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Microcomputer-Based Discovering and Testing of Combinatorial Identities

SERGEI ABRAMOVICH AND ILYA LEVIN
School of Education
Tel Aviv University, Ramat Aviv, 69978 Tel Aviv, Israel

This article presents several ideas for using a spreadsheet to reinforce teaching of enumerative combinatorics in the secondary mathematics classroom. More specifically, a spreadsheet's operational capability provides an environment conducive to students' modelling numbers with combinatorial meaning and testing their own conjectures about rules (identities) which constrain these numbers. We argue that every combinatorial identity can be associated with the visual imagery that introduces students to the art of counting without counting. Finally, it is shown how students, both in specializing and in generalizing, can unexpectedly come to recognize the capacity of the Fibonacci numbers to solve combinatorial problems. The Microsoft Excel 3 for the Macintosh computer is used.

An important aspect of learning mathematics with computers is the possibility to use the latter as a technological tool for facilitating creativity. Probably the most natural way to draw students into challenging and productive mathematics activity is to arrange the environment for recognizing patterns and regularities, making connections and generalizations, and speculating on methods of testing their own conjectures. All these were claimed as a major goal of the school mathematics curriculum (N.C.T.M., 1989).

Computers today strongly influence mathematics teaching and learning, and provide support by which, as noted by Noss (1988), students can explore and develop relationships within mathematics contents that are just beyond their grasp with pencil-and-paper technology. Moreover, computers change the traditional role of a teacher who has the opportunity to integrate students' learning of mathematics with their doing mathematics, at different levels beginning from the most elementary. One way to turn a classroom into a place where learners are encouraged to do mathematics is to provide modelling and analysis of the obtained results. Indeed, such a teaching approach could make students themselves both pose and solve problems in a natural way, as new mathematical facts appear not through didactic transmission of erudition by a know-all teacher (leaving open the question of how the teacher him/herself has learned this lore), but as a result of independent discovery and cognitive activity.

In the following we shall discuss teaching ideas concerning the application of a spreadsheet in contriving combinatorial identities. Usually, in the course of enumerative combinatorics, these identities are introduced to students in the form of a finished formula. Most students lack the willingness and motivation to be concerned with already solved problems, so that the teacher has difficulty drawing students into a process of proving combinatorial identities. Even when the task is offered with the right-hand-side removed, the question remains how the teacher knew the expression could be simplified. The spreadsheet, however, provides an ideal medium for making up these formulae by speculating on results of modelling. Proposing the use of a spreadsheet for teaching enumerative combinatorics is the result of the following idea. The basic combinatorial notions of ordered selections and unordered selections (combinations) depend on two positive integral variables and through "recursive definition" (Jacobs, 1992, p.100) can be expressed by equations of partial differences subject to boundary conditions. It is possible that owing to unfamiliarity with the ways of tackling such equations, learners are left with long and tedious computations that allow them to think of nothing but numbers. But this is not necessary. A spreadsheet has the outstanding ability to numerically model equations of partial differences and provides an opportunity to study discrete concepts through the numerical approach. This approach, as will be shown below, can effectively involve students in the making of mathematical connections—the rich source of modelling data can help students find regularities in many special cases and generalize from these. Exploiting these opportunities constitutes an important step toward answering Fey's (1989) challenge regarding more appreciation of the role that ease of numerical computation can play in development of conceptual understanding of topics in discrete mathematics, particularly in combinatorics.

RECURSIVE DEFINITION OF BASIC NOTIONS

The simplest combinatorial notion is that of permutations: the number of ways of listing in order *n* given objects. Denoting this number as P(n), it is not hard to understand that if *n* is small then P(n) can be found without much difficulty. For example, for $n \leq 2$ one can easily "guess" of that one given object can be placed on the list in only one way, and two objects—in two ways. This suggests that the method of finding P(n) for $n>2$ would be to express P(n) in similar terms with a smaller value of *n* or, in other words, to make use of recursive reasoning. In so doing a teacher can guide students to the conclusion that one of the *n* given objects can be placed first on the list in *n* ways and each of these *n* choices results in (n-1) remaining objects which can be placed in P(n-1) different orders, which results in the recursion formula $P(n)=nP(n-1)$, $n=2,3,\ldots$. At this point it would be helpful to emphasize that recursion is a discrete process of defining the current state in terms of the preceding and that therefore a recursive definition would be completed when the starting point (or, in other words, the base clause) of the process is assigned. Using the above "guess" at a base clausend providing the case n=0 by the equality P(0)=1 result eventually in the formal recursive definition of permutations

$$P(n)=nP(n-1),\ n=1,2,\ldots;\ P(0)=1 \qquad (1)$$

Passing to ordered selections, the teacher subsequently introduces the next basic problem, namely that of counting A(n,k)—the number of ways of listing *k* objects chosen from *n* given objects. It may be noticed that the first object on the list can be chosen in *n* ways and then (k-1) of (n-1) remaining objects have to be added to the list. This results in the recursion formula $A(n,k)=nA(n-1,k-1)$, $1\leq k \leq n$. A good way to promote students' skills in recursion techniques consists of encouraging a discussion concerning the boundary conditions for A(n,k).The discussion eventually will lead to the formal recursive definition of ordered selections:

$$A(n,k) = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{if } n=0 \text{ and } k \geq 1 \\ nA(n-1,k-1) & \text{otherwise} \end{cases} \qquad (2)$$

Finally the concept of unordered selections (combinations) C(n,k)—the number of ways of choosing *k* objects from *n* given objects can be introduced as the corollary of (1) and (2) through the following formula

$$C(n,k)= A(n,k)/P(k);\ n=0,1,2,\ldots,\ k=0,1,2,\ldots,\ k\leq n \qquad (3)$$

To spark students' interest the teacher must now provide the necessary assistance in the modelling of combinations C(n,k) within a spreadsheet. Below we shall use the standard notation $\binom{n}{k}$ for combinations .

