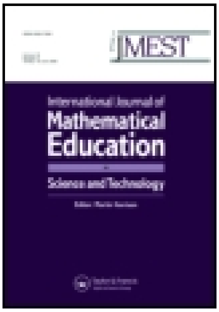


International Journal of Mathematical Education in Science and Technology
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tmes20

Spreadsheets in teaching and learning topics in calculus

Sergei Abramovich [a] & Ilya Levin [b]
[a] Department of Mathematics Education , The University of Georgia , 105 Aderhold Hall, Athens, GA 30602, USA
[b] School of Education , Tel Aviv University , 69978 Ramat Aviv, Tel Aviv, Israel
Published online: 09 Jul 2006.

INT. J. MATH. EDUC. SCI. TECHNOL., 1994, VOL. 25, NO. 2, 263–275

Spreadsheets in teaching and learning topics in calculus

by SERGEI ABRAMOVICH
Department of Mathematics Education, The University of Georgia, 105 Aderhold Hall, Athens, GA 30602, USA

and ILYA LEVIN
School of Education, Tel Aviv University, 69978 Ramat Aviv, Tel Aviv, Israel

(*Received 30 October 1992*)

The use of spreadsheets in introducing students to the concept of a limit of a sequence is demonstrated and the possible computer-based scenario as the enhancement of the teaching/learning process of calculus is exemplified. It is shown how the spreadsheet's operational capability assists visualizing the Bolzano–Cauchy principle of convergence and leads eventually to the possibility of employing computer technology in deciding the convergence of positive series.

1. Introduction

It is hard to overrate the impact of electronic technologies on mathematics education in general [1, 2] and of the use of the spreadsheet program to assist in the study of various topics at different mathematical levels in particular [3–7]. The aim of this article is to illustrate the spreadsheet's effectiveness as a pedagogical tool in calculus lessons when teaching the concept of a limit of a sequence and addressing some related issues.

The concept of a limit of a sequence is of great importance and permeates the whole course of calculus. Teaching experience and research [8–10] show, however, that usually this concept is not clear enough to most students. Unfamiliar in comparison with pre-calculus notions, and thus abstruse when first encountered, the exact definition of a limit of a sequence involves the characters ε and N accompanied by existential and universal quantifiers. This, actually, signifies a move to advanced mathematical thinking [11]. The fact that in average conditions learners have such difficulty in comprehending this definition is caused by the very form of the definition which uses symbols of predicate calculus, with hardly any visual imagery.

The main concern of the work of the authors in this direction, originally presented as a conference paper [12], is the issue of modelling, visualization and exploration. As is known, a spreadsheet makes it possible to model sequences in the form of a column (row) of numbers by replicating a formula which determines a sequence. This provides an opportunity, given a certain sequence, to offer students a visualization of the behaviour of a finite but sufficiently large number of its members in a numerical form that is simple to understand. The spreadsheet is a powerful tool, with its remarkable capacity of recurrent counting and operation, and gives the opportunity to challenge and stimulate students—this, together with the rationale for teaching iteration sequences expounded by Weigand [13] determined the form of a sequence representation in favour of the iteration formula in this article.

We assume that students have basic skill in operating a spreadsheet and defining functions in cells [14]. Thus we can assume students to be able to implement the following computer-based scenario during a lesson, and under teacher guidance:

(1) Development of the spreadsheet template
(2) Observing the behaviour of the sequence on the spreadsheet template through a teacher-student dialogue that will eventually lead to the exact definition of a limit of a sequence
(3) Verification of the definition of a limit of a sequence followed by the formulation of its negation

Before proceeding to discuss the above scenario, one remark advocating the spreadsheet as a pedagogical tool is needed. The point may be raised that a canned program made up by the instructor before the lesson could have more advantages since it does not require students to be skilled in operating the spreadsheet. But this is not so. In the spirit of Lawler [15] the spreadsheet is an open interactive learning environment where students can exercise their own creativity. Moreover spreadsheet-oriented teaching, using this technological tool as an explorer [16], boosts students' constructive thinking [17] by giving them opportunity to engage in exploratory mathematics. In contrast, the use of a canned program actually is an example of passive rendering of mathematics knowledge.

2. Representing the sequence in the spreadsheet

The remarkable feature of the spreadsheet software is that it can replicate formulae entered into cells. So an iteration sequence defined by the iteration formula can be represented in the spreadsheet as follows:

(1) The given first term of a sequence is entered in an arbitrarily chosen cell of a certain column of the spreadsheet.
(2) Following the given iteration formula, the second term of a sequence is entered in the next cell of this column below the chosen cell.
(3) The spreadsheet options Copy and Paste replicate the iteration formula down this column.

