{P}_j} = K$,

(6b) $\mathrm{Ram}(N_{0,1}/K) \subseteq \{\mathfrak{P}_0, \mathfrak{P}_1\}$, $\mathrm{Ram}(N_{jk}/K) \subseteq \{\mathfrak{P}_0, \mathfrak{P}_j\}$, and

(6c) the fields $N_{0,1}, N_{1,1}, \ldots, N_{s,r_s}$ are linearly disjoint over K.

For each $0 \leq j \leq s$ let $N_j = N_{j,1} \cdots N_{j,r_j}$. By (6c), $\mathrm{Gal}(N_j/K) = \mathrm{Gal}(N_{j,1}/K) \times \cdots \times \mathrm{Gal}(N_{j,r_j}/K)$. By (6a), the map res: $\mathrm{Gal}(\hat{N}_{jk}/\hat{K}_{\mathfrak{P}_j}) \to \mathrm{Gal}(N_{jk}/K)$ is an isomorphism, so, by (5), $\mathrm{Gal}(N_{jk}/K) \cong A_{jk}$ for all k between 1 and r_j. This gives an isomorphism $h_j\colon \mathrm{Gal}(N_j/K) \to A_j$ such that $h_j \circ \mathrm{res}_{\hat{N}_j/N_j} \circ \mathrm{res}_{\hat{K}_{\mathfrak{P}_j,\mathrm{sep}}/\hat{N}_j} = h_{\mathfrak{P}_j}$.

By (6c), the fields $N_0, N_1, \ldots, N_s$ are linearly disjoint over K. We set $N = N_0 N_1 \cdots N_s$. Then, $\mathrm{Gal}(N/K) = \mathrm{Gal}(N_0/K) \times \mathrm{Gal}(N_1/K) \times \cdots \times \mathrm{Gal}(N_s/K)$, so $h_0, h_1, \ldots, h_s$ combine to a homomorphism $\bar{h}\colon \mathrm{Gal}(N/K) \to A$. In other words, $\bar{h}(\sigma) = \prod_{j=0}^s h_j(\mathrm{res}_{N/N_j}(\sigma))$ for each $\sigma \in \mathrm{Gal}(N/K)$. It follows that $h = \bar{h} \circ \mathrm{res}_{K_{\mathrm{sep}}/N}$ is a homomorphism from $\mathrm{Gal}(K)$ into A that coincides with $h_{\mathfrak{P}_j}$ on $\mathrm{Gal}(\hat{K}_{\mathfrak{P}_j})$ for $j = 0, 1, \ldots, s$.

By (6b), $\mathrm{Ram}(N/K) \subseteq \{\mathfrak{P}_0, \mathfrak{P}_1, \ldots, \mathfrak{P}_s\}$. Since $\mathrm{Gal}(N) \le \mathrm{Ker}(h)$, this implies that $h(\hat{I}_{\mathfrak{P}}) = \mathbf{1}_A$ for each $\mathfrak{P} \in \mathbb{P}(K) \smallsetminus \{\mathfrak{P}_0, \mathfrak{P}_1, \ldots, \mathfrak{P}_s\}$, as (b) claims. ∎

Remark 1.3: Lemma 15.3 of [JaR18] is trivial if $l = p$. Indeed, in this case the group $\mu_{\mathfrak{p}}$ of roots of unity of order p is trivial, so $A' = \mathrm{Hom}(A, \mu_p) = \mathbf{1}$. However, the proof of [JaR18, Lemma 15.5] breaks down, because $H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A') = \mathbf{1}$, so that group cannot be dual to $H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$ as needed in Part C of that proof. ∎ CHAd input, 264

2. A Choice of an Element in the First Cohomology Group CRS input, 14

Let K be a finite Galois extension of our basic global field K_0 of positive characteristic p. We prove an improved version of Proposition 9.3 of [JaR18]. To this end we need to introduce notation and results from [JaR18] for our context.

Setup 2.1: Completions. We denote the set of all primes of K_0 (resp. K) by $\mathbb{P}(K_0)$ (resp. $\mathbb{P}(K)$). For each $\mathfrak{p} \in \mathbb{P}(K_0)$ we fix a completion $\hat{K}_{0,\mathfrak{p}}$ of K_0 at $\mathfrak{p}$ and fix a separable algebraic closure $\hat{K}_{0,\mathfrak{p},\mathrm{sep}}$ of $\hat{K}_{0,\mathfrak{p}}$ that contains $K_{0,\mathrm{sep}}$. Then, $K_{0,\mathfrak{p}} = K_{0,\mathrm{sep}} \cap \hat{K}_{0,\mathfrak{p}}$ is a Henselian closure of K_0 at $\mathfrak{p}$. By Krasner's Lemma, $K_{0,\mathrm{sep}}\hat{K}_{0,\mathfrak{p}} = \hat{K}_{0,\mathfrak{p},\mathrm{sep}}$. Hence, we may identify $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$ with $\mathrm{Gal}(K_{0,\mathfrak{p}})$ via restriction. CRSa input, 22

Next let x be a primitive element of K/K_0 and set $f = \mathrm{irr}(x, K_0)$. Then, there is a decomposition $f(X) = \prod_{\mathfrak{P}|\mathfrak{p}} f_{\mathfrak{P}}(X)$ of $f(X)$ into irreducible polynomials over $\hat{K}_{0,\mathfrak{p}}$, where $\mathfrak{P}$ ranges over all prime divisors of K that lie over $\mathfrak{p}$. For each such $\mathfrak{P}$ we choose a root $x_{\mathfrak{P}}$ of $f_{\mathfrak{P}}$ in K and set $\hat{K}_{\mathfrak{P}} = \hat{K}_{0,\mathfrak{p}}(x_{\mathfrak{P}})$. Then, the map $x \mapsto x_{\mathfrak{P}}$ extends to a K-automorphism $\lambda_{\mathfrak{P}}$ that extends, with the same name, to an embedding $\lambda_{\mathfrak{P}}\colon K_{0,\mathrm{sep}} \to \hat{K}_{0,\mathfrak{p},\mathrm{sep}}$ that leaves K_0 invariant. We denote the fixed field of $\lambda_{\mathfrak{P}}^{-1}(\mathrm{Gal}(\hat{K}_{\mathfrak{P}}))$ in $K_{0,\mathrm{sep}}$ by $K_{\mathfrak{P}}$. It is a Henselian closure of K at $\mathfrak{P}$ (that does not necessarily contain $K_{0,\mathfrak{p}}$). ∎

Setup 2.2: Commutative diagram. Let A be a (multiplicative) $\mathrm{Gal}(K_0)$-module with right action. Following Setup 2.1, we consider the following diagram: CRSb input, 56

(1)
$$\begin{tikzcd} H^1(\mathrm{Gal}(K), A) \arrow[r, "\mathrm{Res}"] \arrow[d, "\mathrm{cor}"'] & \prod_{\mathfrak{P}|\mathfrak{p}} H^1(\mathrm{Gal}(\hat{K}_{\mathfrak{P}}), A) \arrow[d, "\mathrm{Cor}"] \\ H^1(\mathrm{Gal}(K_0), A) \arrow[r, "\mathrm{res}_{\mathfrak{p}}"] & H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A). \end{tikzcd}$$

In this diagram

(2a) the identification of $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$ with the subgroup $\mathrm{Gal}(K_{0,\mathfrak{p}})$ of $\mathrm{Gal}(K_0)$ also makes A a $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$-module,

(2b) for each prime $\mathfrak{P}$ of K over $\mathfrak{p}$, we let $\mathrm{Gal}(\hat{K}_{\mathfrak{P}})$ act on A by the rule $a^\tau = a^{\lambda_{\mathfrak{P}}^{-1}(\tau)}$ for $a \in A$ and $\tau \in \mathrm{Gal}(\hat{K}_{\mathfrak{P}})$, in particular, if $\mathrm{Gal}(K)$ acts trivially on A, then so does $\mathrm{Gal}(K_{\mathfrak{P}})$ and therefore also $\mathrm{Gal}(\hat{K}_{\mathfrak{P}})$, (2c) the map cor: $H^1(\mathrm{Gal}(K), A) \to H^1(\mathrm{Gal}(K_0), A)$ is the corestriction map for the open subgroup $\mathrm{Gal}(K)$ of $\mathrm{Gal}(K_0)$,

(2d) the map $\mathrm{res}_{\mathfrak{p}}$: $H^1(\mathrm{Gal}(K_0), A) \to H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$ is the restriction map for the closed subgroup $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$ of $\mathrm{Gal}(K_0)$,

(2e) the map Res is an abbreviation for the system of maps $(\mathrm{res}_{\mathfrak{P}})_{\mathfrak{P}|\mathfrak{p}}$, where for each $\mathfrak{P}|\mathfrak{p}$ the map $\mathrm{res}_{\mathfrak{P}}$: $H^1(\mathrm{Gal}(K), A) \to H^1(\mathrm{Gal}(\hat{K}_{\mathfrak{P}}), A)$ is defined for each homogeneous cochain η: $\mathrm{Gal}(K)^2 \to A$ by $\mathrm{res}_{\mathfrak{P}}(\eta) = \eta_{\mathfrak{P}}$, where $\eta_{\mathfrak{P}}(\sigma_0, \sigma_1) = \eta(\sigma_0^{\lambda_{\mathfrak{P}}^{-1}}, \sigma_1^{\lambda_{\mathfrak{P}}^{-1}})$ for $\sigma_1, \sigma_1 \in \mathrm{Gal}(\hat{K}_{\mathfrak{P}})$,

(2f) the map Cor is defined for each tuple $(h_{\mathfrak{P}})_{\mathfrak{P}|\mathfrak{p}} \in \prod_{\mathfrak{P}|\mathfrak{p}} H^1(\mathrm{Gal}(\hat{K}_{\mathfrak{P}}), A)$ by

$$\mathrm{Cor}((h_{\mathfrak{P}})_{\mathfrak{P}|\mathfrak{p}}) = \prod_{\mathfrak{P}|\mathfrak{p}} \mathrm{cor}_{\mathfrak{P}}(h_{\mathfrak{P}}),$$

where for each $\mathfrak{P}|\mathfrak{p}$ the map $\mathrm{cor}_{\mathfrak{P}}$: $H^1(\mathrm{Gal}(\hat{K}_{\mathfrak{P}}), A) \to H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$ is the corestriction map for the open subgroup $\mathrm{Gal}(\hat{K}_{\mathfrak{P}})$ of $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$.

