The Adjoint Representation

Let G be a Lie group and $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra. Throughout these notes we take $A, B, C \in G$ and $X, Y, Z \in \mathfrak{g}$. We describe here the adjoint representation.

The inner automorphism (conjugation) induced by A:

$$\mathrm{Inn}_A(B) = ABA^{-1} \tag{1.1}$$

Then the map

$$\mathrm{Inn} : G \to \mathbf{GL}(G) \quad ; \quad A \mapsto \mathrm{Inn}_A \tag{1.2}$$

is an homomorphism sending $A \in G$ to an automorphism on G.

Differentiating with respect to the second argument, we get a map

$$\mathrm{Ad} = d\mathrm{Inn}_e : G \to \mathbf{GL}(\mathfrak{g}) \quad ; \quad A \mapsto \mathrm{Ad}_A \tag{1.3}$$

This map sends $A \in G$ to an automorphism on $\mathfrak{g}$:

$$\mathrm{Ad}_A : \mathfrak{g} \to \mathfrak{g} \quad ; \quad \mathrm{Ad}_A(X) = AXA^{-1} \tag{1.4}$$

Finally, differentiating with respect to the first argument, we get a map

$$\mathrm{ad} = d\mathrm{Ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \quad ; \quad X \mapsto \mathrm{ad}_X \tag{1.5}$$

defined by $\mathrm{ad}_X(Y) = [X, Y]$ for $X, Y \in \mathfrak{g}$ (where $[\ ,\ ]$ are the Lie brackets of $\mathfrak{g}$).

$\mathrm{Inn} : G \to \mathbf{GL}(G)$	$\mathrm{Inn}_A : G \to G$
Lie group homomorphism: $\mathrm{Inn}_{AB} = \mathrm{Inn}_A \circ \mathrm{Inn}_B$	Lie group automorphism: $\mathrm{Inn}_A(B \cdot C) = \mathrm{Inn}_A(B) \cdot \mathrm{Inn}_A(C)$, $(\mathrm{Inn}_G)^{-1} = \mathrm{Inn}_{G^{-1}}$
$\mathrm{Ad} : G \to \mathbf{GL}(\mathfrak{g})$	$\mathrm{Ad}_A : \mathfrak{g} \to \mathfrak{g}$
Lie group homomorphism: $\mathrm{Ad}_{AB} = \mathrm{Ad}_A \circ \mathrm{Ad}_B$	Lie algebra automorphism: Ad_A is linear, and $(\mathrm{Ad}_A)^{-1} = \mathrm{Ad}_{A^{-1}}$, and $\mathrm{Ad}_A([X, Y]) = [\mathrm{Ad}_A(X), \mathrm{Ad}_A(Y)]$
$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$	$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$
Lie algebra homomorphism: ad is linear, and $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$	Lie algebra derivation: ad_X is linear, and $\mathrm{ad}_X([Y, Z]) = [\mathrm{ad}_X(Y), Z] + [Y, \mathrm{ad}_X(Z)]$

$$\begin{aligned}
&\mathbf{GL}(G)=\mathrm{Aut}(G) &&\Rightarrow \quad \text{automorphism group of } G\\
&\mathbf{GL}(\mathfrak{g})=\mathrm{Aut}(\mathfrak{g}) &&\Rightarrow \quad \text{automorphism group of } \mathfrak{g}\\
&\mathfrak{gl}(\mathfrak{g})=\mathrm{Der}(\mathfrak{g}) &&\Rightarrow \quad \text{derivation algebra of } \mathfrak{g}
\end{aligned}$$

Matrix group case: Let $G\subset\mathbf{GL}_n$ be a matrix Lie group. Then $\mathrm{Ad}_{e^X}=e^{\mathrm{ad}_X}$ (see Baker-Hausdorff_Lemma) which we write also as $\mathrm{Ad}\left(e^X\right)=e^{\mathrm{ad}(X)}$ for convenience. From the definition (1.5) we have

$$\mathrm{ad}_X=\left[\frac{d}{dt}\mathrm{Ad}\left(e^{tX}\right)\right]_{t=0}$$

We calculate it explicitly

$$\mathrm{ad}_X(Y)=\left[\frac{d}{dt}\mathrm{Ad}\left(e^{tX}\right)(Y)\right]_{t=0}=\left[\frac{d}{dt}e^{tX}Ye^{-tX}\right]_{t=0}=\left[\left(e^{tX}XYe^{-tX}-e^{tX}YXe^{-tX}\right)\right]_{t=0}=XY-YX$$

and thus $\mathrm{ad}_X(Y)=XY-YX=[X,Y]$.

Weight: consider the character group $\mathcal{X}(T)=\left\{\lambda:T\to\mathbb{C}^\times\right\}$ and its differential algebra $d\mathcal{X}(T)=\left\{d\lambda:\mathfrak{t}\to\mathbb{C}\mid\lambda\in\mathcal{X}(T)\right\}\subseteq\mathfrak{t}^*$. We identify λ with $d\lambda$ and call it *weight*.

Weights decomposition: let $R:G\to\mathrm{GL}(V)$ be a linear representation and $dR:\mathfrak{g}\to\mathfrak{gl}(V)$ its differential representation. Let $T\subset G$ be a torus and $\mathfrak{t}=\mathrm{Lie}\,T$. Then

$$V=\bigoplus_{\lambda\in\Phi_R}V_R \quad \text{where} \quad V_R=\left\{\vec{v}\in V\mid\forall\tau\in\mathfrak{t}:(dR)(\tau)\vec{v}=(d\lambda)(\tau)\vec{v}\right\}$$

where $\Phi_R\subset\mathcal{X}(T)$ is the system of weights of the restriction $R|_T$.

Roots: Take the adjoint representation $R=\mathrm{Ad}:G\to\mathrm{GL}(\mathfrak{g})$, then for $\lambda\in d\mathcal{X}(T)$,

$$\begin{aligned}
\mathfrak{g}_\lambda&=\left\{x\in\mathfrak{g}\mid\forall\tau\in\mathfrak{t}:\mathrm{ad}_\tau(x)=[\tau,x]=\lambda(\tau)x\right\}\\
\mathfrak{g}_0&=\left\{x\in\mathfrak{g}\mid\forall\tau\in\mathfrak{t}:\mathrm{ad}_\tau(x)=[\tau,x]=0\right\}
\end{aligned}$$

The nonzero weights of $\Phi_{\mathrm{Ad}}(T)$ are called *roots*. The root system is denoted by $\Delta(T)$. Hence $\Phi_{\mathrm{Ad}}(T)=\Delta(T)\cup\{0\}$ and $\mathfrak{g}=\mathfrak{g}_0\bigoplus_{\alpha\in\Delta(T)}\mathfrak{g}_\alpha$ is root decomposition.