On the Sum of Angles in a Star Polygon

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8 January 2025

Given a convex polygon of odd number $m := 2n+1$ of vertices, we choose one of the vertices A_1, and number the rest of the vertices clockwise as $A_2, A_3, \dots, A_m$. Then we construct the diagonal from A_1 to $A_{n+1 \bmod m}$, a second diagonal from $A_{n+1 \bmod m}$ to $A_{2 \bmod m}$[1] a third diagonal from $A_2 \mod m$ to $A_{2+n \bmod m}$, where by $k \mod m$ we mean here the residue of k divided by m (e.g. $8 \mod 7 = 1$) and so on. The last diagonal connects A_{2n-1} to A_1, that is, it returns to the point we started with. We call the polygon generated in this way a **star polygon**. The simplest star polygon is the triangle for which a basic theorem in geometry says that the sum of angles of which is 180^o. We prove that this is the case for each star polygon. [2]

For this we need the following well known basic lemma:

{SUMa}

Lemma 0.1. *The sum of the angles of a convex polygon P of $m \geq 3$ vertices is $(m-2)180^o$.*

Proof. We repeat the well known proof.

As above, let $A_1, A_2, \dots, A_m$ be consecutive edges of P. The diagonals $A_1A_3, A_1A_4, \dots, A_1A_{m-2}$ divide P into $m-2$ triangles that cover the whole area of the polygon and intersect one another at most at the edges. The sum of the angles in each of those triangles is 180^o. Hence, the sum of the angles of P is $(m-2)180^o$, as claimed. □

Lemma 0.2. *Every edge of the star polygon P intersects all other edges.*

Proof. Consider for example the edge A_1A_{n+1} (this is a general case, because each edge can be obtained from another edge by an appropriate rotation). This edge splits the polygon into two parts and the verticies of each other edge do not lie on the same side of A_1A_{n+1}, because the difference of the indices of the vertices of the latter edge is n. □

Theorem 0.3. *The sum of the angles in a star polygon P of odd edges is* 180^o.

[1] When we write $a \equiv b \mod m$ in this note, we mean the smallest non negative integer a which is congruent to b modulo m.

[2] I proved the main result of this paper during 1957 while I studied in the 10th class of The Hebrew Gymnasium, Jerusalem, Israel. The paper was published in January 1963 in the Hebrew Magazin “Gilyonot Mathematica lanoar halomed ulechovevim”. The current version is a translation (and working out) into English of the original paper.

Proof. Denote the sum of the angles of P by D. Let $A_1, A_2, \ldots, A_m$ be consequitive vertices of P. Then consider the angles

$$\alpha_1 := \measuredangle A_n A_1 A_{n+1},\ \alpha_2 := \measuredangle A_{n+1} A_2 A_{n+2},\ ...$$

and so on. Let $\beta_1 = \measuredangle A_n A_1 A_2,\ \beta_2 = \measuredangle A_{n+1} A_2 A_3,$ ... and so on. Finally let $\gamma_1 = \measuredangle A_{n+1} A_1 A_m$, $\gamma_2 = \measuredangle A_{n+2} A_2 A_{m-1}$, and so on.

Using that the sum of the angles in a triangle is 180^o, we have

$$\begin{aligned} \alpha_1 &= 180^o - (\alpha_{n+1} + \beta_{n+1}) - (\gamma_{n+2} + \alpha_{n+2}) \\ \alpha_2 &= 180^o - (\alpha_{n+2} + \beta_{n+2}) - (\gamma_{n+3} + \alpha_{n+3}) \\ &\ldots \\ \alpha_m &= 180^0 - (\alpha_{n+m} + \beta_{n+m}) - (\gamma_{n+m+1} + \alpha_{n+m+1}), \end{aligned} \qquad (1) \quad \{\texttt{angl}\}$$

where $\alpha_{k+m} = \alpha_k$ and $\beta_{k+m} = \beta_k$ for each $1 \leq k \leq m$. By definition $\sum_{i=1}^m \alpha_i = D$. Adding both sides of equalities (1), we get

$$D = 180^0 m - (\sum_{i=1}^m \alpha_i + \sum_{i=1}^m \beta_i + \sum_{i=1}^m \gamma_i + \sum_{i=1}^m \alpha_i),$$

so, by Lemma 0.1, $D = 180^0 m - [180^0(m-2) + D]$. Hence,

$$2D = 180^0 m - 180^0 m + 360^0,$$

therefore, $D = 180^0$, as claimed. $\square$