By [JaR18, Lemma 5.3], (1) is a commutative diagram. ∎

Remark 2.3: For a subset V of $\mathbb{P}(K_0)$ we denote the maximal Galois extension of [CHAe input, 124] K_0 which is unramified away from V by $K_{0,V}$. In other words, $K_{0,V}$ is the maximal Galois extension of K in which only primes $\mathfrak{p} \in V$ are ramified. Thus, if K is a finite Galois extension of K_0 and $\mathrm{Ram}(K/K_0) \subseteq V$, then $K \subseteq K_{0,V}$. It follows that if A is a multiplicative $\mathrm{Gal}(K/K_0)$-module, then the action of $\mathrm{Gal}(K/K_0)$ on A can be naturally lifted to an action of $\mathrm{Gal}(K_{0,V}/K_0)$ on A through the restriction map res: $\mathrm{Gal}(K_{0,V}/K_0) \to \mathrm{Gal}(K/K_0)$.

Let A be a (multiplicative) $\mathrm{Gal}(K_{0,V}/K_0)$-module. For each $\mathfrak{p} \in \mathbb{P}(K_0)$ we embed $K_{0,\mathrm{sep}}$ into $\hat{K}_{0,\mathfrak{p},\mathrm{sep}}$. Then, res: $\mathrm{Gal}(K_{0,V}\hat{K}_{0,\mathfrak{p}}/\hat{K}_{0,\mathfrak{p}}) \to \mathrm{Gal}(K_{0,V}/K_{0,V} \cap \hat{K}_{0,\mathfrak{p}})$ is an isomorphism that we use to identify the two groups. We write $\bar{\sigma}$ for the restriction of an element $\sigma \in \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$ to $K_{0,V}\hat{K}_{0,\mathfrak{p}}$. Then, for each $a \in A$, we define $a^\sigma = a^{\bar{\sigma}}$. This defines A also as a $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$-module.

Next we consider an element $x \in H^1(\mathrm{Gal}(K_{0,V}/K_0), A)$ and choose a crossed homomorphism χ: $\mathrm{Gal}(K_{0,V}/K_0) \to A$ that represents x. Then, we denote the compositum of the maps

$$\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}) \to \mathrm{Gal}(K_{0,V}\hat{K}_{0,\mathfrak{p}}/\hat{K}_{0,\mathfrak{p}}) \to \mathrm{Gal}(K_{0,V}/K_{0,V} \cap \hat{K}_{0,\mathfrak{p}}) \to \mathrm{Gal}(K_{0,V}/K_0) \xrightarrow{\chi} A,$$

where the first two maps are the corresponding restriction maps and the third is the inclusion map, by $\chi_{\mathfrak{p}}$.

The map $\chi \to \chi_{\mathfrak{p}}$ is compatible with the actions of $\mathrm{Gal}(K_{0,V}/K_0)$ and $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$ on A, so $\chi_{\mathfrak{p}}$ is a crossed homomorphism. We denote the cohomology class of $\chi_{\mathfrak{p}}$ by $\mathrm{res}_{\mathfrak{p}}(x)$. Then, $\mathrm{res}_{\mathfrak{p}}$: $H^1(\mathrm{Gal}(K_{0,V}/K_0), A) \to H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$ is a natural homomorphism. ∎

Definition 2.4: Let $\mathfrak{p}$ be a prime of K_0 and let h: $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}) \to A$ be a homomorphism of groups. We say that h is **unramified** if $h(\hat{I}_\mathfrak{p}) = \mathbf{1}$. We say that a homomorphism h: $\mathrm{Gal}(K_0) \to A$ is **unramified at** $\mathfrak{p}$ if $h|_{\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})}$ is unramified. This is the case if and only if $\mathfrak{p}$ is unramified in the fixed field of $\mathrm{Ker}(h)$ in $K_{0,\mathrm{sep}}$. RAMi input, 178

Now let A be a finite $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}})$-module and let $x \in H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$. We say that x is **unramified** if $\chi(\hat{I}_\mathfrak{p}) = \mathbf{1}$ for each (alternatively, for one) crossed homomorphism χ: $\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}) \to A$ that represents x. ∎

LEMMA 2.5: *Let K_0 be a global field of positive characteristic p, K a finite Galois extension of K_0, r a positive integer, $C_{p,1}, \ldots, C_{p,r}$ isomorphic copies of C_p, and $A = C_{p,1} \times \cdots \times C_{p,r}$ a simple* $\mathrm{Gal}(K/K_0)$*-module. Let T be a finite set of primes of K_0 that contains* $\mathrm{Ram}(K/K_0)$*. For each $\mathfrak{p} \in T$ consider $y_\mathfrak{p} \in H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$. Let $\mathfrak{q}$ be a prime in $\mathbb{P}(K_0) \smallsetminus T$. Then, there exists $z \in H^1(\mathrm{Gal}(K_0), A)$ such that* CHAf input, 196

(a) $\mathrm{res}_\mathfrak{p}(z) = y_\mathfrak{p}$ *for each* $\mathfrak{p} \in T$ *and*

(b) $\mathrm{res}_\mathfrak{p}(z)$ *is unramified for each* $\mathfrak{p} \in \mathbb{P}(K_0) \smallsetminus (T \cup \{\mathfrak{q}\})$.

Proof: We set $T' = T \cup \{\mathfrak{q}\}$ and let $K_{0,T'}$ be the maximal Galois extension of K_0 which is unramified away from T'. Since $\mathrm{Ram}(K/K_0) \subseteq T \subset T'$, Remark 2.3 implies that $K \subseteq K_{0,T'}$. Hence, again by Remark 2.3, A can be considered also as an $\mathrm{Gal}(K_{0,T'}/K_0)$-module. By [NSW15, Thm. 9.2.5] applied to T and T' rather than to T and S, there exists $y \in H^1(\mathrm{Gal}(K_{0,T'}/K_0), A)$ such that $\mathrm{res}_\mathfrak{p}(y) = y_\mathfrak{p}$ for each $\mathfrak{p} \in T$. Let $\inf: H^1(\mathrm{Gal}(K_{0,T'}/K_0), A) \to H^1(\mathrm{Gal}(K_0), A)$ be the inflation map and set $z = \inf(y) \in H^1(\mathrm{Gal}(K_0), A)$. Then, by Remark 2.3, $\mathrm{res}_\mathfrak{p}(z) = \mathrm{res}_\mathfrak{p}(y) = y_\mathfrak{p}$ for each $\mathfrak{p} \in T$.

If $\mathfrak{p} \in \mathbb{P}(K_0) \smallsetminus T'$, then $\mathfrak{p}$ is unramified in $K_{0,T'}$, so the inertia subgroup $I_\mathfrak{p}$ of $\mathrm{Gal}(K_\mathfrak{p})$ is contained in $\mathrm{Gal}(K_{0,T'})$. Let χ: $\mathrm{Gal}(K_{0,T'}/K_0) \to A$ be a crossed homomorphism that represents y. Then, $\psi = \chi \circ \mathrm{res}_{K_{0,\mathrm{sep}}/K_{0,T'}}$ is a crossed homomorphism that represents z. Hence, for each $\sigma \in I_\mathfrak{p}$ and with $\bar{\sigma}$ being the restriction of σ to $K_{0,T'}$ we have $\psi(\sigma) = \chi(\bar{\sigma}) = \chi(1) = 1$. Hence, z is unramified at $\mathfrak{p}$, as desired. ∎

LEMMA 2.6: *Let K_0 be a global field of positive characteristic p, K a finite Galois extension of K_0, and L a finite Galois extension of K_0 that contains K such that L/K is an abelian p-extension. Let r be a positive integer and $A = C_p^r$ a simple* $\mathrm{Gal}(K/K_0)$*-module. Let n be a positive integer such that $p \nmid n$. Let T be a finite subset of $\mathbb{P}(K_0)$ that contains* $\mathrm{Ram}(K/K_0)$*. For each $\mathfrak{p} \in T$, let $y_\mathfrak{p} \in H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$.* CHAg input, 251

Then, there exist a prime $\mathfrak{q} \in \mathbb{P}(K_0) \smallsetminus T$ and an element $x \in H^1(\mathrm{Gal}(K_0), A)$ such that

(a) *for each $\mathfrak{p} \in T$ we have* $\mathrm{res}_\mathfrak{p}(x) = y_\mathfrak{p}$,

(b) *for each $\mathfrak{p} \in \mathbb{P}(K_0) \smallsetminus (T \cup \{\mathfrak{q}\})$ the element* $\mathrm{res}_\mathfrak{p}(x)$ *of* $H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$ *is unramified, and*

(c) $\mathfrak{q}$ *totally splits in* $L(\zeta_n)$ *and* $\mathrm{res}_{\mathfrak{q}}(x)\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{q}}) \to A$ *is a homomorphism whose image is contained in a subgroup of* A *which is isomorphic to* C_p.

(d) *Moreover, let* G *and* $\bar{G}$ *be finite groups such that* $A \le \bar{G}$ *and let* $\lambda\colon G \to \bar{G}$ *be an epimorphism. Then, there exists a homomorphism* $x'_{\mathfrak{q}}\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{q}}) \to G$ *such that* $\lambda \circ x'_{\mathfrak{q}} = \mathrm{res}_{\mathfrak{q}}(x)$.

Proof: We lift the action of $\mathrm{Gal}(K/K_0)$ on A to an action of $\mathrm{Gal}(K_0)$ on A with trivial action of $\mathrm{Gal}(K)$. Then, we use the Chebotarev density theorem to choose a prime $\mathfrak{p}_0 \in \mathbb{P}(K_0) \smallsetminus T$ that totally splits in $L(\zeta_n)$ and use Lemma 2.5 to choose an element $z \in H^1(\mathrm{Gal}(K_0), A)$ such that

(3a) $\mathrm{res}_{\mathfrak{p}}(z) = y_{\mathfrak{p}}$ for each $\mathfrak{p} \in T$ and

(3b) $\mathrm{res}_{\mathfrak{p}}(z)$ is unramified for each $\mathfrak{p} \in \mathbb{P}(K_0) \smallsetminus (T \cup \{\mathfrak{p}_0\})$.

The rest of the proof breaks up into four parts.

Part A: *Definition of* $\eta_{\mathfrak{p}}$. For each $\mathfrak{p} \in T \cup \{\mathfrak{p}_0\}$ let $\eta_{\mathfrak{p}} \in H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$ be defined as follows:

(4) $\eta_{\mathfrak{p}} = 1$ for $\mathfrak{p} \in T$ and $\eta_{\mathfrak{p}_0} = \mathrm{res}_{\mathfrak{p}_0}(z)^{-1}$.