CONSTRUCTION OF THE BASIC SPREADSHEET

We assume that students have elementary skills in operating a spreadsheet and defining functions in cells (Cobb, McGuffey, & Dodge, 1991). The teacher can exploit this by providing only technical assistance in modelling numbers $\binom{n}{k}$. To this end, in accordance with Relations 1 and 2, two auxiliary spreadsheets which implement numbers P(k) and A(n,k) (referred to below as **TP** and **TA** respectively) are programmed as follows.

The spreadsheet TP. In column **A** (beginning from cell **A2**) non-negative integral values of *k* are defined. The base clause for P(k), that is number 1, is defined in the cell **B2**. The spreadsheet function **=$B2*$A3** is defined in cell **B3** and computes the value of P(1). This function is copied down to cell **B16** by using the Copy and Paste commands from the Option menu.

Note that here and below when formatting the entries (to get integers rather than floating point numbers) students first set a spreadsheet to Integer by choosing the line Numbers from the Format menu to display its dialog box, and by switching on first the Integer regime and then OK.

The spreadsheet TA. In row **1** (beginning from cell **B1**) and in column **A** (beginning from cell **A2**) non-negative integral values of *n* and *k* are defined respectively. In row **2** and column **B** boundary conditions for A(n,k) are defined (that is number 1 is defined in **B2:P2**, number 0 is defined in **B3:B16**). The spreadsheet function **=B2*C$1** is defined in cell **C3** and computes the value of A(1,1). This function is replicated to cell **P16** by using the Copy and Paste commands.

The basic spreadsheet **TC** which implements numbers $\binom{n}{k}$ in accordance with Formula 3 is programmed as follows. In row **1** (beginning from cell **B1**) and in column **A** (beginning from cell **A2**) non-negative integral values of *n* and *k* are defined respectively. All other cells of this spreadsheet must consist of ratios of corresponding cells of spreadsheets **TA** and **TP**. To this end the spreadsheet function **=TA!B2/TP!$B2** is defined in cell **B2** and computes the value of $\binom{0}{0}$ This function is replicated to cell **P16** by using the Copy and Paste commands. The spreadsheet **TC** (Pascal's

k\n	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	0	0	1	3	6	10	15	21	28	36	45	55	66	78	91
3	0	0	0	1	4	10	20	35	56	84	120	165	220	286	364
4	0	0	0	0	1	5	15	35	70	126	210	330	495	715	1001
5	0	0	0	0	0	1	6	21	56	126	252	462	792	1287	2002
6	0	0	0	0	0	0	1	7	28	84	210	462	924	1716	3003
7	0	0	0	0	0	0	0	1	8	36	120	330	792	1716	3432
8	0	0	0	0	0	0	0	0	1	9	45	165	495	1287	3003
9	0	0	0	0	0	0	0	0	0	1	10	55	220	715	2002
10	0	0	0	0	0	0	0	0	0	0	1	11	66	286	1001
11	0	0	0	0	0	0	0	0	0	0	0	1	12	78	364
12	0	0	0	0	0	0	0	0	0	0	0	0	1	13	91
13	0	0	0	0	0	0	0	0	0	0	0	0	0	1	14
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Figure 1 (a). The spreadsheet TC filled with numbers

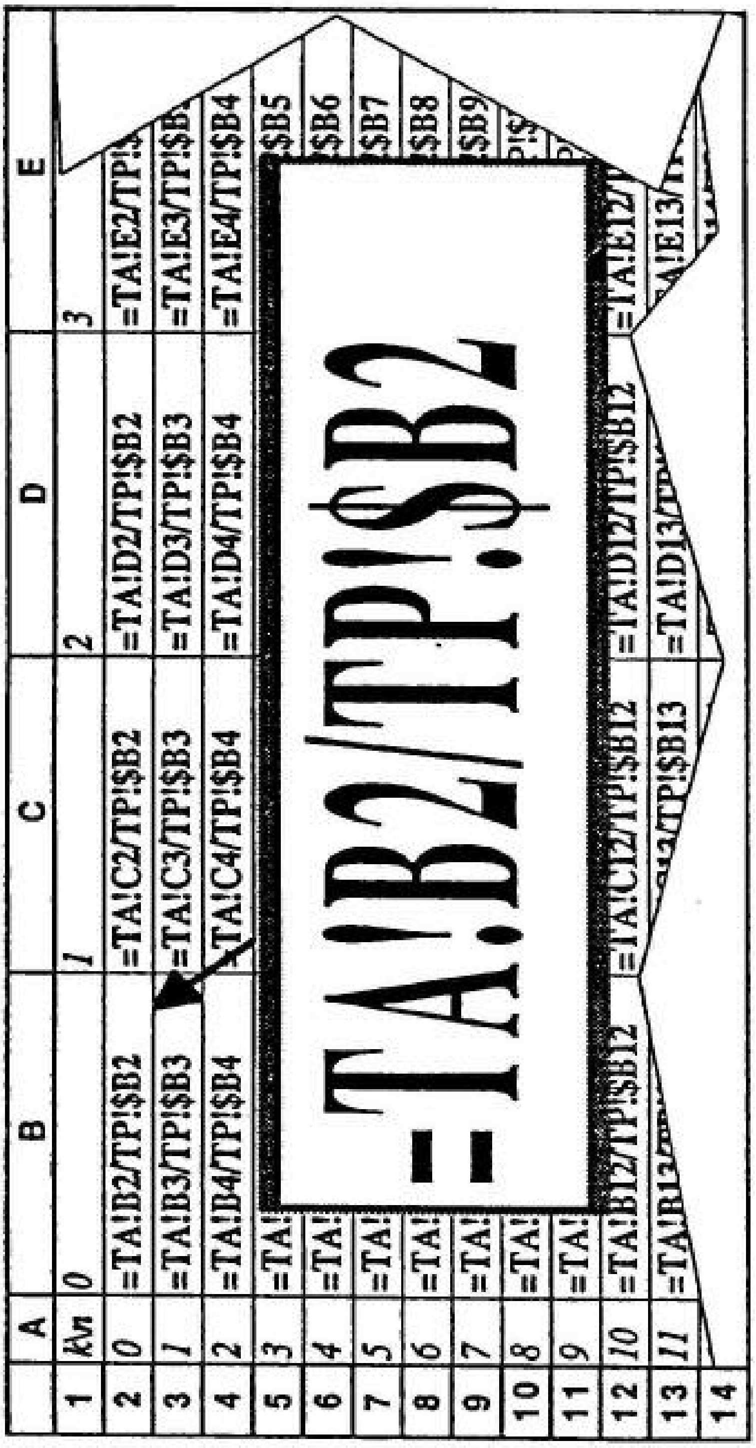

Figure 1 (b). The spreadsheet TC filled with formulas triangle represented as a rectangular array) filled with numbers $\binom{n}{k}$ and its "screen-snap" with formulas which compute these numbers (binomial coefficients), are shown in Figures 1(a), and 1(b) respectively.