After these steps have been taken the column is immediately filled up with the terms of a sequence.

3. Definition of a limit of a sequence

Consider the sequence

$$X_{n+1}=\frac{X_n}{3}+2 \tag{1}$$

Note that formula (1) does not express the sequence in a closed form as a function whose independent variable varies over the set of all positive integers. Formula (1) only permits one to compute any member of the sequence in terms of a previous one. Hence a teacher should specify the value of the first term (initial value). The very choice of an initial value is arbitrary in general. So let

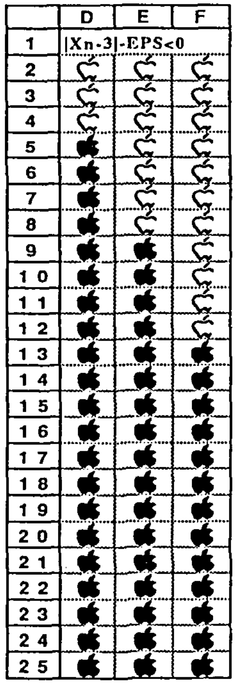
$$X_1=10 \tag{2}$$

	A	B
1	n	Xn
2	1	10
3	2	5.33333333
4	3	3.77777778
5	4	3.25925926
6	5	3.08641975
7	6	3.02880658
8	7	3.00960219
9	8	3.00320073
10	9	3.00106691
11	10	3.00035564
12	11	3.00011855
13	12	3.00003952
14	13	3.00001317
15	14	3.00000439
16	15	3.00000146
17	16	3.00000049
18	17	3.00000016
19	18	3.00000005
20	19	3.00000002
21	20	3.00000001
22	21	3
23	22	3
24	23	3
25	24	3

(*a*)

	C
1	|Xn-3|
2	7
3	2.3333333
4	0.7777778
5	0.2592593
6	0.0864198
7	0.0288066
8	0.0096022
9	0.0032007
10	0.0010669
11	0.0003556
12	0.0001185
13	3.952E-05
14	1.317E-05
15	4.391E-06
16	1.464E-06
17	4.878E-07
18	1.626E-07
19	5.42E-08
20	1.807E-08
21	6.023E-09
22	2.008E-09
23	6.692E-10
24	2.231E-10
25	7.435E-11

(*b*)



(*c*)

Figure 1.

At the very beginning a spreadsheet with only two columns, say **A** and **B**, is needed (see Figure 1(*a*)). In column **A** current numbers of the terms of the sequence are defined. Then, the values of these terms will be located in column **B**. To do this one must define the sequence X_n in column **B**. The first term (2) is defined in cell **B2**. Following iteration formula (1) the spreadsheet function $=(\mathbf{B2})/3+2$ is defined in cell **B3** and computes the value of X_2. This function is copied down into cell **B25**. As a result the computed values of the sequence X_n immediately occur on the screen. The results of modelling indicate that the third term already has 3 as its integral part. Furthermore, while the integral part does not change, the figures in the fraction part flash across the lines of the template and gradually vanish.

The following imaginary dialogue between a teacher and a student is an example of a desirable teaching/learning process of the concept of a limit of a sequence by using the possibility of modelling, visualization and exploration within a spreadsheet template.

Teacher: What is the behaviour of sequence X_n while n increases?

Student: Sequence X_n approaches number 3 while n increases.

Teacher: What does it mean that sequence X_n approaches 3?

Student: X_n gets closer and closer to 3.

Teacher: What describes the degree of closeness of two numbers a and b, or in other words the distance between them?

Student: The closeness of two numbers a and b, or the distance between them, is either $a–b$ or $b–a$, whichever is not negative, i.e. the absolute value $|a–b|$ represents the distance between a and b.

Here a teacher must briefly interrupt the dialogue in order to create a third column **C** in the spreadsheet (see Figure 1 (*b*)) for computing the absolute value $|X_n - 3|$ as the degree of closeness between X_n and 3 in accordance with the student's definition of closeness. To this end the spreadsheet function =**ABS(B2**-3) enters into cell **C]** and is copied down into cell **C25**.

After the computations in the column **C** have been carried out, the dialogue continues again.

Teacher: What does it mean that sequence X_n more and more approaches the number 3?

Student (having examined column **C**):
The value that expresses the degree of their closeness, i.e. the absolute value $|X_n - 3|$ gets smaller and smaller.

Teacher: What do you mean by 'smaller and smaller'?

Student: However small this absolute value may seem to us, at the following step it becomes even smaller.

Teacher: What do you mean by 'however small it seems to us'? Is it true that everyone has his own concept of smallness?