Claim: For each $\mathfrak{p} \in T \cup \{\mathfrak{p}_0\}$, the element $\eta_{\mathfrak{p}}$ lies in the image of the map

$$\text{(5)} \qquad \mathrm{Cor}\colon \prod_{\mathfrak{P}|\mathfrak{p}} H^1(\mathrm{Gal}(\hat{K}_{\mathfrak{P}}), A) \to H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A).$$

Indeed, the claim holds for $\mathfrak{p} \in T$, because by (4), $\eta_{\mathfrak{p}} = 1$ for $\mathfrak{p} \in T$ and $\mathrm{Cor} = \prod_{\mathfrak{P}|\mathfrak{p}} \mathrm{cor}_{\mathfrak{P}}$ is a homomorphism of groups. Since $\mathfrak{p}_0$ totally splits in $L(\zeta_n)$, it totally splits in K. Hence, by [JaR18, Lemma 6.2], Cor is surjective. In particular, $\eta_{\mathfrak{p}}$ lies in the image of Cor, as claimed.

Part B: *Shifting the* $\eta_{\mathfrak{p}}$*'s.* We use Part A to choose for each $\mathfrak{p} \in T \cup \{\mathfrak{p}_0\}$ and for every $\mathfrak{P} \in \mathbb{P}(K)$ over $\mathfrak{p}$ an element $\tilde{\eta}_{\mathfrak{P}} \in H^1(\mathrm{Gal}(\hat{K}_{\mathfrak{P}}), A)$ such that

$$\text{(6)} \qquad \eta_{\mathfrak{p}} = \prod_{\mathfrak{P}|\mathfrak{p}} \mathrm{cor}_{\mathfrak{P}}(\tilde{\eta}_{\mathfrak{P}}).$$

Since $\mathrm{Gal}(K)$ acts trivially on A, the group $\mathrm{Gal}(\hat{K}_{\mathfrak{P}})$ acts trivially on A (by (2b)), hence $\tilde{\eta}_{\mathfrak{P}}\colon \mathrm{Gal}(\hat{K}_{\mathfrak{P}}) \to A$ is a homomorphism for each $\mathfrak{P}|\mathfrak{p}$ [JaR18, Subsection 6.1]. Likewise,

(7) $z' = z|_{\mathrm{Gal}(K)}\colon \mathrm{Gal}(K) \to A$ is a homomorphism.

Let L' be the fixed field of $\mathrm{Ker}(z')$ in K_{sep}. Then, L' is a finite abelian p-extension of K, hence so is LL'. Hence, by [JaR18, Remark 4.1], the Galois closure L'' of LL' over K_0 is also a finite abelian p-extension of K. We use the Chebotarev density theorem to choose $\mathfrak{q} \in \mathbb{P}(K_0) \smallsetminus (T \cup \{\mathfrak{p}_0\})$ that totally splits in $L''(\zeta_n)$. Let $\mathfrak{Q}$ be the prime of K that lies over $\mathfrak{q}$ such that $\lambda_{\mathfrak{Q}}$ is the inclusion map [JaR18, Subsection 1.4] and let $h_{\mathfrak{Q}}\colon \mathrm{Gal}(\hat{K}_{\mathfrak{Q}}) \to A$ be a homomorphism with $h_{\mathfrak{Q}}(\mathrm{Gal}(\hat{K}_{\mathfrak{Q}})) = C_p \times \mathbf{1} \times \cdots \times \mathbf{1}$ (e.g. $h_{\mathfrak{Q}}(\sigma) = (\iota(\sigma), 1, \ldots, 1)$, where $\iota\colon \mathrm{Gal}(\hat{K}_{\mathfrak{Q}}) \to \mathrm{Gal}(N/\hat{K}_{\mathfrak{Q}})$ is the restriction map with N being the unique unramified extension of $\hat{K}_{\mathfrak{Q}}$ of degree p). For each $\mathfrak{Q}' \in \mathbb{P}(K)$ over $\mathfrak{q}$ such that $\mathfrak{Q}' \neq \mathfrak{Q}$ let $h_{\mathfrak{Q}'}\colon \mathrm{Gal}(\hat{K}_{\mathfrak{Q}'}) \to A$ be the trivial homomorphism.

Let T'_K be the non-empty set of primes of K that lie over $T \cup \{\mathfrak{p}_0\}$. By Proposition 1.2, there exists a homomorphism $h\colon \mathrm{Gal}(K) \to A$ such that

(8a) $\mathrm{res}_{\mathfrak{P}}(h) = \tilde{\eta}_{\mathfrak{P}}$ for every $\mathfrak{P} \in T'_K$,

(8b) for each $\mathfrak{Q}' \in \mathbb{P}(K)$ over $\mathfrak{q}$ we have $\mathrm{res}_{\mathfrak{Q}'}(h) = h_{\mathfrak{Q}'}$, in particular $\mathrm{res}_{\mathfrak{Q}'}(h)(\mathrm{Gal}(\hat{K}_{\mathfrak{Q}'})) \leq C_p \times \mathbf{1} \times \cdots \times \mathbf{1}$, and

(8c) $\mathrm{res}_{\mathfrak{P}}(h)(\hat{I}_{\mathfrak{P}}) = \mathbf{1}_A$ for each $\mathfrak{P} \in \mathbb{P}(K) \smallsetminus (T'_K \cup \{\mathfrak{Q}\})$.

Let $u = \mathrm{cor}(h) \in H^1(\mathrm{Gal}(K_0), A)$, where cor is the corestriction map that appears in diagram (1). By the commutativity of that diagram [JaR18, Lemma 5.3], we have for each $\mathfrak{p} \in \mathbb{P}(K_0)$ that

$$(9) \quad \mathrm{res}_{\mathfrak{p}}(u) = \mathrm{res}_{\mathfrak{p}}(\mathrm{cor}(h)) = \mathrm{Cor}(\mathrm{Res}(h)) = \mathrm{Cor}\big((\mathrm{res}_{\mathfrak{P}}(h))_{\mathfrak{P}|\mathfrak{p}}\big) = \prod_{\mathfrak{P}|\mathfrak{p}} \mathrm{cor}_{\mathfrak{P}}(\mathrm{res}_{\mathfrak{P}}(h)).$$

In particular, for $\mathfrak{p} \in T \cup \{\mathfrak{p}_0\}$ we have that

$$(9') \qquad \mathrm{res}_{\mathfrak{p}}(u) = \prod_{\mathfrak{P}|\mathfrak{p}} \mathrm{cor}_{\mathfrak{P}}(\mathrm{res}_{\mathfrak{P}}(h)) \overset{(8a)}{=} \prod_{\mathfrak{P}|\mathfrak{p}} \mathrm{cor}_{\mathfrak{P}}(\tilde{\eta}_{\mathfrak{P}}) \overset{(6)}{=} \eta_{\mathfrak{p}}.$$

Part C: *We prove that the image of* $\mathrm{res}_{\mathfrak{q}}(u)\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{q}}) \to A$ *is contained in a subgroup of* A *which is isomorphic to* C_p. Indeed, let $\mathfrak{Q}'$ be a prime of K over $\mathfrak{q}$. Since $\mathfrak{q}$ totally splits in K (by its choice), $\mathrm{Gal}(\hat{K}_{0,\mathfrak{q}}) = \mathrm{Gal}(\hat{K}_{\mathfrak{Q}'})$ [JaR18, Subsection 1.5]. For each $\hat{\sigma} \in \mathrm{Gal}(\hat{K}_{0,\mathfrak{q}})$ we observe that $\sigma = \hat{\sigma}^{\lambda_{\mathfrak{Q}'}^{-1}} \in \mathrm{Gal}(K_{\mathfrak{Q}'}) \leq \mathrm{Gal}(K)$ [JaR18, Subsection 1.4]. Since $\mathrm{Gal}(K)$ acts trivially on A, we have, by [JaR18, Convention (2b) of Section 5], that $a^{\hat{\sigma}} = a^{\sigma} = a$ for each $a \in A$. Hence, by [JaR18, Subsection 6.1], $\mathrm{res}_{\mathfrak{q}}(u)\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{q}}) \to A$ is a homomorphism.

Again, since $\mathfrak{q}$ totally splits in K, we have $\mathrm{cor}_{\mathfrak{Q}'}(\mathrm{res}_{\mathfrak{Q}'}(h)) = \mathrm{res}_{\mathfrak{Q}'}(h)$ for each $\mathfrak{Q}'|\mathfrak{q}$ [JaR18, Statement (8) of Section 5]. By (8b), $\mathrm{res}_{\mathfrak{Q}'}(h)(\mathrm{Gal}(\hat{K}_{\mathfrak{Q}'})) \leq C_p \times \mathbf{1} \times \cdots \times \mathbf{1}$. Hence, by (9) with $\mathfrak{q}$ replacing $\mathfrak{p}$, we have $\mathrm{res}_{\mathfrak{q}}(u)(\mathrm{Gal}(\hat{K}_{0,\mathfrak{q}})) \leq C_p \times \mathbf{1} \times \cdots \times \mathbf{1}$, as claimed.

Part D: *We prove that the element* $x = uz$ *of* $H^1(\mathrm{Gal}(K_0), A)$ *satisfies the conditions (a)–(d) of the lemma.*

Proof of (a): For each $\mathfrak{p} \in T$ we have that

$$\mathrm{res}_{\mathfrak{p}}(x) = \mathrm{res}_{\mathfrak{p}}(u)\mathrm{res}_{\mathfrak{p}}(z) \overset{(9'),(3a)}{=} \eta_{\mathfrak{p}} y_{\mathfrak{p}} \overset{(4)}{=} y_{\mathfrak{p}},$$

as Condition (a) claims.

Proof of (b): Let $\mathfrak{p} \in \mathbb{P}(K_0) \smallsetminus (T \cup \{\mathfrak{q}\})$. If $\mathfrak{p} = \mathfrak{p}_0$, then by (9) and (4),

$$\mathrm{res}_{\mathfrak{p}}(x) = \mathrm{res}_{\mathfrak{p}}(u)\mathrm{res}_{\mathfrak{p}}(z) \overset{(9')}{=} \eta_{\mathfrak{p}} \cdot \mathrm{res}_{\mathfrak{p}}(z) \overset{(4)}{=} \mathrm{res}_{\mathfrak{p}}(z)^{-1} \cdot \mathrm{res}_{\mathfrak{p}}(z) = 1.$$

Hence, $\mathrm{res}_{\mathfrak{p}}(x)$ is unramified [JaR18, Subsection 9.1]. If $\mathfrak{p} \neq \mathfrak{p}_0$, then by (3b), $\mathrm{res}_{\mathfrak{p}}(z)$ is unramified, so $\mathrm{res}_{\mathfrak{p}}(z)|_{\hat{I}_{\mathfrak{p}}} = 1$. Since $\mathrm{Ram}(K/K_0) \subseteq T$, we have that $\mathfrak{p}$ is unramified in K. By (8c), $\mathrm{res}_{\mathfrak{P}}(h)(\hat{I}_{\mathfrak{P}}) = \mathbf{1}_A$ for each $\mathfrak{P}|\mathfrak{p}$ and by (9) $\mathrm{res}_{\mathfrak{p}}(u) = \prod_{\mathfrak{P}|\mathfrak{p}} \mathrm{cor}_{\mathfrak{P}}(\mathrm{res}_{\mathfrak{P}}(h))$. Hence, by [JaR18, Lemma 6.3], $\mathrm{res}_{\mathfrak{p}}(u)|_{\hat{I}_{\mathfrak{p}}} = 1$, so $\mathrm{res}_{\mathfrak{p}}(x)|_{\hat{I}_{\mathfrak{p}}} = \mathrm{res}_{\mathfrak{p}}(u)|_{\hat{I}_{\mathfrak{p}}}\mathrm{res}_{\mathfrak{p}}(z)|_{\hat{I}_{\mathfrak{p}}} = 1$. Therefore, $\mathrm{res}_{\mathfrak{p}}(x)$ is unramified, as asserted by (b).