Because the computer calculates and recalculates so quickly the teacher can provoke what-if questions, promoting a discussion about changes in the base clause for P(k). Entering different values for P(0) into cell **B2** of the spreadsheet **TP** enables students to experience the importance of the correct assignment of initial data for the description of a recursion process.

Note that students may compare results of modelling of binomial coefficients $\binom{n}{k}$ given through recursive definition with those of given by the well-known formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad 0 \le k \le n$$

which is not hard to ascertain when counting A(n,k) and P(k) through combinatorial reasoning. This formula, however, will not be of use below.

MODELLING AS A METHOD OF DISCOVERING COMBINATORIAL IDENTITIES

Now moving to the major goal of the lesson, the teacher asks students to recognize patterns and regularities through examination of the results of modelling. The template of the spreadsheet **TC** that consists of numbers $\binom{n}{k}$ provides students with an exciting pursuit in contriving **combinatorial** identities.

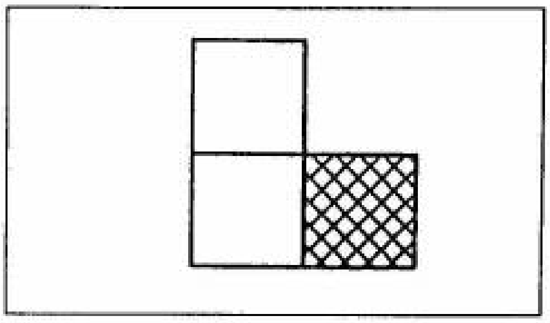

Figure 2. Visual imagery of the rule of three numbers (L-shaped triple of cells)

Observing the template shown in Figure 1(a) students may first discover regularities within each L-shaped triple of numbers (see Figure 2) which provides the simple rule: The sum of two numbers equals the third.

Affirming this rule through special cases students set down the following connections:

$$\binom{1}{0}+\binom{1}{1}=\binom{2}{1},\binom{2}{1}+\binom{2}{2}=\binom{3}{2},\binom{3}{1}+\binom{3}{2}=\binom{4}{2},\binom{3}{2}+\binom{3}{3}=\binom{4}{3}$$

etc.

Trying to attain generalization then results in developing the following conjecture:

$$\binom{n-1}{k}+\binom{n-1}{k-1}=\binom{n}{k} \qquad (4)$$

The teacher could use the recursive definition and the rule of sum in order to provide students with the combinatorial proof of identity (4). Really, combinations $\binom{n}{k}$ may be divided into those which include a given object, say the first, and those which do not. The number of those of the first kind is

$$\binom{n-1}{k-1}$$

since fixing one element of a combination reduces both n and k by one and the number of those of the second kind is

$\binom{n-1}{k}$. The rule of sum yields (4).

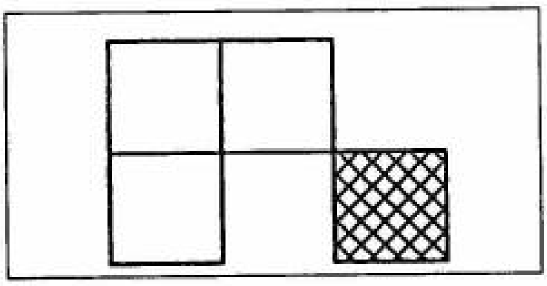

Figure 3. Visual imagery of the rule of four numbers, type 1 (R-shaped chain of cells)

Becoming aware of their ability to invent formulae they never knew before and, possibly, experiencing "the thrill of discovery—a thrill that memorization cannot match" (Miller, 1991, p.96), students are highly motivated in doing this work. The teacher can exploit this by encouraging them to look for new patterns and connections, for example among sets of four numbers, *walking* along different four-step paths within the template. Most likely students then arrive at an R-shaped chain of cells, shown in Figure 3, providing the three-element analogue of the last rule: The sum of three numbers equals the fourth.

Applying this analogous rule to different R-shaped chains of cells may lead students to the new identity:

$$\binom{n-1}{k-1}+\binom{n-2}{k-1}+\binom{n-2}{k}=\binom{n}{k} \qquad (5)$$

Note that students could uncover Identity 5 geometrically as a consequence of the rule of three cells (numbers). Indeed, applying the L-shaped rule to itself, i.e., substituting the bottom left cell in Figure 2 with two cells results in the R-shaped combination of cells (see Figure 3). Once students have understood this, they can replace the top cell similarly in accordance with the L-shaped rule. Now the rule of four numbers assumes the shape shown in Figure 4.