Student: Of course, everyone has his own concept of smallness.

Teacher: Well, in accordance with our sense of smallness let us point out three values of a certain number, say ε, $\varepsilon = 0{\cdot}5$, $\varepsilon = 0{\cdot}005$ and $\varepsilon = 0{\cdot}00005$ and check whether the condition $|X_n - 3| < \varepsilon$ holds true or not.

At this point, the teacher introduces the complementary columns (see Figure 1(c)) for calculating values (TRUE or FALSE) of the predicate $|X_n - 3| < \varepsilon$. (The symbols and Ḉ indicate TRUE and FALSE respectively.) To this end, the spreadsheet functions =**IF($C2**$-0{\cdot}5<0$, "", "Ḉ"), =**IF($C2**$-0{\cdot}005<0$, "", "Ḉ"), =**IF($C2**$-0{\cdot}00005<0$, "", "Ḉ"), are defined in cells **D2**, **E2**, **F2**, respectively, and are copied down as shown in Figure 1(*c*).

With recourse to the spreadsheet one may note that when $\varepsilon = 0{\cdot}5$ the values of this predicate become TRUE beginning from $n = 4$; when $\varepsilon = 0{\cdot}005$, beginning from $n = 8$; and when $\varepsilon = 0{\cdot}00005$, beginning from $n = 12$. It turns out that for any given positive number ε one can always come across such a number N (which depend on ε such that when ε is smaller then N is larger) that for any $n > N$ the inequality $|X_n - 3| < \varepsilon$ will be true (though we are unable to try all εs, working with the spreadsheet permits us to describe our impressions). In such a case it is usually said that number 3 is the limit of the sequence X_n.

After this detailed preliminary discussion of the results of modelling, students are able to give the in fact already stated exact definition of a limit of a sequence.

Definition 1. L is called the limit of the sequence X_n if for any given positive number ε there exists a positive integer number N such that $|X_n - L| < \varepsilon$ for all n greater than N.

4. Verification of the definition of a limit

The next question that the teacher could propose to students is as follows: may any other number, say 4, be the limit of sequence (1), (2)? To answer this question students are asked to check the above definition 1 for number 4. This results in choosing an arbitrary positive number (the words 'for any' allow for this possibility), say $\varepsilon = 0{\cdot}8$, and then trying to seek for this ε a number N such that the absolute value $|X_n - L|$ by $n = N$ has become—and for all n greater than N remains—less than ε.

	A	B	C	D
1	n	Xn	|Xn-4|	|Xn-4|-0.8<0
2	1	10	6	Ç
3	2	5.333333	1.333333	Ç
4	3	3.777778	0.222222	
5	4	3.259259	0.740741	
6	5	3.08642	0.91358	Ç
7	6	3.028807	0.971193	Ç
8	7	3.009602	0.990398	Ç
9	8	3.003201	0.996799	Ç
10	9	3.001067	0.998933	Ç
11	10	3.000356	0.999644	Ç
12	11	3.000119	0.999881	Ç
13	12	3.00004	0.99996	Ç
14	13	3.000013	0.999987	Ç
15	14	3.000004	0.999996	Ç
16	15	3.000001	0.999999	Ç

Figure 2.

The spreadsheet testing the inequality

$$|X_n - 4| < 0{\cdot}8 \qquad (3)$$

is shown in Figure 2 and indicates that inequality (3) holds true only with $n=3$ and $n=4$, but is impossible to retain (3) when n increases. However, though such an N for $\varepsilon = 0{\cdot}8$ is not to be found, it might be possible to do this for other values of ε. Fortunately, there is no further need to seek ε because it is sufficient to specify only one ε for which the definition of a limit of a sequence does not hold true as this definition requires implementing inequality (3) which, though beginning from a certain number $n=N$, must take any given positive ε. In doing so the teacher leads students to the conclusion that number 4 is not the limit of sequence (1), (2).

In order to formulate what it means that number L is not the limit of the sequence X_n it would be helpful to emphasize that having arranged within the spreadsheet the verification of inequality (3) for sequence (1), (2), students learnt that if in some row of the template the symbol has appeared (inequality (3) holds true) then necessarily there exists a row below containing the symbol Ç (i.e. inequality (3) is not valid). In other words we were able to specify a number $\varepsilon = 0{\cdot}8$ such that for any N (the number of the row) there exists $n > N$ (the row with a larger number) for which inequality (3) is not valid, i.e. $|X_n - 4| \geqslant 0{\cdot}8$ with this n. And this is like saying that number 4 is not the limit of sequence (1), (2). At this point it is easy to formulate

Definition 2. L is not the limit of the sequence X_n if there exists an $\varepsilon > 0$ such that for every $N > 0$ there is a number $n > N$ such that the inequality $|X_n - L| \geqslant \varepsilon$ holds true.