Proof of (c): Since $\mathfrak{q}$ totally splits in $L''(\zeta_n)$ (Part B), it also totally splits in $L(\zeta_n)$. By Part C, $\mathrm{res}_{\mathfrak{q}}(u)$ is a C_p-homomorphism.

Since $\mathfrak{q}$ totally splits in K, we have $\mathrm{Gal}(\hat{K}_{0,\mathfrak{q}}) = \mathrm{Gal}(\hat{K}_{\mathfrak{Q}'})$ for each prime $\mathfrak{Q}'$ of K over $\mathfrak{q}$. In particular, this is the case for $\mathfrak{Q}$. Since $\mathrm{Gal}(K)$ acts trivially on A, the group $\mathrm{Gal}(\hat{K}_{0,\mathfrak{q}})$ acts trivially on A. Hence, $\mathrm{res}_{\mathfrak{q}}(z) \in H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{q}}), A)$ is a homomorphism [JaR18, Subsection 9.1]. Moreover, with $z' = z|_{\mathrm{Gal}(K)}$ being the homomorphism introduced in (7), we have, by the choice of $\mathfrak{Q}$, that $\mathrm{res}_{\mathfrak{q}}(z) = \mathrm{res}_{\mathfrak{Q}}(z')$. Again, by the choice of $\mathfrak{q}$, the prime $\mathfrak{Q}$ totally splits in L'. Hence, $L' \subseteq \hat{K}_{\mathfrak{Q}}$. Since $\mathrm{Gal}(L') = \mathrm{Ker}(z')$ (Part B), the homomorphism $\mathrm{res}_{\mathfrak{q}}(z)\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{q}}) \to A$ is trivial. Therefore, $\mathrm{res}_{\mathfrak{q}}(x) = \mathrm{res}_{\mathfrak{q}}(u)$, so by the preceding paragraph, $\mathrm{res}_{\mathfrak{q}}(x)$ is a C_p-homomorphism, as (c) claims.

Proof of (d): If $\mathrm{res}_{\mathfrak{q}}(x)$ is the trivial homomorphism, then the trivial homomorphism $x'_{\mathfrak{q}}\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{q}}) \to G$ satisfies $\lambda \circ x'_{\mathfrak{q}} = \mathrm{res}_{\mathfrak{q}}(x)$. Otherwise, by (c), $\mathrm{Im}(\mathrm{res}_{\mathfrak{q}}(x)) = \langle \bar{g} \rangle$, where $\bar{g}$ is an element of $\bar{G}$ of order p. Since $\lambda\colon G \to \bar{G}$ is surjective, there exists $g \in G$ with $\lambda(g) = \bar{g}$. Now recall that G is finite and let $\mathrm{ord}(g) = p^k m$, where $k \geq 1$ and $p \nmid m$. In particular, $\langle \bar{g}^m \rangle = \langle \bar{g} \rangle$. Replacing g by g^m and $\bar{g}$ by $\bar{g}^m$, we may assume that $\mathrm{ord}(g) = p^k$. Let $\hat{K}_{0,\mathfrak{q}}^{(p)}$ be the maximal pro-p extension of $\hat{K}_{0,\mathfrak{q}}$. Then, there exists an epimorphism $\bar{x}_{\mathfrak{q}}\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{q}}^{(p)}/\hat{K}_{0,\mathfrak{q}}) \to \langle \bar{g} \rangle$ such that $\bar{x}_{\mathfrak{q}} \circ \mathrm{res} = \mathrm{res}_{\mathfrak{q}}(x)$, where res: $\mathrm{Gal}(\hat{K}_{0,\mathfrak{q}}) \to \mathrm{Gal}(\hat{K}_{0,\mathfrak{q}}^{(p)}/\hat{K}_{0,\mathfrak{q}})$ is the restriction map. By [Rib70, p. 257, Cor. 3.4], $\mathrm{Gal}(\hat{K}_{0,\mathfrak{q}}^{(p)}/\hat{K}_{0,\mathfrak{q}})$ is a free pro-p group. Hence, there exists an epimorphism

$\overline{x'_{\mathfrak{q}}}\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{q}}^{(p)}/\hat{K}_{0,\mathfrak{q}}) \to \langle g\rangle$ such that $\lambda|_{\langle g\rangle}\circ\overline{x'_{\mathfrak{q}}} = \bar{x}_{\mathfrak{q}}$.

$$\begin{tikzcd}
 & \mathrm{Gal}(\hat{K}_{0,\mathfrak{q}}) \arrow[d, "\mathrm{res}"] \arrow[ddl, bend right, "x'_{\mathfrak{q}}"'] \arrow[dd, bend left=60, "\mathrm{res}_{\mathfrak{q}}(x)"] \\
 & \mathrm{Gal}(\hat{K}_{0,\mathfrak{q}}^{(p)}/\hat{K}_{0,\mathfrak{q}}) \arrow[d, "\bar{x}_{\mathfrak{q}}"] \arrow[dl, "\overline{x'_{\mathfrak{q}}}"'] \\
\langle g\rangle \arrow[r, "\lambda|_{\langle g\rangle}"] \arrow[d] & \langle\bar{g}\rangle \arrow[d] \\
G \arrow[r, "\lambda"] & \bar{G}
\end{tikzcd}$$

Hence, the epimorphism $x'_{\mathfrak{q}} = \overline{x'_{\mathfrak{q}}}\circ\mathrm{res}$ satisfies $\lambda|_{\langle g\rangle}\circ x'_{\mathfrak{q}} = \mathrm{res}_{\mathfrak{q}}(x)$. ∎

3. The Case $l \neq p$

TAME input, 13

Let l be a prime number, r a positive integer, and $h\colon G \to A$ a homomorphism of groups. We say that h is an **l-homomorphism** if $\mathrm{Im}(h)$ is contained in a subgroup of A which is isomorphic to C_l.

The following lemma replaces [JaR18, Lemma 8.3] for homomorphisms λ with kernel whose order is a multiple of p.

LEMMA 3.1: *Let K_0 be a global field of positive characteristic p. Let $\lambda\colon G \to \bar{G}$ be an epimorphism of finite groups. Let $l \neq p$ be a prime number. Set $|\mathrm{Ker}(\lambda)| = ep^s$ and let n be a multiple of el with $p \nmid n$. Consider $\mathfrak{p} \in \mathbb{P}(K_0)$ such that $\zeta_n \in \hat{K}_{0,\mathfrak{p}}$. Let $\bar{\psi}_{\mathfrak{p}}\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}) \to \bar{G}$ be a ramified C_l-homomorphism (thus, $\bar{\psi}_{\mathfrak{p}}(\hat{I}_{\mathfrak{p}}) \neq \mathbf{1}$). Then, there exists a homomorphism $\psi_{\mathfrak{p}}\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}) \to G$ such that $\lambda\circ\psi_{\mathfrak{p}} = \bar{\psi}_{\mathfrak{p}}$.* TAMe input, 23

Proof: Let $N_{\mathfrak{p}}$ be the fixed field of $\mathrm{Ker}(\bar{\psi}_{\mathfrak{p}})$ in $\hat{K}_{0,\mathfrak{p},\mathrm{sep}}$. Since $\mathrm{Im}(\bar{\psi}_{\mathfrak{p}}) \leq C_l$ and $\bar{\psi}_{\mathfrak{p}}(\hat{I}_{\mathfrak{p}}) \neq \mathbf{1}$, we have $\mathrm{Im}(\bar{\psi}_{\mathfrak{p}}) = C_l$. Hence, $N_{\mathfrak{p}}/\hat{K}_{0,\mathfrak{p}}$ is a ramified C_l-extension and we identify $\mathrm{Gal}(N_{\mathfrak{p}}/\hat{K}_{0,\mathfrak{p}})$ with $\mathrm{Im}(\bar{\psi}_{\mathfrak{p}})$. Since $l \neq p$, the ramification of $N_{\mathfrak{p}}/\hat{K}_{0,\mathfrak{p}}$ is tame. Since $\zeta_n \in \hat{K}_{0,\mathfrak{p}}$ and $l|n$, we have $\zeta_l \in \hat{K}_{0,\mathfrak{p}}$. By [CaF67, p. 32, Prop. 1(i)], there exists a prime element π of $\hat{K}_{0,\mathfrak{p}}$ with $N_{\mathfrak{p}} = \hat{K}_{0,\mathfrak{p}}(\sqrt[l]{\pi})$. Let $\bar{\sigma}$ be a generator of $\mathrm{Gal}(N_{\mathfrak{p}}/\hat{K}_{0,\mathfrak{p}})$ and choose $\sigma \in G$ with $\lambda(\sigma) = \bar{\sigma}$.

Replacing σ by σ^m for an appropriate positive integer m with $l \nmid m$, we may assume that $d = \mathrm{ord}(\sigma) = l^i$ for some positive integer i. Let $\lambda' = \lambda|_{\langle\sigma\rangle}$. Then, $l^{i-1} = |\mathrm{Ker}(\lambda')|$ divides $|\mathrm{Ker}(\lambda)| = ep^s$. Since $l \neq p$, we have that $l^{i-1}|e$, so l^i divides el which divides n. Since $\zeta_n \in \hat{K}_{0,\mathfrak{p}}$, we have $\zeta_{l^i} \in \hat{K}_{0,\mathfrak{p}}$. Thus, $N'_{\mathfrak{p}} = \hat{K}_{0,\mathfrak{p}}(\sqrt[l^i]{\pi})$ is a (tamely and totally ramified) cyclic extension of $\hat{K}_{0,\mathfrak{p}}$ of degree l^i that contains $N_{\mathfrak{p}}$. Since $N_{\mathfrak{p}}$ is the fixed field of $\mathrm{Ker}(\bar\psi_{\mathfrak p})$, there exists an epimorphism $\bar\varphi_{\mathfrak p}\colon \mathrm{Gal}(N'_{\mathfrak p}/\hat K_{0,\mathfrak p}) \to \mathrm{Gal}(N_{\mathfrak p}/\hat K_{0,\mathfrak p})$ such that $\bar\psi_{\mathfrak p} = \bar\varphi_{\mathfrak p} \circ \mathrm{res}_{\hat K_{0,\mathfrak p,\mathrm{sep}}/N'_{\mathfrak p}}$.