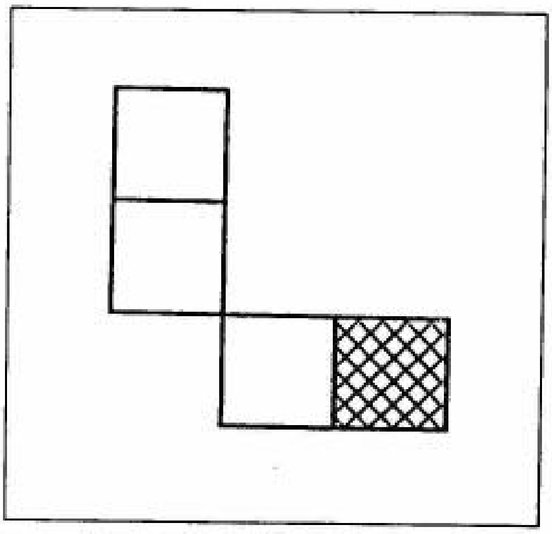

Figure 4. Visual imagery of the rule of four numbers, type 2

Continuing in this vein students may recognize recurring numbers within each column. For example, a few special cases provide the equalities

$$\binom{1}{0}=\binom{1}{1},\binom{2}{0}=\binom{2}{2},\binom{3}{1}=\binom{3}{2},\binom{4}{1}=\binom{4}{3},\binom{5}{1}=\binom{5}{4},\binom{5}{2}=\binom{5}{3}$$

etc. Then obviously the question arises: How to write down the general formula which includes these equalities as special cases? It is quite easy to arrive at the conclusion that cells containing the recurring numbers have the same first coordinate while the sum of their second coordinates equals the first. This results in the following conjecture:

$$\binom{n}{k}=\binom{n}{n-k} \tag{6}$$

Identity 6, equating the number of ways of choosing k objects from n given objects to that of choosing n-k objects from n is quite obvious. Indeed, the choice of k objects from n is equivalent to the choice of n-k remaining objects. By the way, the numerical approach enables a teacher to show students how Identity 6 could be proved through the method of mathematical induction. To this end it should be noticed that the template of combinations (see Figure 1(a)) provides an ideal medium for creating proofs that use mathematical induction through an understanding of how the numerical evidence has been generated. Actually, with regard to Identity 6, observing the column n=1 students read two nearest recurring numbers as the base of the induction. Then they recognize that the fact of the existence in a certain column, say in the n-th column, of the template of two recurring numbers results in another two recurring numbers in the column immediately to the right, that is the column with the number (n+1). This observation provides students with the numerical evidence of the recursion clause which constitutes the essence of mathematical induction. Now it is not hard to prove the conjecture about Identity 6 through this method. Actually, the base clause for Identity 6 can be seen on the template in the column which corresponds to n=1. Next, in order to complete the recursion clause students assume Identity 6 to be true for a certain value of n. This assumption together with the already proved Identity 4 yields

$$\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}=\binom{n}{n-k}+\binom{n}{n-k+1}=\binom{n+1}{n+1-k}$$

Identity 6 has been proved.

MODELLING RECURSION RELATIONS AS A METHOD OF TESTING COMBINATORIAL IDENTITIES

The teacher can also offer students different computer-based approaches for testing their conjectures through numerical evidence, each relevant to a particular type of proposed formula. The first approach (referred to below as the R-method) is based on the possibility of modelling recursion relations within a spreadsheet, and can be applied to test conjectures about Identities 4 and 5.

Starting with Identity 4 the teacher explains that this identity can be rewritten in the form of the following recursion relation (first-order equation of partial differences)

$$C(n,k)=C(n-1,k)+C(n-1,k-1) \quad (7)$$

subject to the boundary condition

$$C(0,k)=0,\ k\geq 1;\ C(n,0)=1,\ n\geq 0 \quad (8)$$

The spreadsheet **T-Id4** which implements the function C(n,k) as the solution of Equation 7 under Condition 8 is programmed as follows. The contents of rows **1**, **2** and columns **A**, **B** of the spreadsheet **T-Id4** coincide with those of the spreadsheet **TC**. The spreadsheet function **=B2+B3** is defined in cell **C3** and computes the sum C(0,0)+C(0,1). This function is replicated to cell **M12** by using the Copy and Paste commands. As a result the template is immediately filled up with numbers $\binom{n}{k}$. This completes the numerical proof of the students' conjecture about Identity 4. The template of the spreadsheet **T-Id4** and a "screen-snap" with the above spreadsheet formula which is embedded into it are shown in Figure 5.

The R-method can be applied to test the conjecture about Identity 5 through modelling the following second-order (in variable n) recursion relation

$$C(n,k)=C(n-1,k-1)+C(n-2,k-1)+C(n-2,k) \quad (9)$$

To specify the solution of Equation 9 it should be emphasized that when dealing with the second-order recursion relation in variable n students must define numbers C(n,k) not only for n=0 and k=0 but for n=1 as well. Modelling equation (9) results in the template filled up with numbers $\binom{n}{k}$.

k\n	0	1	2	3	4	5	6	7	8	9	10	11
0	1	1	1	1	1	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6	7	8	9	10	11
2	0	0	1	3	6	10	15	21	28	36	45	55
3	0	0	0	1	4	10	20	35	56	84	120	165
4	0				1	5	15	35	70	126	210	330
5	0				0	1	6	21	56	126	252	462
6	0				0	0	1	7	28	84	210	462
7	0	0	0	0	0	0	0	1	8	36	120	330
8	0	0	0	0	0	0	0	0	1	9	45	165
9	0	0	0	0	0	0	0	0	0	1	10	55
10	0	0	0	0	0	0	0	0	0	0	1	11

=B2+B3

Figure 5. The spreadsheet T-Id4

USE OF A SPREADSHEET FUNCTION INDEX IN TESTING COMBINATORIAL IDENTITIES

It should be useful to draw students' attention to the fact that Identity 6, being written down in the form of recursion relation (difference equation)

$$C(n,k)=C(n,n-k) \tag{10}$$

turns out to be of the order $|2k-n|$. To specify the solution of Equation 10, $|2k-n|$ boundary conditions in the second variable are required, i.e., $|2k-n|$ rows of a spreadsheet must be known and this, of course, is impossible. Proposing, therefore a different method to attack the problem the teacher introduces students to a new technique (referred to below as the I-method) of modelling numbers C(n,n-k) that uses the function **=INDEX** (*index range, row, column*). The above three arguments of this spreadsheet function have the following meaning (Cobb, McGuffey, & Dodge, 1991): *index range* indicates the given spreadsheet's array and its *range*, i.e., it refers to the upper left cell and the lower right cell; *row* and *column* indicate the coordinates of a certain cell within this spreadsheet's array and whose content we need to compute. Entering the function **INDEX** into an arbitrary cell of a spreadsheet results in computing the content of the cell referred to by its arguments *row* and *column* .

