Cognitive Complexity of Boolean Problems

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Abstract

Assessing the level of complexity of Boolean problems takes into account factors of: a) complexity of an initial representation of a Boolean concept; b) characteristics of a specific type of basic blocks used for implementing the Boolean concept; and c) optimization criteria (such as: hardware minimization, power minimization, testability, security etc.). While the above factors are widely known and well studied, cognitive complexity of Boolean problems was never considered as a factor. At the same time, the problem of human concept learning is an established research area. The problem of Boolean concepts learning by humans is especially interesting since it studies human understanding of modern digital environment. The cognitive complexity is the main issue in this field. Our paper focuses on the problem of assessing the cognitive complexity of Boolean problems; reviews known research results and proposes a conceptual framework for future research in the corresponding field.

1. Introduction

The problem of complexity of Boolean problems has always been associated with the problem of synthesis of digital systems [18,23]. It is understood, since the interest to the Boolean problems was born together with computers and grew together with total computerization. Therefore it is not surprising that up to now, engineers were the ones who defined the measures for evaluating the complexity. The interest in synthesis problems was based on an initial assumption of the principle possibility of a "thinking automaton", which stems from the revolutionary ideas of Alan Turing. Therefore, the synthesis task, i.e. the task of building a scheme according to an initial Boolean description, was considered as main and natural.

Today, the situation has changed. The task of creating a machine capable of thinking as a human is considered neither a real practical technological problem, nor a hot research topic anymore. The most important thing is that the task itself looses its significance at the level of social consciousness.

A similar situation can be noticed also in the pattern recognition task [2]. Traditionally, the pattern recognition is considered as a part of machine learning that deals with algorithms for effective understanding specific patterns by computers but not by humans. Recently, along with the traditional machine learning, Human Concept Learning [7] becomes the subject of intensive development.

Success of modern technologies puts a human personality in the center, and the contemporary problem is actually how she/he develops her/his abilities to exist in a new informational society full of intellectual artifacts. Nowadays, the above modern problem comprises, for example, a phenomenon of "data intensive science" [10], so that efforts of engineers and scientists are invested to ensure forming, processing, transmitting and absorbing of colossal volumes of content. In these new conditions, the problem of complexity of any concepts - and Boolean concepts as well - becomes more and more a cognitive problem rather than a technological one.

It is quite logical that scientists have become interested in a simply formulated question: *are there properties of Boolean functions that would allow a human to effectively (i.e., with minimal efforts) learn (understand) tasks corresponding to such functions?* In 1999, J. Feldman published a work [7] where he showed that the cognitive complexity of a Boolean concept is proportional to the number of literals in the minimized (AND-OR-NOT) Boolean expression corresponding to the Boolean concept. The pioneer work of Feldman in the field of Boolean Cognitive Complexity actually accelerated scientific activity in that field.

In quite a short time after the mentioned Feldman's work, a number of works were published which criticized and even refuted the result (criterion) obtained by Feldman. The criticism might be explained, for example, by the fact that symmetric functions, which could be recognized by a human quite easily, have high complexity by Feldman (it should be mentioned, however, that the symmetric functions have low value of the Kolmogorov complexity [12, 13]).

Another factor for criticizing the Feldman's criterion was its insensibility to humans' ability to think in a non-traditional basis. For example, if a function has a simple expression in the XOR basis, it may simplify perception of the function by a human, so that the function is recognized quite easily. Some works proposed novel and more accurate methods for estimating the Boolean cognitive complexity [8, 21, 22, 14]

In this context, we find the following three research directions of a special interest:

1) Influence of properties of Boolean concepts on cognitive complexity of Boolean functions;

2) Cognitive complexity of one and the same Boolean concept when used for solving problems of different types. Two types of problems are of a special interest – recognition [2] and reverse engineering [6,11].

3) Approximation of Boolean problems as a tool used by a human for reducing cognitive complexity of problems.

2. Cognitive Complexity of Boolean Functions

In this section we briefly introduce significant milestones in the field of Cognitive Complexity.

Concepts are the atoms of thought and they are therefore at the nucleus of cognition science [9]. People begin to acquire concepts from infancy and continue to acquire and plan new concepts throughout their entire lives [15, 16]. One way to create a new concept is by utilizing existing concepts in different combinations. One of the problems in learning concepts is determining the concept's subjective difficulty. An important aspect of concept learning theory is the ability to predict the level of difficulty in learning different types of concepts. In this respect, Boolean concepts are the fundamental topic in the literature. Boolean concepts can be defined by a Boolean expression composed of basic logic operations: negation, disjunction, and conjunction. These types of Boolean concepts have been studied extensively by [20, 3, 4, 17]. These studies were focused on Boolean concepts with three binary variables. Shepard, Hovland, and Jenkins (SHJ) in [20] defined 70 possible Boolean concepts that can be categorized into six subcategories. The six structurally equivalent subcategories can be described in a geometrical representation using cubes (Figure 1). This notion of congruence has been firstly introduced in [1] and became prevalent in the literature on the theory of switching circuits. It was introduced into psychology in [20].