Finally, we choose a generator τ of $\mathrm{Gal}(N'_{\mathfrak p}/\hat K_{0,\mathfrak p})$ such that $\bar\varphi_{\mathfrak p}(\tau) = \bar\sigma$ and define a homomorphism $h\colon \mathrm{Gal}(N'_{\mathfrak p}/\hat K_{0,\mathfrak p}) \to G$ by setting $h(\tau) = \sigma$. Then, the homomorphism $\psi_{\mathfrak p} = h \circ \mathrm{res}_{\hat K_{0,\mathfrak p,\mathrm{sep}}/N'_{\mathfrak p}}$ satisfies $\lambda \circ \psi_{\mathfrak p} = \bar\psi_{\mathfrak p}$,

$$\begin{tikzcd}
& \mathrm{Gal}(\hat K_{0,\mathfrak p}) \arrow[dl, "\mathrm{res}"'] \arrow[d, "\mathrm{res}"] \arrow[dd, bend left, "\bar\psi_{\mathfrak p}"] \\
\mathrm{Gal}(N'_{\mathfrak p}/\hat K_{0,\mathfrak p}) \arrow[r, "\bar\varphi_{\mathfrak p}"] \arrow[d, "h"'] & \mathrm{Gal}(N_{\mathfrak p}/\hat K_{0,\mathfrak p}) \arrow[d] \\
G \arrow[r, "\lambda"] & \bar G
\end{tikzcd}$$

as desired. ∎

Following [JaR18, Subsection 1.6], we fix a finite subset $S_0(K)$ of $\mathbb{P}(K)$ such that $I_K = I_{K,S}K^\times$ and $C_K = I_{K,S}/K_S$ for each finite subset S of $\mathbb{P}(K)$ that contains $S_0(K)$. Here, I_K is the idele group of K, C_K is the idele class group of K, $I_{K,S}$ is the group of S-ideles of K, and K_S is the group of S-units in K. We call $S_0(K)$ the **basic set** of K. Its existence follows from [Neu99, Prop. VI.1.4].

LEMMA 3.2: *Let K_0 be a global field of positive characteristic p, K a finite Galois extension of K_0, and L a finite Galois extension of K_0 that contains K such that L/K is an abelian l-extension with $l \neq p$ and $\zeta_l \notin K$. Let r be a positive integer and $A = C_l^r$ a simple $\mathrm{Gal}(K/K_0)$-module. Let n be a positive integer such that $l|n$ and $p \nmid n$. Let T be a finite subset of $\mathbb{P}(K_0)$ that contains $\mathrm{Ram}(K/K_0)$ and $S_0(K) \subseteq T_K$. For each $\mathfrak p \in T$, let $y_{\mathfrak p} \in H^1(\mathrm{Gal}(\hat K_{0,\mathfrak p}), A)$.* TAMb input, 120

Then, there exist distinct primes $\mathfrak q_1, \ldots, \mathfrak q_r \in \mathbb{P}(K_0) \smallsetminus T$ and an element $x \in H^1(\mathrm{Gal}(K_0), A)$ such that

(a) *for each $\mathfrak p \in T$ we have $\mathrm{res}_{\mathfrak p}(x) = y_{\mathfrak p}$,*
(b) *for each $\mathfrak p \in \mathbb{P}(K_0) \smallsetminus (T \cup \{\mathfrak q_1, \ldots, \mathfrak q_r\})$ the element $\mathrm{res}_{\mathfrak p}(x)$ of $H^1(\mathrm{Gal}(\hat K_{0,\mathfrak p}), A)$ is unramified, and*
(c) *for $i = 1, \ldots, r$ the prime $\mathfrak q_i$ totally splits in $L(\zeta_n)$ and $\mathrm{res}_{\mathfrak q_i}(x)\colon \mathrm{Gal}(\hat K_{0,\mathfrak q_i}) \to A$ is a C_l-homomorphism.*
(d) *Moreover, let G and $\bar G$ be finite groups such that $A \leq \bar G$ and let $\lambda\colon G \to \bar G$ be an epimorphism. Suppose that $|\mathrm{Ker}(\lambda)| = ep^s$ such that $el|n$ but $p \nmid en$. Then, for $i = 1, \ldots, r$ there exists a homomorphism $x'_{\mathfrak q_i}\colon \mathrm{Gal}(\hat K_{0,\mathfrak q_i}) \to G$ such that $\lambda \circ x'_{\mathfrak q_i} = \mathrm{res}_{\mathfrak q_i}(x)$.*

Proof: Proposition 9.3 of [JaR18] provides the lemma short of Conclusion (d). That conclusion holds by Lemma 3.1 if $\mathrm{res}_{\mathfrak{q}_i}(x)$ is ramified and by [JaR18, Lemma 8.1] if $\mathrm{res}_{\mathfrak{q}_i}(x)$ is unramified. ∎

4. A Proper Solution of an Embedding Problem with Bounded Ramification ELEMp input, 12

Proposition 4.5 below certifies the existence of a proper solution to each finite embedding problem over K_0 with local data whose kernel is a simple $\mathrm{Gal}(K_0)$-module $A = C_l^r$, where l is a prime number. Moreover, the number of the new primes of K_0 that ramify in the solution field is r if $l \neq p$ and 1 if $l = p$.

Setup 4.1: Let K_0 be a global field of positive characteristic p and let l be a prime number. We consider a finite Galois extension K of K_0 and an embedding problem CHAi input, 21

$$(1) \qquad (\rho\colon \mathrm{Gal}(K_0) \to \Gamma,\ \bar{\alpha}\colon \bar{G} \to \Gamma),$$

where $\Gamma = \mathrm{Gal}(K/K_0)$, $\bar{G}$ is a finite group, $\bar{\alpha}$ is an epimorphism, and $\rho = \mathrm{res}_{K_{0,\mathrm{sep}}/K}$. Let r be a positive integer, $A = \mathrm{Ker}(\bar{\alpha})$.

We assume that $\zeta_l \notin K$ and note that this assumption is automatically satisfied if $l = p$. We also assume that $A = C_l^r$ and the action of Γ on A defined by $a^{\bar{\alpha}(\bar{g})} = \bar{g}^{-1}a\bar{g}$ makes A a simple (multiplicative) Γ-module. We lift the action of Γ on A via ρ to an action of $\mathrm{Gal}(K_0)$ on A. Then, A is a simple $\mathrm{Gal}(K_0)$-module on which $\mathrm{Gal}(K)$ trivially acts.

Finally, we denote the finite group of roots of unity in K by $\mu(K)$. ∎

Remark 4.2: Equivalent classes of homomorphisms. We say that two homomorphisms $\psi, \psi'\colon \mathrm{Gal}(K_0) \to \bar{G}$ that satisfy $\bar{\alpha}\circ\psi = \rho = \bar{\alpha}\circ\psi'$ are **A-equivalent** if there exists $a \in A$ such that $\psi'(\sigma) = a^{-1}\psi(\sigma)a$ for each $\sigma \in \mathrm{Gal}(K_0)$. We denote the equivalence class of ψ by $[\psi]$. Then, we denote the set of all equivalence classes by $\mathcal{H}om_{\Gamma,\rho,\bar{\alpha}}(\mathrm{Gal}(K_0), \bar{G})$. Observe that if $[\psi'] = [\psi]$ and ψ is surjective (resp. unramified, totally split, trivial), then so is ψ' [JaR18, Subsection 7.4]. We therefore say that an equivalence class of $\mathcal{H}om_{\Gamma,\rho,\bar{\alpha}}(\mathrm{Gal}(K_0), \bar{G})$ **totally splits**, is **unramified at $\mathfrak{p}$**, **surjective**, or **trivial** if one (alternatively, every) representative of that class has the corresponding property. We denote the subset of all $[\psi] \in \mathcal{H}om_{\Gamma,\rho,\bar{\alpha}}(\mathrm{Gal}(K_0), \bar{G})$ with $[\psi]$ surjective by $\mathcal{H}om_{\Gamma,\rho,\bar{\alpha}}(\mathrm{Gal}(K_0), \bar{G})_{\mathrm{sur}}$. Finally, there is a natural action of $H^1(\mathrm{Gal}(K_0), A)$ on $\mathcal{H}om_{\Gamma,\rho,\bar{\alpha}}(\mathrm{Gal}(K_0), \bar{G})$. If $x \in H^1(\mathrm{Gal}(K_0), A)$, $\chi\colon \mathrm{Gal}(K_0) \to A$ is a crossed homomorphism that represents x, and $[\psi] \in \mathcal{H}om_{\Gamma,\rho,\bar{\alpha}}(\mathrm{Gal}(K_0), \bar{G})$, then $[\psi]^x = [\psi \cdot \chi]$ [JaR18, Subsection 10.3]. Moreover, by [JaR18, Lemma 10.4], $\mathcal{H}om_{\Gamma,\rho,\bar{\alpha}}(\mathrm{Gal}(K_0), \bar{G})$ becomes a principal homogeneous space over $H^1(\mathrm{Gal}(K_0), A)$ under this action. EQUi input, 54

Similarly, $\mathcal{H}om_{\Gamma,\rho_\mathfrak{p},\bar\alpha}(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}),\bar G)$ is the set of all equivalence classes $[\psi_\mathfrak{p}]$ of homomorphisms $\psi_\mathfrak{p}\colon \mathrm{Gal}(\hat K_{0,\mathfrak{p}})\to\bar G$ that satisfy $\bar\alpha\circ\psi_\mathfrak{p}=\rho_\mathfrak{p}$, where $\rho_\mathfrak{p}=\rho|_{\mathrm{Gal}(\hat K_{0,\mathfrak{p}})}$.
$\blacksquare$

The proof of the following lemma is a verbatim repetition of the proof of [JaR18, Lemma 11.2]. In particular note that the assumption $l\neq\mathrm{char}(K_0)$ that appears in [JaR18, Setup 11.1] is not used in the proof of [JaR18, Lemma 11.2].