Therefore students can model numbers C(n,n-k) within a spreadsheet by entering into its cell (n,k) the function **INDEX** whose first argument corresponds to the index range of the **TC** array, and numbers of row and column equal *n-k* and *n* respectively (in accordance with Equation 10).

The spreadsheet **T-Id6**, which models numbers

$$\binom{n}{n-k}$$

by transforming the spreadsheet **TC** through the I-method, is programmed as follows. The contents of rows **1**, **2** and columns **A**, **B** of the spreadsheet **T-Id4** coincide with those of the spreadsheet **TC**. The spreadsheet function **=INDEX(TC!$C$3:$M$12,C$1-$A3,C$1)** is defined in cell **C3** and computes the value of $\binom{1}{0}$. This function is replicated to cell **J10** by using the

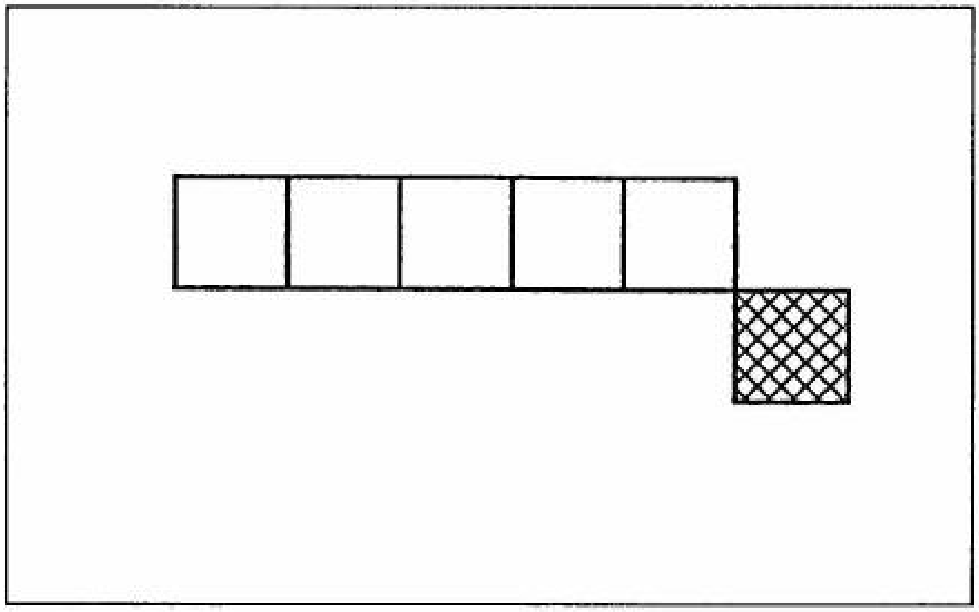

Figure 6. Horizontally-shaped chain of cells

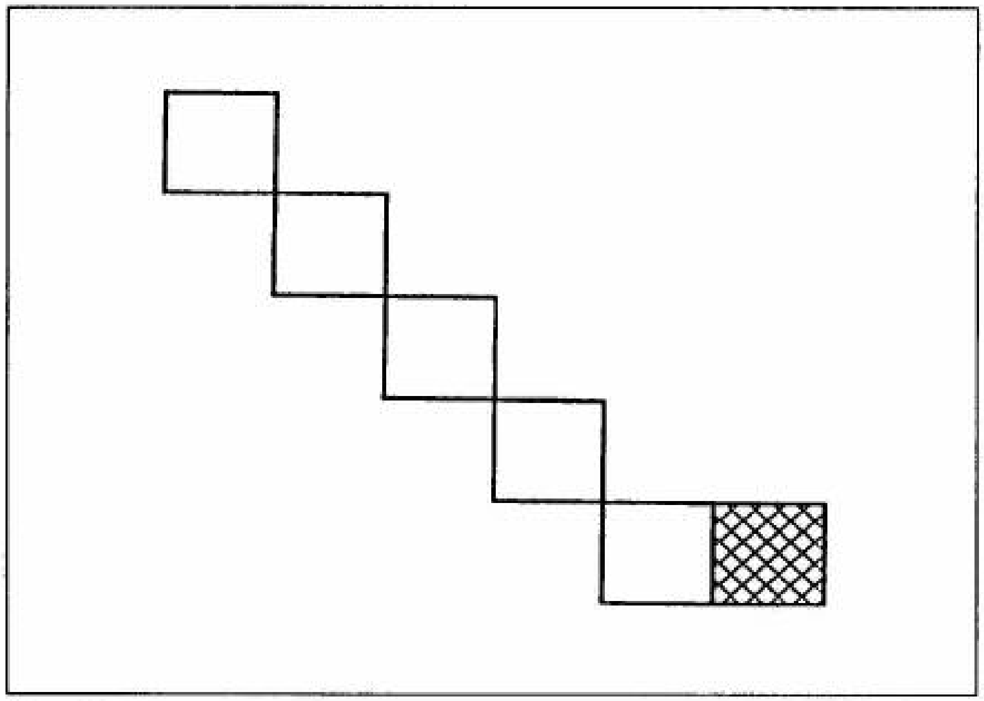

Figure 7. Diagonally-shaped chain of cells

Copy and Paste commands. As a result the template is immediately filled up with numbers $\binom{n}{k}$. In other words, by modelling numbers

$$\binom{n}{n-k}$$

we arrived at numbers $\binom{n}{k}$, thus confirming Identity 6 through numerical evidence.

IDENTITIES WITH EVOLVING LEFT-HAND SIDE

To pursue the discussion on recognizing patterns in more complicated shapes the teacher could sketch the shapes shown in Figures 6 and 7, asking for new links among numbers $\binom{n}{k}$. The horizontally-shaped chain of cells (see Figure 6) will prompt students to discover that adding up the contents of two, three, four, and so forth, cells along each row of the template, starting from the first non-zero element, results in the sum indicated in the cell below.