By comparing definition 1 and definition 2 one can observe that substituting in definition 1 the words 'for any...ε' with 'there exists an ε', the words 'there exists...N' with 'for every N', the words 'for all n' with 'there is...n' and the inequality $|X_n - L| < \varepsilon$ with $|X_n - L| \geqslant \varepsilon$, results in definition 2.

This completes the scenario of the discussion of the definition of a limit of a sequence. However, having completed our initial objective, the learning process can be extended with the help of the acquired skills.

5. Variation of the first term and coefficients of a linear iteration sequence

One of the most important practical applications of the theory of sequences is in the discussion of the iteration processes used in computing. Generally, the analysis of iteration processes is concerned with deciding whether a certain sequence converges and if so, to what limit it tends. Moreover, it is of importance here to know how fast the convergence is and whether some iteration procedures (at least from a given class) will provide answers more quickly than others. It should be recalled that we chose the first term (2) arbitrarily. So, it is natural to start with the variation of X_1. Students detect from the outset that such a change has no influence upon the limit. As students usually like to 'play' with large numbers, the teacher can exploit this by asking the following question:

> Let us multiply the starting point (2) by the factor of 10^6. How many steps will the computer require to 'achieve' the sequence (1) to the limit?

Most likely, students' guesses will be wrong as long as they have not made previous calculations. However, quick calculations on the computer will show that the first term's value extension by six orders leads to a difference of thirteen steps. Otherwise, the value of a first term does not affect the speed of convergence of sequence (1) to the limit.

Next, students may turn to the family of linear iteration sequences

$$X_{n+1} = aX_n + b \qquad (4)$$

which determines the iteration process depending on real parameters a and b. One possible goal of investigating this on the basis of the spreadsheet could be to establish which of these parameters is responsible for which feature of the process. It is easy to compose experiments which result in the influence of only the multiplicative parameter a, both on the existence of the limit and on the speed of convergence while the additive parameter b contributes only to the value of the limit to which sequence (4) tends. These explorations, when performed under teacher guidance, will lead students to the conclusion that, given a real number b and the starting point X_1, there exists a limit of sequence (4) with $|a|<1$, and the smaller $|a|$, the faster the convergence. The teacher can demonstrate this to students by using the graphical analysis of the iteration process [18] as well as by transforming X_n into a progression [19]. One more visual approach in deciding the convergence of sequence (4) which does not involve the concept of a limit will be shown below in section 7.

To experiment with parameters the spreadsheet is programmed as follows (see Figure 3). In row **1** different values of the parameter a are defined by tabulating a from $-1{\cdot}5$ (cell **B1**) with step 1 (cell **B4**) up to $1{\cdot}5$ (cell **E1**), or in other words the spreadsheet function **=B1+$B$4** is defined in cell **C1** and is copied right into cell **E1**. In cell **B2** the value of parameter b is defined. In column **A** beginning from cell **A5** current numbers of the terms of the sequence are defined whose values are defined in column **B** by entering the first term into cell **B5**, spreadsheet function **=B$1*B5 +$B$2** into cell **B6** and replicating this function into cell **E25**.

	A	B	C	D	E
1	a	-1.5	-0.5	0.5	1.5
2	b	2			
3	X1	10			
4	STEP	1			
5	1	10	10	10	10
6	2	-13	-3	7	17
7	3	21.5	3.5	5.5	27.5
8	4	-30.25	0.25	4.75	43.25
9	5	47.375	1.875	4.375	66.88
10	6	-69.063	1.0625	4.1875	102.3
11	7	105.59	1.4688	4.09375	155.5
12	8	-156.39	1.2656	4.04688	235.2
13	9	236.59	1.3672	4.02344	354.8
14	10	-352.88	1.3164	4.01172	534.2
15	11	531.32	1.3418	4.00586	803.3
16	12	-794.98	1.3291	4.00293	1207
17	13	1194.5	1.3354	4.00146	1812
18	14	-1789.7	1.3323	4.00073	2721
19	15	2686.5	1.3339	4.00037	4083
20	16	-4027.8	1.3331	4.00018	6127
21	17	6043.7	1.3335	4.00009	9192
22	18	-9063.6	1.3333	4.00005	13790
23	19	13597	1.3334	4.00002	20686
24	20	-20394	1.3333	4.00001	31032
25	21	30593	1.3333	4.00001	46550

Figure 3.