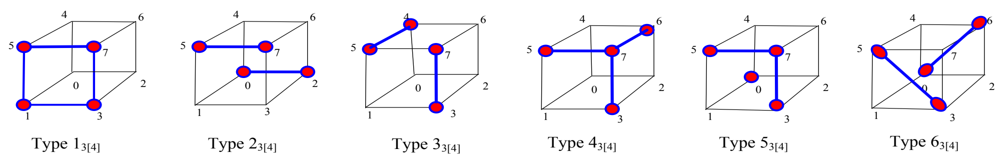

Figure 1: Shepard, Hovland, and Jenkins (SHJ) category types.

Concept subcategories [20] with structural equivalence belong to the same category and are defined as a Type. The study results pertaining to the six types of concepts presented in Fig. 1 showed that the concepts belonging to Type 1 are the simplest to learn and the subgroup of concepts belonging to Type 6 are the most difficult, according to the following order:

Type 1 < Type 2 < (Type 3, Type 4, Type 5) < Type 6.

The results of [20] are highly influential since SHJ proposed two informal hypotheses; the first is that the number of literals in the minimal expression predicts the level of difficulty. The second hypothesis is that ranking the difficulty among the concepts in each type depends on the number of binary variables in the concept. It is interesting to note that these two relations were originally introduced by Shannon and Lupanov in the context of (asymptotic) complexity of Boolean functions. (Refer the introduction to complexity in [23]).

Feldman [7], based on the conclusions from the SHJ study, defined a quantitative relationship between the level of difficulty of learning Boolean concepts and the concept's Boolean complexity. In [7], *D[P]* indicates a family of Boolean functions with D variables and weight P. In his study, Feldman examined the *3[2], 3[3], 3[4], 4[2], 4[3]* concept family. Feldman also addressed the family of concepts where the number of combinations with "1" differs from the number of combinations with "0". The complexity measure of a Boolean concept as defined by Feldman is the number of literals in the minimal SOP expression. The complexities of functions from each type are given in Table 1.

Determining the Boolean concept's complexity as a minimal number of literals in a minimal expression creates a number of problems. The first: since a) the complexity is defined as the number of literals in the minimal expression and b) the expressions can be minimized by using different techniques, a uniform complexity measure cannot be obtained. For example: according to Feldman's heuristics, Types 3, 4, 5 are of the same complexity. Contrary to Feldman, in the correct minimal expression, concepts from Types 2 and 3 have the same complexity.

Table 1: Six SHJ concepts and their complexities

The SHJ six concepts	minimal expression	complexity	minimal descriptions (with XOR)	complexity	structural complexity
$Type1_{3[4]}$	$\overline{a}$	1	$\overline{a}$	1	1.66
$Type2_{3[4]}$	$ab+\overline{a}\,\overline{b}$	4	$\overline{a\oplus b}$	2	2.00
$Type3_{3[4]}$	$\overline{a}\overline{c}+\overline{b}c$	4	$\overline{a}\overline{c}+\overline{b}c$	4	2.14
$Type4_{3[4]}$	$\overline{c}\left(\overline{a}+\overline{b}\right)+\overline{a}\overline{b}$	5	$\overline{c}\left(\overline{a}+\overline{b}\right)+\overline{a}\overline{b}$	5	2.14
$Type5_{3[4]}$	$\overline{a}\left(\overline{b}\overline{c}\right)+abc$	6	$\overline{a\oplus\left(bc\right)}$	3	2.34
$Type6_{3[4]}$	$a\left(\overline{b}c+\overline{c}b\right)+\overline{a}\left(\overline{b}\overline{c}+bc\right)$	10	$\overline{a\oplus b\oplus c}$	3	4.00

The second problem: studies show that XOR as an operator is learned and acquired as a concept in human thought to the same degree or even more easily than OR [5]. By using the XOR, some of the Boolean expressions can be simplified significantly and therefore, the cognitive complexity decreases in comparison with the Feldman's complexity (without XOR). In light of the above problems, Feldman and Vigo suggested alternative approaches for estimation of the cognitive Boolean complexity.