Moving along such chains with an evolving upper component results in the following general identity.

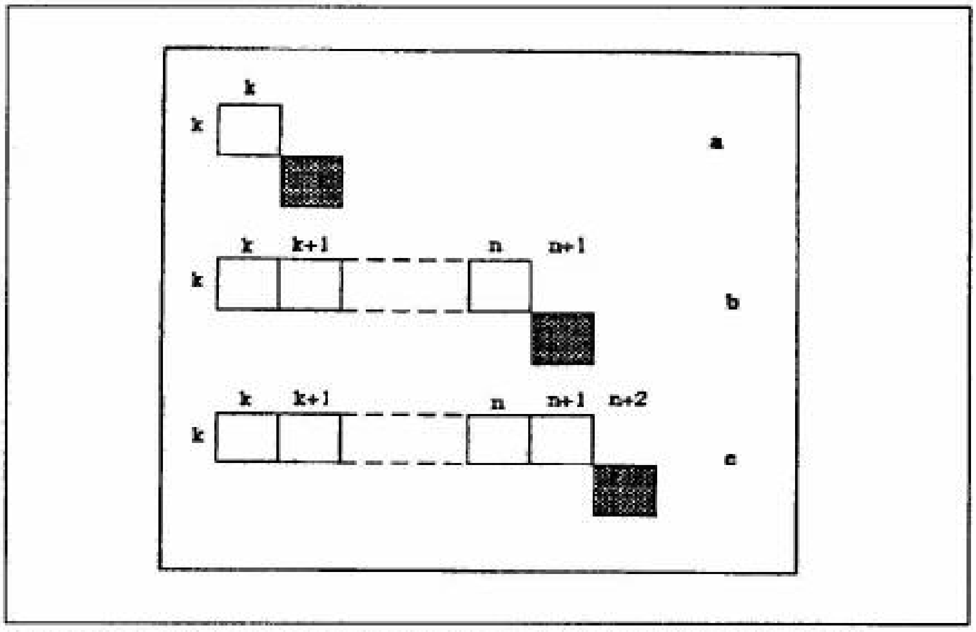

Figure 8 (a, b, c). The geometrical interpretation of the proof of Identity 11

$$\binom{k}{k}+\binom{k+1}{k}+\ldots+\binom{n}{k}=\binom{n+1}{k+1} \tag{11}$$

Note that the visual imagery associated with Identity 11 provides students with the geometrical interpretation of its proof through mathematical induction. Really, the base clause for Identity 11 constitutes the fact that in the template **TC** every left cell with the unity has the diagonally-adjacent cell also with the unity (see Figure 8(a)). The recursion clause is geometrically quite obvious (see Figures 8(b), and 8(c)) since, by virtue of the rule of the L-shaped triple, the assumption that the sum of *n* cells equals the sum indicated in the adjacent cell remains true when *n* is replaced by n+1.

Now it is not hard to complete the proof of Identity 11 through mathematical induction with respect to *n*. Actually, the equality

$$\binom{k}{k}=\binom{k+1}{k+1}$$

constitutes the base clause of the induction. Assuming Identity 11 to be true for a certain value of *n* results in the following transformation of the sum of (n+1) numbers:

$$\binom{k}{k}+\binom{k+1}{k}+\ldots+\binom{n}{k}+\binom{n+1}{k}=\binom{n+1}{k+1}+\binom{n+1}{k}=\binom{n+2}{k+1}$$

This completes the proof of Identity 11.

With regard to the diagonally-shaped chain (see Figure 7), after setting down the following equalities:

$$\binom{0}{0}=\binom{1}{0},$$

$$\binom{1}{0}+\binom{2}{1}=\binom{3}{1},$$

$$\binom{2}{0}+\binom{3}{1}+\binom{4}{2}=\binom{5}{2},$$

$$\binom{3}{0}+\binom{4}{1}+\binom{5}{2}+\binom{6}{3}=\binom{7}{3}$$

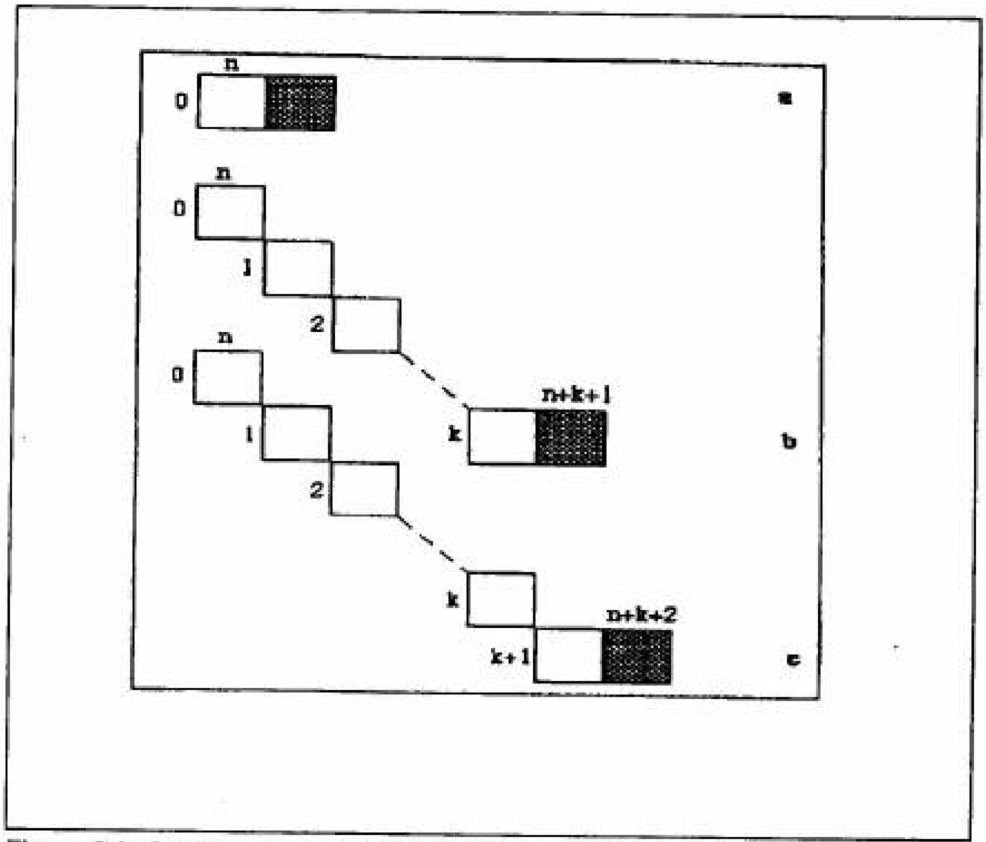

Figure 9 (a, b, c). The geometrical interpretation of the proof of Identity 12

students realize that these equalities relate to the cases k=n=0, k=n=1, k=n=2, k=n=3 respectively. Generalizing from these special cases leads students to the conclusion that the evolving sum for the general case with arbitrarily chosen *k* and *n* has the following performance

$$\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\ldots+\binom{n+k}{k}=\binom{n+k+1}{k} \qquad (12)$$

The geometrical interpretation of the proof of Identity 12 through mathematical induction (with respect to *k*) is shown in Figure 9(a) (the base clause: k=0) and in Figures 9(b), and 9(c) (the recursion clause).

This visual imagery enables students to carry out the proof of Identity 12 by using, as above, the rule of the L-shaped triple of numbers. Actually the transfer from *k* to k+1 can be performed as follows:

$$\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\ldots+\binom{n+k}{k}+\binom{n+k+1}{k+1}=\binom{n+k+1}{k}+\binom{n+k+1}{k+1}=\binom{n+k+2}{k+1}$$

Note that students could arrive at Identities 11 and 12 geometrically, when evolving an arbitrary L-shaped triple of cells on a template by successive application to itself of the rule of the L-shaped triple. Really, by starting from both the bottom left cell and the top cell alike one will easily obtain the shapes generating Identities 11 and 12 respectively thus demonstrating the art of counting without counting.

A FEW MORE DISCOVERIES: FIBONACCI NUMBERS AND THE GOLDEN RATIO

A natural sequel to these studies is to investigate the top right and bottom left diagonals of the template (see Figure 10).

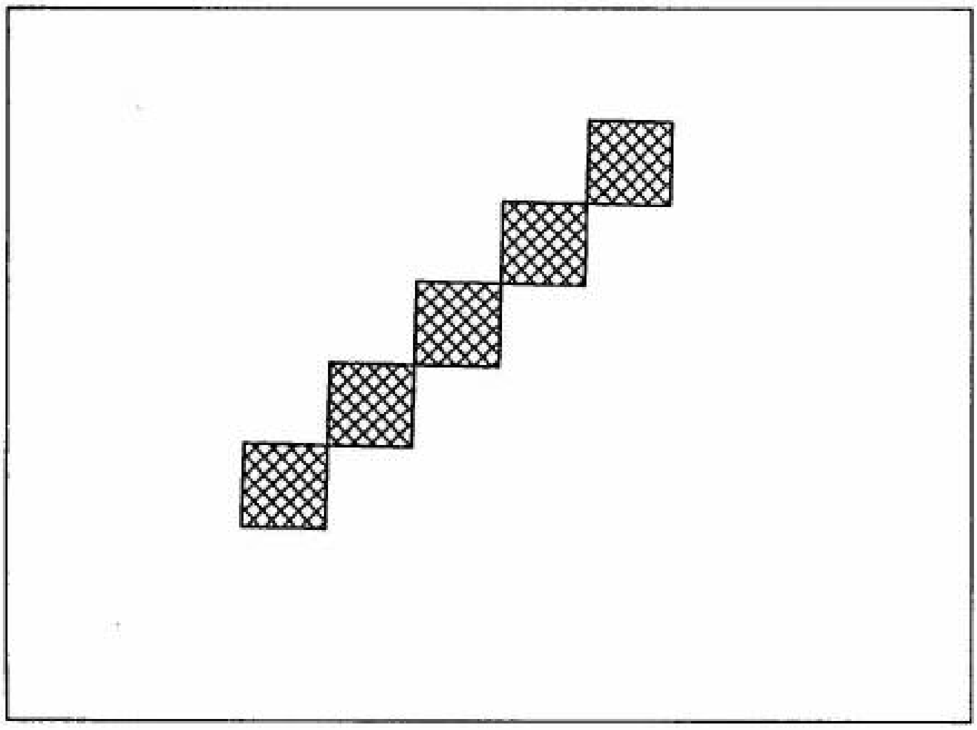

Figure 10. Top right and bottom left chain of cells

The teacher may ask students to write down the sums of non-zero elements along each diagonal in terms of $\binom{n}{k}$ beginning from cell n=0, k=0. In doing so students will arrive at the following sequence of sums:

$$F_1=\binom{0}{0}, F_2=\binom{1}{0}, F_3=\binom{2}{0}+\binom{1}{1}, F_4=\binom{3}{0}+\binom{2}{1}, F_5=\binom{4}{0}+\binom{3}{1}+\binom{2}{2}, F_6=\binom{5}{0}+\binom{4}{1}+\binom{3}{2},$$

etc.