6. Modelling sequences with different behaviour to prevent misconceptions

It has been noted by many authors [11] that teaching the concept of a limit of a sequence through specific examples such as monotonic sequences could cause students' misconception which, in turn, may lead to erroneous operations with limits. For instance, when asked to find the limit of a sequence students sometimes act on a mistaken assumption that such a limit exist. Proceeding from the slant that one picture is worth a thousand words a teacher could introduce to students sequences with different and quite unexpected behaviours using a spreadsheet as a tool of visualization. Thus, modelling the quadratic sequence

$$X_{n+1}=\frac{d}{2}+\frac{X_n^2}{2} \qquad X_1=\frac{d}{2} \tag{5}$$

for different values of the real parameter d by tabulating d with appropriate step, provides the visualization of different types (divergent, convergent, oscillating, periodic, chaotic) of sequence (5) behaviours as shown in Figure 4. Particularly, column **D** indicates that it is irrelevant for the existence of a limit whether the values of X_n are located on one side of the limit since, as stated in the definition, it is only essential that the variable should finally differ from its limit by an arbitrarily small amount. Finally, this example calls students' attention to the importance of the theorem of the existence of limit for a bounded monotonic sequences.

The spreadsheet in Figure 4 is programmed similarly to the spreadsheet in Figure 3. Values of the parameter d enter into row **1**, values of the sequence (5) are provided by the spreadsheet function =**(B2** ∧ 2)/2+**(B$1)**/2 defined in the cell **B3** and replicated into cell **G19**.

	A	B	C	D	E	F	G
1	n\d	-5.5	-4	-2.5	-1	0.5	2
2	1	-2.75	-2	-1.25	-0.5	0.25	1
3	2	1.03125	0	-0.46875	-0.375	0.28125	1.5
4	3	-2.21826	-2	-1.140137	-0.42969	0.289551	2.125
5	4	-0.28966	0	-0.600044	-0.40768	0.29192	3.25781
6	5	-2.70805	-2	-1.069974	-0.4169	0.292609	6.30667
7	6	0.91677	0	-0.677578	-0.4131	0.29281	20.8871
8	7	-2.32977	-2	-1.020444	-0.41467	0.292869	219.134
9	8	-0.03608	0	-0.729347	-0.41402	0.292886	24011
10	9	-2.74935	-2	-0.984026	-0.41429	0.292891	2.9E+08
11	10	1.02946	0	-0.765846	-0.41418	0.292893	4.2E+16
12	11	-2.22011	-2	-0.95674	-0.41423	0.292893	8.6E+32
13	12	-0.28556	0	-0.792324	-0.41421	0.292893	3.7E+65
14	13	-2.70923	-2	-0.936111	-0.41422	0.292893	7E+130
15	14	0.91995	0	-0.811848	-0.41421	0.292893	2E+261
16	15	-2.32684	-2	-0.920451	-0.41421	0.292893	#NUM!
17	16	-0.0429	0	-0.826385	-0.41421	0.292893	#NUM!
18	17	-2.74908	-2	-0.908544	-0.41421	0.292893	#NUM!
19	18	1.02872	0	-0.837274	-0.41421	0.292893	#NUM!

Figure 4.

7. Visualization of the Bolzano–Cauchy principle of convergence

The celebrated theorem states:

> In order that the variable X_n has a finite limit it is necessary and sufficient that for any number $\varepsilon > 0$ there exists a number N such that the inequality
>
> $$|X_n - X_m| < \varepsilon \tag{6}$$
>
> is valid, provided $n > N$ and $m > N$.

. As can be seen, given the variable, inequality (6) serves as the example of an *inequality depending on two positive integral variables. Similarly to the ability to* numerically model equations of partial differences [20], a spreadsheet can model inequality (6) by calculating its values (TRUE or FALSE) for any positive number ε. This has the advantage of enabling students to discover whether the sequence converges or not without involving the concept of the limit, whose existence they want to prove.

Modelling inequalities of type (6) can be realized on the spreadsheet quite easily. By way of an example a teacher could propose that students develop the spreadsheet implementing inequality (6) for sequence (4) with the starting point (2). This spreadsheet (see Figure 5(*a*) where $a = 0{\cdot}6$, $b = 2$, $\varepsilon = 0{\cdot}3$) is programmed as follows.

In row **4** beginning from cell **B4** and in column **A** beginning from cell **A5**, positive integral values of n and m are defined respectively. In cell **B5** the first term (2) is defined. *The spreadsheet function* **=B5*B$2+$B3** *is defined in cell* **B6 (C5)** and copied down (write) into cell **B24 (P5)**. The spreadsheet function **=IF(ABS(C$5 −$B6)<**EPS, "", "Ç") is defined in cell **C6** and calculate the value of the predicate $|x_2 - x_2| < \varepsilon$. *This function is replicated into cell* **P24**.