LEMMA 4.3: *Under Setup 4.1, let n be a positive integer with $\gcd(n,|\mu(K)|)=1$ and $p\nmid n$, let m be the minimal number of generators of $\mathrm{Gal}(K(\zeta_n)/K)$, and let T be a finite set of primes of K_0.* CHAj input, 108

Then, there exist distinct primes $\mathfrak{p}_1,\dots,\mathfrak{p}_m,\mathfrak{q}_0\in\mathbb{P}(K_0)\smallsetminus T$ that totally split in K such that for each $\mathfrak{p}\in\{\mathfrak{p}_1,\dots,\mathfrak{p}_m,\mathfrak{q}_0\}$ there exist $[\varphi_\mathfrak{p}]\in\mathcal{H}om_{\Gamma,\rho_\mathfrak{p},\bar\alpha}(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}),\bar G)$ with the following property: if an element $[\bar\psi]\in\mathcal{H}om_{\Gamma,\rho,\bar\alpha}(\mathrm{Gal}(K_0),\bar G)$ satisfies $[\bar\psi_\mathfrak{p}]=[\varphi_\mathfrak{p}]$ in $\mathcal{H}om_{\Gamma,\rho_\mathfrak{p},\bar\alpha}(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}),\bar G)$ for each $\mathfrak{p}\in\{\mathfrak{p}_1,\dots,\mathfrak{p}_m,\mathfrak{q}_0\}$, then
(a) *$[\bar\psi]$ is unramified at $\mathfrak{p}_1,\dots,\mathfrak{p}_m,\mathfrak{q}_0$,*
(b) *if $\bar N$ is the fixed field of $\mathrm{Ker}(\bar\psi)$ in $K_{0,\mathrm{sep}}$, then $\gcd(n,|\mu(\bar N)|)=1$, and*
(c) *$[\bar\psi]$ is surjective.*

The following local-global principle is [NSW15, p. 565, Lemma 9.5.6].

LEMMA 4.4: *Under Setup 4.1 and the assumption $l\nmid|\mu(K)|$,* NFSa input, 144

$$\mathcal{H}om_{\Gamma,\rho,\bar\alpha}(\mathrm{Gal}(K_0),\bar G)\neq\emptyset\iff\prod_\mathfrak{p}\mathcal{H}om_{\Gamma,\rho_\mathfrak{p},\bar\alpha}(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}),\bar G)\neq\emptyset.$$

PROPOSITION 4.5: *In addition to the data introduced in Setup 4.1 let $\lambda\colon G\to\bar G$ be an epimorphism of finite groups. Write $|\mathrm{Ker}(\lambda)|=ep^s$ and let n be a positive integer such that $p\nmid en$. Moreover, we assume that $el|n$ if $l\neq p$. Let T be a finite set of primes of K_0 that contains $\mathrm{Ram}(K/K_0)$ and $S_0(K)\subseteq T_K$. Suppose that $\gcd(n,|\mu(K)|)=1$ and $\prod_\mathfrak{p}\mathcal{H}om_{\Gamma,\rho_\mathfrak{p},\bar\alpha}(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}),\bar G)\neq\emptyset$. For each $\mathfrak{p}\in T$ let $[\varphi_\mathfrak{p}]\in\mathcal{H}om_{\Gamma,\rho_\mathfrak{p},\bar\alpha}(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}),\bar G)$.* CHAl input, 155

Then, there exist a finite set $R\subseteq\mathbb{P}(K_0)\smallsetminus T$ and $[\bar\psi]\in\mathcal{H}om_{\Gamma,\rho,\bar\alpha}(\mathrm{Gal}(K_0),\bar G)_{\mathrm{sur}}$ such that
(a) *$[\bar\psi_\mathfrak{p}]=[\varphi_\mathfrak{p}]$ in $\mathcal{H}om_{\Gamma,\rho_\mathfrak{p},\bar\alpha}(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}),\bar G)$ for each $\mathfrak{p}\in T$,*
(b) *$|R|=r$ if $l\neq p$ and $|R|=1$ if $l=p$ and $[\bar\psi]$ is unramified at $\mathfrak{p}\in\mathbb{P}(K_0)\smallsetminus(T\cup R)$, so if $\bar N$ is the solution field of $\bar\psi$ (i.e. the fixed field of $\mathrm{Ker}(\bar\psi)$), then $\mathrm{Ram}(\bar N/K_0)\subseteq T\cup R$,*
(c) *for each $\mathfrak{p}\in\mathbb{P}(K_0)\smallsetminus T$ we have $\mathcal{H}om_{\bar G,\bar\psi_\mathfrak{p},\lambda}(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}),G)\neq\emptyset$, and*
(d) *$\gcd(n,|\mu(\bar N)|)=1$.*

Proof: We break up the proof into several parts.

PART A: *The surjectivity and the number of roots of unity.* Let m be the minimal number of generators of $\mathrm{Gal}(K(\zeta_n)/K)$. We choose distinct primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_m, \mathfrak{q}_0 \in \mathbb{P}(K_0) \smallsetminus T$ and elements $[\varphi_\mathfrak{p}] \in \mathcal{H}om_{\Gamma,\rho_\mathfrak{p},\bar\alpha}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), \bar{G})$ for $\mathfrak{p} \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m, \mathfrak{q}_0\}$ that satisfy the conclusion of Lemma 4.3. Thus, if $[\bar\psi] \in \mathcal{H}om_{\Gamma,\bar\alpha,\rho}(\mathrm{Gal}(K_0), \bar{G})$ satisfies $[\bar\psi_\mathfrak{p}] = [\varphi_\mathfrak{p}]$ for each $\mathfrak{p} \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m, \mathfrak{q}_0\}$, then

(2a) $[\bar\psi]$ is unramified at $\mathfrak{p}_1, \ldots, \mathfrak{p}_m, \mathfrak{q}_0$,

(2b) the fixed field $\bar{N}$ of $\mathrm{Ker}(\bar\psi)$ in $K_{0,\mathrm{sep}}$ satisfies $\gcd(n, |\mu(\bar{N})|) = 1$, and

(2c) $[\bar\psi]$ is surjective.

PART B: *Strategy of the proof.* Since $\prod_\mathfrak{p} \mathcal{H}om_{\Gamma,\rho_\mathfrak{p},\bar\alpha}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}, \bar{G}) \neq \emptyset$, Lemma 4.4 yields an element $[\psi_0] \in \mathcal{H}om_{\Gamma,\rho,\bar\alpha}(\mathrm{Gal}(K_0), \bar{G})$. We are going to find an $x \in H^1(\mathrm{Gal}(K_0), A)$ such that $[\bar\psi] = [\psi_0]^x$ satisfies the conclusions (a), (b), (c), and (d) of the proposition.

To this end let N_0 be the fixed field of $\mathrm{Ker}(\psi_0)$ in $K_{0,\mathrm{sep}}$. Then, $\rho(\mathrm{Gal}(N_0)) = \bar\alpha(\psi_0(\mathrm{Gal}(N_0))) = \mathbf{1}$, so $\mathrm{Gal}(N_0) \leq \mathrm{Ker}(\rho) = \mathrm{Gal}(K)$, hence $K \subseteq N_0$. Moreover, $\psi_0|_{\mathrm{Gal}(K)}$ induces an embedding $\psi_0'\colon \mathrm{Gal}(N_0/K) \to \bar{G}$ such that $\bar\alpha(\psi_0'(\mathrm{Gal}(N_0/K))) = \mathbf{1}$, so $\mathrm{Gal}(N_0/K)$ is isomorphic to a subgroup of A. Hence, N_0/K is an elementary abelian l-extension.

PART C: *The sets T^* and T^{**}.* We set $T^* = T \cup \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m, \mathfrak{q}_0\}$ and let $\mathfrak{r}_1, \ldots, \mathfrak{r}_s$ be the primes that belong to $\mathbb{P}(K_0) \smallsetminus T^*$ at which ψ_0 ramifies. Then, we set $T^{**} = T^* \cup \{\mathfrak{r}_1, \ldots, \mathfrak{r}_s\}$ and have that

(3) ψ_0 is unramified at each $\mathfrak{p} \in \mathbb{P}(K_0) \smallsetminus T^{**}$.

Next we observe that since $\mathrm{Ram}(K/K_0) \subseteq T$, each $\mathfrak{p} \in \{\mathfrak{r}_1, \ldots, \mathfrak{r}_s\}$ is unramified in K, so $\rho_\mathfrak{p}\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}) \to \Gamma$ is unramified [JaR18, Subsection 7.4]. Hence, by [JaR18, Lemma 8.1],

(4) there exists an unramified element $[\varphi_\mathfrak{p}] \in \mathcal{H}om_{\Gamma,\rho_\mathfrak{p},\bar\alpha}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), \bar{G})$.

Now we consider the system $([\varphi_\mathfrak{p}] \in \mathcal{H}om_{\Gamma,\rho_\mathfrak{p},\bar\alpha}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), \bar{G}))_{\mathfrak{p}\in T^{**}}$. For each $\mathfrak{p} \in T^{**}$, [JaR18, Lemma 10.4] supplies a unique element $y_\mathfrak{p} \in H^1(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), A)$ that satisfies

$$[\psi_{0,\mathfrak{p}}]^{y_\mathfrak{p}} = [\varphi_\mathfrak{p}]. \tag{5}$$

By Setup 4.1, $A \cong C_l^r$ is a simple $\mathrm{Gal}(K_0)$-module on which $\mathrm{Gal}(K)$ acts trivially. Let $r' = r$ if $l \neq p$ and $r' = 1$ if $l = p$. If $l \neq p$, then by Lemma 3.2, applied to T^{**} rather than to T, there exist an element $x \in H^1(\mathrm{Gal}(K_0), A)$ and primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_{r'} \in \mathbb{P}(K_0) \smallsetminus T^{**}$ such that

(6a) $\mathrm{res}_\mathfrak{p}(x) = y_\mathfrak{p}$ for each $\mathfrak{p} \in T^{**}$,

(6b) $\mathrm{res}_\mathfrak{p}(x)$ is unramified at each $\mathfrak{p} \in \mathbb{P}(K_0) \smallsetminus (T^{**} \cup \{\mathfrak{q}_1, \ldots, \mathfrak{q}_{r'}\})$,

(6c) for $i = 1, \ldots, r'$ the prime $\mathfrak{q}_i$ totally splits in $N_0(\zeta_n)$ and $\mathrm{res}_{\mathfrak{q}_i}(x)\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{q}_i}) \to A$ is a C_l-homomorphism, and

(6d) there exists a homomorphism $x'_{\mathfrak{q}_i}\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{q}_i}) \to G$ such that $\lambda \circ x'_{\mathfrak{q}_i} = \mathrm{res}_{\mathfrak{q}_i}(x)$.

If $l = p$, then $r' = 1$ and Lemma 2.6 gives x and $\mathfrak{q}_1$ that satisfy Condition (6).