They might guess that each term of this sequence beginning from the third equals the sum of its two precedents. To prove this conjecture the teacher asks students to write down three arbitrary subsequent terms

$$F_{n-1}=\binom{n-1}{0}+\binom{n-2}{1}+\binom{n-3}{2}+\ldots,\ F_n=\binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\binom{n-3}{3}+\ldots,$$

$$F_{n+1}=\binom{n+1}{0}+\binom{n}{1}+\binom{n-1}{2}+\binom{n-2}{3}+\ldots,$$

and to apply Identity 4 to F_{n+1}. This yields

$$F_{n+1}=\binom{n+1}{0}+\binom{n-1}{1}+\binom{n-1}{0}+\binom{n-2}{2}+\binom{n-2}{1}+\binom{n-3}{3}+\binom{n-3}{2}+\ldots,$$

Comparing F_{n+1} written in the expanded form with F_{n-1} and F_n results in the following recursion relation

$$F_{n+1}=F_n+F_{n-1} \tag{13}$$

which by virtue of equalities

$$F_1=1,\ F_2=1 \tag{14}$$

represents the recursive definition of the celebrated Fibonacci numbers. These numbers possess a variety of remarkable properties. One property—constituting a link between the Fibonacci numbers and combinations—is the following:

$$F_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \ldots + \binom{n-k}{k} \quad (15)$$

where $k = \frac{n}{2}$ if n is even and $k = \frac{n+1}{2}$ if n is odd.

The importance of the present teaching sequence is that using a spreadsheet has made it possible for students to discover Formula 15 by *themselves*. The teacher may provide students with a combinatorial interpretation of this formula by formulating the following problem: *An n-storied building is given and you must color it with a fixed color in such a way that no consecutive stories are colored the same—how many different ways of coloring are possible?* To answer this question the combinatorial

n	*Fn*	*Fn/Fn+1*
1	1	0.500000000
2	2	0.666666667
3	3	0.600000000
4	5	0.625000000
5	8	0.615384615
6	13	0.619047619
7	21	0.617647059
8	34	0.618181818
9	55	0.617977528
10	89	0.618055556
11	144	0.618025751
12	233	0.618037135
13	377	0.618032787
14	610	0.618034448
15	987	0.618033813
16	1597	0.618034056
17	2584	0.618033963
18	4181	0.618033999
19	6765	0.618033985
20	10946	0.618033990
21	17711	0.618033988
22	28657	0.618033989
23	46368	#DIV/0!

Figure 11. The spreadsheet **TF**: Fibonacci numbers and the golden ratio

meaning of combinations can be used which will lead students to the conclusion that there exist

$$\binom{n-k}{k}$$

permitted ways of coloring an (n-1)-storied building so that exactly *k* stories turn out to be colored with the fixed color. Therefore, in accordance with Formula 15, one, two, three, four, etc.-storied buildings can be colored in F_3 (=2), F_4 (=3), F_5 (=5), F_6 (=8), etc. ways.

At last, students may be offered to model the sequence of sums F_n by implementing Equation 13 subject to Condition 14 within a spreadsheet. To this end positive integral values of *n* are defined in column **A** of the spreadsheet **TF** (see Figure 11). The base clause for F_n is defined in cells **B2** and **B3**. The spreadsheet function =**B2+B3** is defined in cell **B4** and computes the value of F. This function is copied down by using the Copy and Paste commands. As a result the sequence of Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, ... immediately occurs on the screen.

Finally, by entering into cell **C2** the spreadsheet function = **B2/B3** and by replicating this function the teacher introduces students to a behavior of the ratio

$$\frac{F_n}{F_{n+1}}$$

as *n* increases. (Note that here students must switch the General regime in the line Numbers from the Format menu). This ratio seems to approach the number 0.6180339... which is referred to after Leonardo da Vinci as the golden ratio, a number that often appears in the study of geometry, architecture and art.

DISCUSSION

Following are some remarks designed to answer the challenge of Kaput (1991): ''Will the technology help us do better what we have been trying to do ?'' (p.548).

1. With regard to implementing a spreadsheet as a technological tool in the classroom to enhance the teaching/learning of combinatorics—the branch of mathematics concerned with the theory of enumeration—it

has already been mentioned above that the common method of attacking such problems is through "recursive definition". That is why a spreadsheet whose recurrent nature emphasizes its methodological effectiveness, as applied to the subjects and concepts involved, may help instructors in their efforts to reinforce teaching and to promote learning.

2. The advantage of the numerical approach to basic concepts of algebra was demonstrated by Demana and Leitzel (1988). The same is true for combinatorics. A spreadsheet permits to observe at once the array of numbers $\binom{n}{k}$, of as large a size as students wish, whilst it at the same time provides an interactive learning environment where learners can exercise their own creativity (Lawer, 1987). The rich source of modelling data can help students find regularities in many special cases and generalize from these cases.
3. The feature of the spreadsheet that contributes to its pedagogical usefulness is its capability to make students investigate through visualization. The spreadsheet-oriented teaching of combinatorics makes it possible to support learning relationships among integers with combinatorial meaning, with visual imagery. As is shown above (see also Comtet's (1974) diagrams and Hilton and Pedersen's (1987) patterns in the Pascal triangle) different shapes arising from a template provide definite rules (combinatorial identities) which constrain these numbers. The simple transformation of these shapes enables students not only to invent one identity after another, and thereby to do mathematics but, better still, to do mathematics without computations. This actually opens up the possibility of making the study of enumerative combinatorics an exciting challenge for independent investigation.
4. The last section of the article demonstrates merely one example of linking integers with combinatorial meaning to different discrete topics. The rich store of these numbers that results from the spreadsheets' computational capability provides an excellent opportunity for students to arrive unexpectedly at the Fibonacci numbers, so that they may recognize the capacity of these numbers to solve combinatorial problems. This, in turn, helps teachers to make students experience the contiguity of different concepts of discrete mathematics and to appreciate its integrity—a significant source in the development of mathematical knowledge.

To conclude, it should be noted that teaching to contrive combinatorial identities with associated visual imagery can introduce students to the art of counting without counting. While working with the above method, however, combinatorics becomes not only the visual area of discrete mathematics, but it also exemplifies the topic which contributes to students' becoming aware of their personal ability to do mathematics. The mathematical technique needed for these activities is not complex yet the results are powerful because they help students to appreciate mathematics, to communicate and to reason mathematically. We believe moreover that the proposed approach will actually help teachers to come up with skillful, enjoyable, and stimulating instruction.

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Note

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