The spreadsheet shown in Figure 5 (*b*) corresponds to the following triple: $a = 1{\cdot}5$, $b = 2$, $\varepsilon = 0{\cdot}01$.

With recourse to Figure 5(*a*) and 5(*b*) one may note that when $\varepsilon = 0{\cdot}3$ $a = 0{\cdot}6$, $b = 2$ the values of inequality (6) become TRUE beginning from $n = 7$ and $m = 7$ (in the Bolzano–Cauchy theorem $N = 7$), when $a = 1{\cdot}5$, $b = 2$ irrespective of ε the values of inequality (6) appear not to form the rectangular pattern filled with symbols 🍎 (such an N does not exist).

The students should be encouraged to experiment with various values for a and ε and be challenged to explore the dependence among a, ε and N when the visualization of the principle of convergence has been originated from modelling on the template. These experiments, actually, can give students an appreciation of the recalculation process by providing a visual image of statements like 'the smaller ε, the greater N', 'the greater ε, 'the smaller N', the smaller a in absolute value, the faster the convergence', and so forth.

8. Applying the visualization of the Bolzano–Cauchy theorem in deciding the convergence of positive series

Finally, we would like to describe a possible computer-based scenario in deciding the convergence (or divergence) of a positive series, which a teacher could arrange during a lesson. This layout includes the following points:

(1) Representation of the sequence of partial sums of the given series in the form of an iteration sequence.

(2) Modelling the Bolzano–Cauchy principle of convergence with respect to this sequence.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	EPS	0.3														
2	a	0.6														
3	b	2														
4	n\m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
5	1	1	2.6	3.56	4.14	4.48	4.69	4.81	4.89	4.93	4.96	4.98	4.99	4.99	4.99	5
6	2	2.6	🍎	Ć	Ć	Ć	Ć	Ć	Ć	Ć	Ć	Ć	Ć	Ć	Ć	Ć
7	3	3.56	Ć	🍎	Ć	Ć	Ć	Ć	Ć	Ć	Ć	Ć	Ć	Ć	Ć	Ć
8	4	4.14	Ć	Ć	🍎	Ć	Ć	Ć	Ć	Ć	Ć	Ć	Ć	Ć	Ć	Ć
9	5	4.48	Ć	Ć	Ć	🍎	🍎	Ć	Ć	Ć	Ć	Ć	Ć	Ć	Ć	Ć
10	6	4.69	Ć	Ć	Ć	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	Ć	Ć	Ć
11	7	4.81	Ć	Ć	Ć	Ć	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎
12	8	4.89	Ć	Ć	Ć	Ć	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎
13	9	4.93	Ć	Ć	Ć	Ć	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎
14	10	4.96	Ć	Ć	Ć	Ć	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎
15	11	4.98	Ć	Ć	Ć	Ć	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎
16	12	4.99	Ć	Ć	Ć	Ć	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎
17	13	4.99	Ć	Ć	Ć	Ć	Ć	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎
18	14	4.99	Ć	Ć	Ć	Ć	Ć	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎
19	15	5	Ć	Ć	Ć	Ć	Ć	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎
20	16	5	Ć	Ć	Ć	Ć	Ć	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎
21	17	5	Ć	Ć	Ć	Ć	Ć	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎
22	18	5	Ć	Ć	Ć	Ć	Ć	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎
23	19	5	Ć	Ć	Ć	Ć	Ć	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎
24	20	5	Ć	Ć	Ć	Ć	Ć	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎	🍎

Figure 5 (*a*).