PART D: *The solution $\bar{\psi}$.* We consider the element $[\bar{\psi}] = [\psi_0]^x$ of $\mathcal{H}om_{\Gamma,\rho,\bar{\alpha}}(\mathrm{Gal}(K_0), \bar{G})$. For each $\mathfrak{p} \in T^{**}$ we have that

$$[\bar{\psi}_{\mathfrak{p}}] = [\psi_{0,\mathfrak{p}}]^{\mathrm{res}_{\mathfrak{p}}(x)} \overset{(6a)}{=} [\psi_{0,\mathfrak{p}}]^{y_{\mathfrak{p}}} \overset{(5)}{=} [\varphi_{\mathfrak{p}}] \tag{7}$$

in $\mathcal{H}om_{\Gamma,\rho_{\mathfrak{p}},\bar{\alpha}}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), \bar{G})$. In particular, (7) holds for each $\mathfrak{p} \in T$, so Conclusion (a) of the proposition holds.

In addition, by Part A, $[\bar{\psi}]$ satisfies Conditions (2a), (2b), and (2c). In particular, by (2b), $\gcd(n, |\mu(\bar{N})|) = 1$, so Conclusion (d) holds. By (2c), $\bar{\psi}$ is an epimorphism. We prove that $\bar{\psi}$ also satisfies Conclusions (b) and (c) of the proposition.

PROOF OF (b): We set $R = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_{r'}\}$, so, by Part C, $|R| = r$ if $l \neq p$ and $|R| = 1$ if $l = p$. Let $\mathfrak{p} \in \mathbb{P}(K_0) \smallsetminus (T \cup R)$. If $\mathfrak{p} \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m, \mathfrak{q}_0\}$, then $[\varphi_{\mathfrak{p}}] \overset{(7)}{=} [\bar{\psi}_{\mathfrak{p}}]$. Hence, by (2a), $[\varphi_{\mathfrak{p}}]$ is unramified. If $\mathfrak{p} \in \{\mathfrak{r}_1, \ldots, \mathfrak{r}_s\}$, then by (4), $[\varphi_{\mathfrak{p}}]$ is unramified. Hence, by (7), $\bar{\psi}$ is unramified at $\mathfrak{p}$ in both cases [JaR18, Subsection 7.4].

Finally, if $\mathfrak{p} \in \mathbb{P}(K_0) \smallsetminus (T^{**} \cup R)$, then by (3) and (6b), both $[\psi_{0,\mathfrak{p}}]$ and $\mathrm{res}_{\mathfrak{p}}(x)$ are unramified. Hence, by [JaR18, Lemma 10.5], $[\bar{\psi}_{\mathfrak{p}}] = [\psi_{0,\mathfrak{p}}]^{\mathrm{res}_{\mathfrak{p}}(x)}$ is unramified. Thus, Condition (b) holds.

PROOF OF (c): Consider $\mathfrak{p} \in \mathbb{P}(K_0) \smallsetminus T$. If $\mathfrak{p} \notin R$, then by (b), $[\bar{\psi}_{\mathfrak{p}}]$ is unramified. Hence, by [JaR18, Lemma 8.1], $\bar{\psi}_{\mathfrak{p}}$ can be lifted to an unramified element of $\mathcal{H}om_{\bar{G},\bar{\psi}_{\mathfrak{p}},\lambda}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), \bar{G})$. If $\mathfrak{p} \in R$, then by (6c), $\mathfrak{p}$ totally splits in $N_0(\zeta_n)$, hence also in N_0. Therefore, $\psi_{0,\mathfrak{p}}\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}) \to \bar{G}$ is the trivial homomorphism [JaR18, Subsection 7.4, second paragraph]. Also, by (6c), $\mathrm{res}_{\mathfrak{p}}(x)\colon \mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}) \to A$ is a C_l-homomorphism, hence $\mathrm{res}_{\mathfrak{p}}(x)$ represents its own cohomology class. Therefore, $[\bar{\psi}_{\mathfrak{p}}] = [\psi_{0,\mathfrak{p}}]^{\mathrm{res}_{\mathfrak{p}}(x)} = [\psi_{0,\mathfrak{p}} \cdot \mathrm{res}_{\mathfrak{p}}(x)] = [\mathrm{res}_{\mathfrak{p}}(x)]$.

By (6d), $\mathrm{res}_{\mathfrak{p}}(x)$ can be lifted to a G-homomorphism $x'_{\mathfrak{p}}$. Hence, also $\bar{\psi}_{\mathfrak{p}}$ has the same property. This implies that $\mathcal{H}om_{\bar{G},\bar{\psi}_{\mathfrak{p}},\lambda}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), G) \neq \emptyset$, as (c) states. ∎

5. Finite Embedding Problems with Solvable Kernel

ALLG input, 12

Using induction, we combine the results obtained so far and prove the main result of this work: Every finite embedding problem over a global field with solvable kernel that satisfies a certain restriction on the roots of unity has a proper solution with bounded ramification that satisfies local conditions.

LEMMA 5.1: *Suppose that a group G acts (from the right) on a finite non-trivial solvable group H such that H and $\mathbf{1}$ are the only G-invariant normal subgroups of H. Then,* SIMp input, 21 *there exists a prime number l and a positive integer r such that $H = C_l^r$ is a simple G-module.*

Proof: Since H is non-trivial and solvable, H has a proper normal subgroup M such that H/M is abelian. Then, M^σ is normal in H for each $\sigma \in G$. Hence, $N = \bigcap_{\sigma\in G} M^\sigma$ is a proper normal G-invariant subgroup of H. By assumption, $N = \mathbf{1}$. Hence, the map $h \mapsto (hM^\sigma)_{\sigma\in G}$ is an embedding of H into the abelian group $\prod_{\sigma\in G} H/M^\sigma$. It follows that H is abelian.

Since H is non-trivial, there exist a prime number l and an element $a \in H$ of order l. Then, $I = \langle a^\sigma \mid \sigma \in G\rangle$ is a non-trivial normal G-invariant subgroup of H, so $I = H$. Hence, the order of each element of H is l. This implies that $H \cong C_l^r$ for some positive integer r, as claimed. ∎

Definition 5.2: Suppose that a group G acts on a finite solvable group H. Let CHAm input, 54

$$\mathbf{1} = H_m < \cdots < H_2 < H_1 < H_0 = H \tag{1}$$

be a **maximal G-series** of H. In other words, for each $1 \le i \le m$, H_i is a proper normal subgroup of H_{i-1} which is maximal among all proper normal subgroups of H_{i-1} that are G-invariant. Since H_{i-1} is solvable, it follows from Lemma 5.1 that $H_{i-1}/H_i \cong C_{l_i}^{r_i}$ is a simple G-module, where l_i is a prime number and r_i is a positive integer. If $\mathbf{1} = H'_{m'} < \cdots < H'_2 < H'_1 < H'_0 = H$ is another maximal G-series of H, then by the Jordan-Hölder theorem, $m = m'$ and there is a permutation κ of $\{0, 1, \ldots, m\}$ such that $\kappa(0) = 0$, $\kappa(m) = m$, and $H_{i-1}/H_i \cong H'_{\kappa(i-1)}/H'_{\kappa(i)}$ for $i = 1, \ldots, m$ [Rob82, p. 66, Thm. 3.1.4].

It follows that if we write $\Lambda_p(H, G)$ for the number of i's between 1 and m such that $l_i = p$, then $\Lambda_p(H, G)$ is an invariant of the pair (H, G), that is $\Lambda_p(H, G)$ does not depend on the maximal G-series of H we use to define it.

With this in mind, we write $|H| = n \cdot p^s$, where $p \nmid n$ and $s \ge 0$, and let $\Omega_p(H, G) = \Omega(n)\Lambda_p(H, G)$, where $\Omega(n)$ is the number of prime divisors of n counted with multiplicity. In particular,

$$\Omega_p(H, G) \le \Omega(|H|) \text{ and } \Omega_p(H, G) = \Omega(|H|) \text{ if } p \nmid |H|. \tag{2}$$

Observe that if H' is a G-invariant normal subgroup of H, then G acts on H/H' and

$$\Omega_p(H, G) = \Omega_p(H/H', G/H') + \Omega_p(H', G). \tag{3}$$ ∎

Setup 5.3: Let K_0 be a global field of positive characteristic p. We consider a finite Galois extension K of K_0 and an embedding problem CHAn input, 106

$$(\rho\colon \mathrm{Gal}(K_0) \to \Gamma, \alpha\colon G \to \Gamma), \tag{4}$$

where $\Gamma = \mathrm{Gal}(K/K_0)$, G is a finite group, α is an epimorphism, and $\rho = \mathrm{res}_{K_{0,\mathrm{sep}}/K}$. Suppose that $H = \mathrm{Ker}(\alpha)$ is a solvable group. In particular, H is a normal subgroup of G, so G acts on H by conjugation. Thus, each G-invariant subgroup of H is normal in H.

Let H_1 be a maximal G-invariant subgroup of H and let l_1 be a prime number and let r_1 be a positive integer such that $H/H_1 \cong C_{l_1}^{r_1}$ (Lemma 5.1). In particular, H/H_1 is a simple G-module, hence also a simple Γ-module, so also a simple $\mathrm{Gal}(K_0)$-module on which $\mathrm{Gal}(K)$ acts trivially. Moreover, $l_1^{r_1}|H_1| = |H|$ and we have a commutative diagram

(5)
$$\begin{tikzcd}
 & H_1 \arrow[r, equal] \arrow[d] & H_1 \arrow[d] & \mathrm{Gal}(K_0) \arrow[d, "\rho"] & \\
\mathbf{1} \arrow[r] & H \arrow[r] \arrow[d, "\lambda"'] & G \arrow[r, "\alpha"] \arrow[d, "\lambda"'] & \Gamma \arrow[r] \arrow[d, equal] & \mathbf{1} \\
\mathbf{1} \arrow[r] & H/H_1 \arrow[r] & G/H_1 \arrow[r, "\bar{\alpha}"] & \Gamma \arrow[r] & \mathbf{1}
\end{tikzcd}$$

with exact horizontal sequences such that both maps λ are the quotient maps. ∎

THEOREM 5.4: *Let K/K_0 be a finite Galois extension of global fields of positive characteristic p and consider the finite embedding problem (4) with solvable kernel H, where $\Gamma = \mathrm{Gal}(K/K_0)$ and $\rho = \mathrm{res}_{K_{0,\mathrm{sep}}/K}$. Let T be a finite set of primes of K_0 that contains $\mathrm{Ram}(K/K_0)$ and $S_0(K) \subseteq T_K$. Let $|H| = np^s$ with $p \nmid n$ and $\gcd(n, |\mu(K)|) = 1$. Suppose that $\prod_{\mathfrak{p}} \mathcal{H}om_{\Gamma,\rho_{\mathfrak{p}},\alpha}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), G) \neq \emptyset$ (Remark 4.2). For each $\mathfrak{p} \in T$ let $[\varphi_{\mathfrak{p}}] \in \mathcal{H}om_{\Gamma,\rho_{\mathfrak{p}},\alpha}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), G)$.* CHAo input, 153

Then, there exists an element $[\psi] \in \mathcal{H}om_{\Gamma,\rho,\alpha}(\mathrm{Gal}(K_0), G)_{\mathrm{sur}}$ and there exists a set $R \subseteq \mathbb{P}(K_0) \smallsetminus T$ with $|R| = \Omega_p(H, G)$ such that

(a) *$[\psi_{\mathfrak{p}}] = [\varphi_{\mathfrak{p}}]$ in $\mathcal{H}om_{\Gamma,\rho_{\mathfrak{p}},\alpha}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), G)$ for each $\mathfrak{p} \in T$ and*

(b) *$[\psi]$ is unramified at each $\mathfrak{p} \in \mathbb{P}(K_0) \smallsetminus (T \cup R)$, that is the fixed field N of $\mathrm{Ker}(\psi)$ in $K_{0,\mathrm{sep}}$ satisfies $\mathrm{Ram}(N/K_0) \subseteq T \cup R$.*

Proof: Let H_1 be a maximal G-invariant subgroup of H. We break up the rest of the proof into three parts.