In 2006 Feldman [8] introduced his spectral decomposition approach. The basic idea of the approach is that learning from examples involves the extractions of patterns and regularities. The formal model describes how a pattern (expressed in terms of a Boolean rule) may be decomposed algebraically into a "spectrum" of component patterns, each of which has a simpler or more "atomic" regularity. Regularities of a higher degree represent more idiosyncratic patterns while regularities of a lower degree represent simpler patterns in the original decomposed pattern. There are two kinds of simple concepts: those that consist of a constant or a single variable, and those that consist of an implication between the values of two variables. These two basic types of concepts can be algebraically combined to represent more complex linear concepts. It is based on an analysis of a concept's "power spectrum". Thus, any concept can be decomposed into a set of underlying rules, each of differing degrees of complexity, depending on the number of variables. The complexity of underlying rules provides an overall index of the *algebraic complexity* of the concept that characterizes an ability to explain unique phenomena of the learning process. The algebraic complexity makes it possible to rank each concept combination according to its structure.

An alternative approach for calculating the cognitive complexity measure of a Boolean concept was proposed in [22] and called a *structural complexity*. The approach is based on the Boolean derivative [19]. The question that the approach is supposed to address is: What is the internal structure of Boolean concepts from any category that makes them harder to learn than concepts from a different category? For the purpose of quantitative definition of the structural complexity, the degree of categorical invariance has to be calculated.

Vigo's calculation of the invariance of concepts does not specify how individuals learn concepts. He suggests only that cognitive processes could detect invariances by comparing a set of instances to the set yielded by the partial derivative of each variable. For SHJ's six categories, the complexity measures described above appear in Table 1.

3. Research Framework of Boolean Concept Learning

The cognitive complexity of a Boolean concept may be assessed using various methods. The methods differ mathematically and are intended to reflect humans' ability to recognize, describe, analyze and synthesize systems corresponding to the Boolean concepts. The history of development of the Boolean Concept Learning forecast that adequacy of one or another method for determining cognitive complexity will, most probably, be assessed empirically – i.e., similarly to the Feldman's approach.

Moreover, one may assume that any further studies in that field will be purely empirical. The main objective of our work is to claim that the above is wrong, in the meaning that exact methods can be proposed and effectively used in integration with empiric ones.

In our opinion, both the theoretical depth and the practical value of Boolean Concept Learning can be defined by a combination of factors, which relate to:

a) properties of the Boolean concepts;

b) types of the Boolean problems to be solved;

c) techniques used by humans to reduce cognitive complexity by approximation.

Properties of Boolean functions are studied in the Boolean Concept Learning in order to investigate the humans' ability to recognize some known features of the Boolean functions for reducing the cognitive complexity of the problem. Such properties as monotony, linearity, and symmetry of Boolean concepts have to be studied in the above context. Preliminary results obtained by our research team clearly indicate an ability of people to recognize the symmetry property.

Study of Boolean Concept Learning on different types of Boolean problems allows broadening the field of expertise, bringing an additional dimension to the concept of cognitive complexity. There are at least the following known types of Boolean problems: recognition, reverse engineering, fault diagnostics. Our preliminary results indicate a correlation between complexities of problems of different types for the same Boolean concept.

For studying the techniques used by humans for reducing cognitive complexity of Boolean problems by "approximation", a so-called non-exact reasoning approach has to be developed. For example, let us consider mental models of the algorithm of a specific elevator. In our study, the mental models were compared with the real algorithm of the elevator. The preliminary results of the comparison show that people have common cognitive style of "approximation" in constructing their mental models of a device implementing a Boolean function. Methods for measuring: a) the distance between the approximated functionality and the real functionality; b) the gain in cognitive complexity and c) the application to the recognition and reverse engineering are yet to be studied.

Based on the mentioned above, we propose an approach for study of the phenomenon of Boolean Concept Learning. The approach considers the issue based on a three-dimensional model that we propose to build, where axes of the model correspond to the mentioned groups of factors. The first axis of the model will be formed by features of Boolean functions reflected in their properties; the second axis will reflect the type of a Boolean problem (such as: recognition, reverse engineering) and the third one – non-exactness of human reasoning.

4. Conclusions

The paper deals with a specific research field - Boolean Concept Learning, in particular cognitive complexity of Boolean problems, which has become actual in the recent years. The cognitive complexity enriches the tradition concept of Boolean complexity by focusing on human centered issues but not just on technological ones. Traditionally, the field has been considered as cognitive science staying far from exact methods. We have defined three groups of factors, which open new directions for further research in the field. The three defined groups of factors have allowed formulating a model of the subject matter. In view of the proposed model, the phenomenon of Boolean Concept Learning has a perspective to be studied not only by using conventional cognitive science methods, but also by analytical methods. This may bring scientists from the classical Logic Design to research in the new field.

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