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	EPS	0.01														
2	a	1.5														
3	b	2														
4	n\m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
5	1	1	3.5	7.25	12.9	21.3	34	53	81.4	124	188	284	428	645	969	1456
6	2	3.5	●	○	○	○	○	○	○	○	○	○	○	○	○	○
7	3	7.25	○	●	○	○	○	○	○	○	○	○	○	○	○	○
8	4	12.88	○	○	●	○	○	○	○	○	○	○	○	○	○	○
9	5	21.31	○	○	○	●	○	○	○	○	○	○	○	○	○	○
10	6	33.97	○	○	○	○	●	○	○	○	○	○	○	○	○	○
11	7	52.95	○	○	○	○	○	●	○	○	○	○	○	○	○	○
12	8	81.43	○	○	○	○	○	○	●	○	○	○	○	○	○	○
13	9	124.1	○	○	○	○	○	○	○	●	○	○	○	○	○	○
14	10	188.2	○	○	○	○	○	○	○	○	●	○	○	○	○	○
15	11	284.3	○	○	○	○	○	○	○	○	○	●	○	○	○	○
16	12	428.5	○	○	○	○	○	○	○	○	○	○	●	○	○	○
17	13	644.7	○	○	○	○	○	○	○	○	○	○	○	●	○	○
18	14	969.1	○	○	○	○	○	○	○	○	○	○	○	○	●	○
19	15	1456	○	○	○	○	○	○	○	○	○	○	○	○	○	●
20	16	2185	○	○	○	○	○	○	○	○	○	○	○	○	○	○
21	17	3280	○	○	○	○	○	○	○	○	○	○	○	○	○	○
22	18	4922	○	○	○	○	○	○	○	○	○	○	○	○	○	○
23	19	7385	○	○	○	○	○	○	○	○	○	○	○	○	○	○
24	20	11080	○	○	○	○	○	○	○	○	○	○	○	○	○	○

Figure 5 (*b*).

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	EPS	0.1														
2	n\m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	1	3.0028	6.0056	6.7597	7.1351	7.3749	7.5488	7.6845	7.7957	7.89	7.9719	8.0444	8.1096	8.1689	8.2233	8.2736
4	2	6.0056	●	○	○	○	○	○	○	○	○	○	○	○	○	○
5	3	6.7597	○	●	○	○	○	○	○	○	○	○	○	○	○	○
6	4	7.1351	○	○	●	○	○	○	○	○	○	○	○	○	○	○
7	5	7.3749	○	○	○	●	○	○	○	○	○	○	○	○	○	○
8	6	7.5488	○	○	○	○	●	○	○	○	○	○	○	○	○	○
9	7	7.6845	○	○	○	○	○	●	○	○	○	○	○	○	○	○
10	8	7.7957	○	○	○	○	○	○	●	●	○	○	○	○	○	○
11	9	7.89	○	○	○	○	○	○	●	●	●	○	○	○	○	○
12	10	7.9719	○	○	○	○	○	○	○	●	●	●	○	○	○	○
13	11	8.0444	○	○	○	○	○	○	○	○	●	●	●	○	○	○
14	12	8.1096	○	○	○	○	○	○	○	○	○	●	●	●	○	○
15	13	8.1689	○	○	○	○	○	○	○	○	○	○	●	●	●	○
16	14	8.2233	○	○	○	○	○	○	○	○	○	○	○	●	●	●
17	15	8.2736	○	○	○	○	○	○	○	○	○	○	○	○	●	●

Figure 6 (*a*)

	A	B	C	D
1	k	a_k	b_k	$a_k > b_k$
2	2	3.00278	0.5	●
3	3	0.75417	0.33333	●
4	4	0.37535	0.25	●
5	5	0.23987	0.2	●
6	6	0.17384	0.16667	●
7	7	0.13572	0.14286	○
8	8	0.11121	0.125	○
9	9	0.09427	0.11111	○
10	10	0.08191	0.1	○
11	11	0.07253	0.09091	○
88	88	0.01114	0.01136	○
89	89	0.01106	0.01124	○
90	90	0.01098	0.01111	○
91	91	0.01089	0.01099	○
92	92	0.01082	0.01087	○
93	93	0.01074	0.01075	○
94	94	0.01066	0.01064	●
95	95	0.01059	0.01053	●
96	96	0.01052	0.01042	●
97	97	0.01045	0.01031	●
98	98	0.01038	0.0102	●
99	99	0.01031	0.0101	●
100	100	0.01024	0.01	●

Figure 6 (*b*).

(3) Analysis of the spreadsheet template and making conjectures concerning the problem of whether given series will converge or fail to converge.
(4) Inspection of the appropriate series by comparing terms within a spreadsheet for applying a comparison test.
(5) The argument of convergence (divergence) by a comparison test.

The above scenario for the series

$$\sum_{k=2}^{\infty} \frac{1}{(\ln k)^3}$$

which turns out to be the 'major series' for the harmonic series

$$\sum_{k=2}^{\infty} \frac{1}{k}$$

is implemented on the spreadsheet templates shown in Figures 6 (*a*) and 6 (*b*).

9. Conclusions

The rapid development of technological tools in mathematics education affects both teaching and learning processes. On the one hand, as computer software becomes more sophisticated and proves to be well-adapted to particular educational purposes, it captures more and more parts of the curriculum. On the other hand students of all ages and abilities are intrigued by computing devices irrespective of their particular assignment, if only they provide an exciting pursuit. The educational task, then, should be only to stimulate students' readiness to learn mathematical ideas through appropriate computer technology.