PART A: *An embedding problem whose kernel is a simple $\mathrm{Gal}(K_0)$-module.* We consider Diagram (5). If $\mathfrak{p} \in \mathbb{P}(K_0)$ and $[\eta_{\mathfrak{p}}] \in \mathcal{H}om_{\Gamma,\rho_{\mathfrak{p}},\alpha}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), G)$, then $[\lambda \circ \eta_{\mathfrak{p}}] \in \mathcal{H}om_{\Gamma,\rho_{\mathfrak{p}},\bar{\alpha}}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), G/H_1)$. Hence, by assumption, $\prod_{\mathfrak{p}} \mathcal{H}om_{\Gamma,\rho_{\mathfrak{p}},\bar{\alpha}}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), G/H_1) \neq \emptyset$. In particular, for each $\mathfrak{p} \in T$ we have that $[\bar{\varphi}_{\mathfrak{p}}] = [\lambda \circ \varphi_{\mathfrak{p}}] \in \mathcal{H}om_{\Gamma,\rho_{\mathfrak{p}},\bar{\alpha}}(\mathrm{Gal}(\hat{K}_{0,\mathfrak{p}}), G/H_1)$.

By Setup 5.3, $H/H_1 \cong C_{l_1}^{r_1}$ is a simple Γ-module, where l_1 is a prime number and r_1 is a positive integer. Also, $\mathrm{Ker}(\lambda) = H_1$ and $l_1^{r_1}$ divides $|H| = np^s$. If $l_1 \neq p$, then

l_1 divides $|H/H_1|$. Writing $|\mathrm{Ker}(\lambda)| = |H_1| = ep^s$ with $l_1 \nmid e$, we have $ep^s \cdot |H/H_1| = |\mathrm{Ker}(\lambda)| \cdot |H/H_1| = |H| = np^s$, so $el_1|n$. In any case Proposition 4.5 yields a finite set $T_1 \subseteq \mathbb{P}(K_0) \smallsetminus T$ and an element

(6) $[\psi_1] \in \mathcal{H}om_{\Gamma,\rho,\bar\alpha}(\mathrm{Gal}(K_0), G/H_1)_{\mathrm{sur}}$

such that

(7) $|T_1| = \Omega_p(H/H_1, G/H_1)$ and

(8a) $[\psi_{1,\mathfrak{p}}] = [\bar\varphi_\mathfrak{p}]$ in $\mathcal{H}om_{\Gamma,\rho_\mathfrak{p},\bar\alpha}(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}), G/H_1)$ for each $\mathfrak{p} \in T$,

(8b) $[\psi_1]$ is unramified at each $\mathfrak{p} \in \mathbb{P}(K_0) \smallsetminus (T \cup T_1)$, so if N_1 is the fixed field of $\mathrm{Ker}(\psi_1)$, then $\mathrm{Ram}(N_1/K_0) \subseteq T \cup T_1$,

(8c) for each $\mathfrak{p} \in \mathbb{P}(K_0) \smallsetminus T$ we have $\mathcal{H}om_{G/H_1,\psi_{1,\mathfrak{p}},\lambda}(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}), G) \neq \emptyset$, and

(8d) $\gcd(n, |\mu(N_1)|) = 1$.

Part B: *The induction step.* Part A gives rise to an embedding problem

$$(\psi_1\colon \mathrm{Gal}(K_0) \to G/H_1,\ \lambda\colon G \to G/H_1) \tag{9}$$

with finite solvable kernel H_1. Let $n_1 = nl_1^{-r_1}$, $s_1 = s$ if $l_1 \neq p$ and $n_1 = n$, $s_1 = s - r_1$ if $l_1 = p$. Then, $|H_1| = |H|/|H/H_1| = n_1 p^{s_1}$ and $p \nmid n_1$.

For each $\mathfrak{p} \in T$ there exists, by (8a), an element $a_\mathfrak{p} \in H$ such that

$$\psi_{1,\mathfrak{p}}(\sigma) = \lambda(a_\mathfrak{p})^{-1}\bar\varphi_\mathfrak{p}(\sigma)\lambda(a_\mathfrak{p}) = \lambda(a_\mathfrak{p}^{-1})\lambda(\varphi_\mathfrak{p}(\sigma))\lambda(a_\mathfrak{p}) = \lambda(a_\mathfrak{p}^{-1}\varphi_\mathfrak{p}(\sigma)a_\mathfrak{p}) = (\lambda \circ \varphi_\mathfrak{p}^{a_\mathfrak{p}})(\sigma)$$

for each $\sigma \in \mathrm{Gal}(\hat K_{0,\mathfrak{p}})$. Hence,

(10) $[\varphi_\mathfrak{p}^{a_\mathfrak{p}}] \in \mathcal{H}om_{G/H_1,\psi_{1,\mathfrak{p}},\lambda}(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}), G)$ for every $\mathfrak{p} \in T$.

It follows from (8c) and (10) that $\prod_\mathfrak{p} \mathcal{H}om_{G/H_1,\psi_{1,\mathfrak{p}},\lambda}(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}), G) \neq \emptyset$. Moreover, (8d) implies that $\gcd(n_1, |\mu(N_1)|) = 1$.

For each $\mathfrak{p} \in T$ we set $\varphi_{1,\mathfrak{p}} = \varphi_\mathfrak{p}^{a_\mathfrak{p}}$. Then, for each $\mathfrak{p} \in T_1$ we use (8c) to choose $[\varphi_{1,\mathfrak{p}}] \in \mathcal{H}om_{G/H_1,\psi_{1,\mathfrak{p}},\lambda}(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}), G)$.

Since H_1 is solvable and $|H_1| < |H|$, an induction hypothesis on the order of the kernel of the embedding problem gives a set $R_1 \subseteq \mathbb{P}(K_0) \smallsetminus (T \cup T_1)$ with $|R_1| = \Omega_p(H_1, G)$ and an element

(11) $[\psi] \in \mathcal{H}_{G/H_1,\psi_1,\lambda}(\mathrm{Gal}(K_0), G)_{\mathrm{sur}}$

such that

(12a) $[\psi_\mathfrak{p}] = [\varphi_{1,\mathfrak{p}}]$ in $\mathcal{H}om_{G/H_1,\psi_{1,\mathfrak{p}},\lambda}(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}), G)$, for each $\mathfrak{p} \in T \cup T_1$, and

(12b) $[\psi]$ is unramified at each $\mathfrak{p} \in \mathbb{P}(K_0) \smallsetminus (T \cup T_1 \cup R_1)$, that is if N is the solution field of embedding problem (9), then $\mathrm{Ram}(N/K_0) \subseteq T \cup T_1 \cup R_1$.

We set $R = T_1 \cup R_1$. Then,

$$|R| = |T_1| + |R_1| \overset{(7)}{=} \Omega_p(H/H_1, G/H_1) + \Omega_p(H_1, G) \overset{(3)}{=} \Omega_p(H, G).$$

Part C: *Conclusion of the proof.* We prove that $[\psi]$ satisfies the conclusion of the theorem. Indeed, by (5), (11), and (6) we have $\alpha\circ\psi \overset{(5)}{=} \bar\alpha\circ\lambda\circ\psi \overset{(11)}{=} \bar\alpha\circ\psi_1 \overset{(6)}{=} \rho$, so $[\psi]\in \mathcal{H}om_{\Gamma,\rho,\alpha}(\mathrm{Gal}(K_0),G)_{\mathrm{sur}}$.

$$
\begin{tikzcd}
 & & \mathbf{1} \arrow[d] & & \\
 & & H_1 \arrow[d] & \mathrm{Gal}(K_0) \arrow[dl, "\psi"'] \arrow[ddl, "\psi_1"] \arrow[dd, "\rho"] & \\
\mathbf{1} \arrow[r] & H \arrow[r] \arrow[d] & G \arrow[d, "\lambda"'] \arrow[dr, "\alpha"] & & \\
1 \arrow[r] & H/H_1 \arrow[r] & G/H_1 \arrow[r, "\bar\alpha"] \arrow[d] & \Gamma \arrow[r] & \mathbf{1} \\
 & & \mathbf{1} & &
\end{tikzcd}
$$

Moreover, by (12a), for each $\mathfrak{p}\in T$ there exists $b_\mathfrak{p}\in H_1$ such that for each $\sigma\in\mathrm{Gal}(\hat K_{0,\mathfrak{p}})$ we have $\psi_\mathfrak{p}(\sigma)=b_\mathfrak{p}^{-1}\varphi_{1,\mathfrak{p}}(\sigma)b_\mathfrak{p}=b_\mathfrak{p}^{-1}a_\mathfrak{p}^{-1}\varphi_\mathfrak{p}(\sigma)a_\mathfrak{p}b_\mathfrak{p}=(a_\mathfrak{p}b_\mathfrak{p})^{-1}\varphi_\mathfrak{p}(\sigma)(a_\mathfrak{p}b_\mathfrak{p})$. Since $a_\mathfrak{p}\in H$ and $b_\mathfrak{p}\in H_1$, we have $a_\mathfrak{p}b_\mathfrak{p}\in H$. Therefore, $[\psi_\mathfrak{p}]=[\varphi_\mathfrak{p}]$ in $\mathcal{H}om_{\Gamma,\rho_\mathfrak{p},\alpha}(\mathrm{Gal}(\hat K_{0,\mathfrak{p}}),G)$ for each $\mathfrak{p}\in T$, as desired. ∎

Remark 5.5: Theorem 5.4 obtains an especially pleasant form in the case where the kernel H of embedding problem (4) is a p-group. In this case $\Omega_p(H,G)=\Lambda_p(H,G)$ is the length of the maximal G-series of H, so $|R|$ is much smaller than in the general case. ∎ CHAp input, 391

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