In this article the authors try to explicate the effectiveness of a spreadsheet software for both teaching and learning the concept of a limit of a sequence. There may be an objection that the usefulness of the computer as a technological tool for calculus lessons is not completely apparent. Of course, a spreadsheet simply calculates the values of X_n and then shows what number L they *seem* to approach since a computer is capable of examining only a finite collection of members of a sequence irrespective of the time of computing. However, and we should like to stress this, a spreadsheet allows students to construct templates of numbers designed to illustrate how it seems that as n grows large, X_n gets close to L. Moreover, watching this apparent approach and responding simultaneously to the precise and directed questions of a teacher, students are able not only to arrive at comprehension of a fairly abstract concept but, better still, to formulate the exact definition of this concept almost independently. Finally, one more peculiarity of the spreadsheet that contributes to its pedagogical advantage is its analytical investigation and visualization capability. Easily constructed as an explorer, the spreadsheet, in combination with teacher guidance, gives students the opportunity to apprehend the importance of the theory of sequences in applications, by conjecturing and then testing the hypotheses concerning the convergence of positive series. This however exceeds the frame of the present article, and the authors hope to devote a separate paper to this matter.

References

[1] FEY, J. T., 1989, *Educ. Studies Math.*, **20,** 237.
[2] KAPUT, J. J., 1992, Technology and mathematics education. In *Handbook of Research on Mathematics Teaching and Learning* (a project of the N.C.T.M.), edited by D. A. Grouws (New York: Macmillan), pp. 515–556.
[3] ARGANBRIGHT, D. E., 1985, *Mathematical Applications of Electronic Spreadsheets* (New York: McGraw-Hill).
[4] HEALY, L., and SUTHERLAND, R., 1990, *Exploring Mathematics with Spreadsheets* (Oxford: Blackwell).
[5] CAPPONI, B., and BALACHEFF, N., 1989, *Educ. Studies Math.*, **20,** 179.
[6] CLEMENTS, R. R., 1991, *Int. J. Math. Educ. Sci. Technol.*, **22,** 1003.
[7] MAXIM, B. R., and VERHEY, R. F., 1991, Using spreadsheets to introduce recursion and difference equations in high school mathematics. In *Discrete Mathematics Across The Curriculum, K-12 1991 Yearbook*, edited by M. J. Kenney and C. R. Hirsch (Reston, VA: N.C.T.M.), pp. 158–165.
[8] SCHWARZENBERGER, R. L. E., and TALL, D. O., 1978, *Math. Teaching*, **82,** 44.
[9] ROBERT, A., 1982, *Reserches en Didactique des Mathèmatiques*, **3,** 307.
[10] DAVIS, R. B., and VINER, S., 1986, *J. Math. Behaviour*, **5,** 281.
[11] TALL, D. O., 1992, The transition to advanced mathematical thinking: functions, limits, infinity and proof. In *Handbook of Research on Mathematics Teaching and Learning* (a project of the N.C.T.M.), edited by D. A. Grouws (New York: Macmillan), pp. 495–511.
[12] ABRAMOVICH, S., and LEVIN, I., 1992, Use of spreadsheets in teaching theory of limits. In *Book of Abstracts of Short Presentations* (ICME-7), coordination: M. Meilleur (Quèbec: Universitè Laval), p. 39.
[13] WEIGAND, H.-G., 1991, *Educ. Studies Math.*, **22,** 4111.

[14] Cobb, D., McGuffey, A., and Dodge, M., 1991, *Microsoft Excel 3 Companion* (Redmont, WA: Microsoft Press).
[15] Lawler, R. W., 1987, Learning environments: now, then, and someday. In *Artificial Intelligence and Education*, vol. 1, edited by R. W. Lawler (Norwood, NJ: Alex Publishing), pp. 1–25.
[16] Levin, I., and Abramovich, S., 1992, *J. Comput. Math. Sci Teaching*, **11,** 337.
[17] Papert, S., 1991, Situating constructionism. In *Constructionism*, edited by I. Harel and S. Papert (Norwood, NJ: Ablex Publishing), pp. 1–12.
[18] Bannard, D. N., 1991, Making connections through iteration. In *Discrete Mathematics Across The Curriculum, K-12 1991 Yearbook*, edited by M. J. Kenney and C. R. Hirsch (Reston, VA: N.C.T.M.), pp. 178–183.
[19] Spence, L. E., 1990, *Math. Teacher*, **83,** 709.
[20] Abramovich, S., and Levin, I., 1993, *J. Comput. Math. Sci. Teaching*, **12,** 331.