Journal of Economic Theory **105**, 483–502 (2002)
doi:10.1006/jeth.2001.2858

Utility in Case-Based Decision Theory[1]

Itzhak Gilboa

Tel-Aviv University, Tel-Aviv, Israel
igilboa@post.tau.ac.il

David Schmeidler

Tel-Aviv University, Tel-Aviv, Israel; and Ohio State University, Ohio
schmeid@tau.ac.il

and

Peter P. Wakker

University of Amsterdam, Amsterdam, The Netherlands
wakker@fee.uva.nl

Received November 27, 1999; final version received May 18, 2001;
published online March 13, 2002

This paper provides two axiomatic derivations of a case-based decision rule. Each axiomatization shows that, if preference orders over available acts in various contexts satisfy certain consistency requirements, then these orders can be numerically represented by maximization of a similarity-weighted utility function. In each axiomatization, both the similarity function and the utility function are simultaneously derived from preferences, and the axiomatic derivation also suggests a way to elicit these theoretical concepts from in-principle observable preferences. The two axiomatizations differ in the type of decisions that they assume as data. *Journal of Economic Literature* Classification Number: D80.

Key Words: decision under uncertainty; case-based decision theory.

1. INTRODUCTION

Case-based decision theory (CBDT; Gilboa and Schmeidler [4, 6]) postulates that decisions are based on the relative success of actions under similar circumstances in the past. Formally, assume as given a set of *decision problems* P, a set of *acts* A, and a set of *outcomes* R. The decision

[1] The authors gratefully acknowledge helpful comments of a referee. Partial financial support by ISF is also acknowledged.

0022-0531/02 $35.00

maker's memory, M, is assumed to consist of n *cases*, which are triples of the form $(p, a, r) \in C = P \times A \times R$. A case c may be viewed as a story in which problem p was encountered, act a was chosen, and outcome r was consequently experienced. A pair of a problem and an act is also referred to as a *circumstance*. The decision maker has a utility function over outcomes, $u: R \to \mathbb{R}$, and a similarity function over circumstances, $s: (P \times A) \times (P \times A) \to \mathbb{R}_+$, and, given a problem $p \in P$ and memory M, he or she chooses an act that maximizes[2]

$$U(a) = U_{p, M}(a) = \sum_{(q, b, r) \in M} s((p, a), (q, b))\, u(r). \tag{1}$$

This paper addresses the question of axiomatic derivation of the above rule. That is, we seek to find conditions on choices that can be assumed observable and that will be equivalent to the existence of a utility function and a similarity function such that decisions are made so as to maximize the corresponding U. Such a set of conditions, or axioms, may help us better judge this decision rule for descriptive and normative purposes alike. Moreover, to the extent that the axiomatization pins down the utility and similarity functions uniquely, it can serve as a definition of these theoretical terms by (in-principle) observable data.

Gilboa and Schmeidler [5] have derived the similarity function in (1) by assuming a given utility function. That is, they assumed that the outcomes are real numbers and suggested axioms involving addition of these numbers. These axioms make sense only if one interprets the numbers as utility levels. In this sense, their axiomatization is comparable to de Finetti's [2] axiomatization of expected utility. More precisely, if a decision maker finds payoff numbers to be meaningful and if he or she behaves according to de Finetti's axioms, the latter may serve to elicit the decision maker's subjective probabilities. Similarly, if an individual relates to payoff numbers in Gilboa and Schmeidler's model, and should he or she obey their axioms, these axioms may be used to elicit his or her subjective similarity judgment. However, de Finetti's axioms will not be satisfied by an individual who is, say, risk averse with respect to the payoffs. Correspondingly, an individual is likely to treat payoffs in a non-linear way in the case-based framework too and therefore should not be expected to satisfy the axioms presented by Gilboa and Schmeidler for CBDT. In addition, in many applications of models of decision making, particularly in circumstances of uncertainty, outcomes are formally represented by multidimensional vectors, i.e., commodities bundles or production plans. De Finetti's

[2] The analysis that follows also holds when the set of available acts depends on the decision problem. For simplicity of notation, however, we use the same set of acts for all problems.

axioms, as well as Eq. (1) with $R = \mathbb{R}$, are not even defined for these general non-numerical outcomes.

An axiomatization of the decision rule (1) that presupposes neither similarity nor utility will provide us with a more general understanding of the normative and positive aspects of this rule. It will also allow us to define and elicit similarity values for individuals who do not treat numerical outcomes linearly or who face non-numerical outcomes, which can be represented numerically only through a concept such as utility. In a way that is analogous to Savage's [14] joint derivation of utility and of subjective probability,[3] we seek to derive both utility and similarity from axioms that presuppose none of these concepts to be already quantified.

Preference axiomatizations typically rely on choices that are, to some degree, hypothetical. In particular, derivations of case-based decision rules resort to choices that are based on various possible memories. Indeed, a given memory induces a certain act. Since any further observation of the same individual will be based on a different (and longer) memory, a single, concrete sequence of choices will typically not contain enough information to uniquely define a similarity function, let alone both a similarity and a utility function. One therefore often assumes that the data are rich enough to express preferences involving not only the actual memory, but also related, hypothetical memories constructed from it. In particular, hypothetical memories consist of cases that combine circumstances that were actually encountered with various outcomes, in addition to those actually experienced in the respective cases. Indeed, one cannot hope to obtain separation of similarity from utility in Eq. (1) unless some freedom in the construction of cases is allowed.

There are two ways to construct classes of hypothetical memories, rich enough for our purposes. The first allows one to consider memories that are obtained from the actual one only by replacing the actual outcomes with other outcomes from the set R. To obtain the desired result we assume that the set R is rich enough, as, say, an interval of real numbers. The alternative approach, on the other hand, allows one to consider models where the set of outcomes, R, is finite. Especially, one may restrict the model to outcomes that have been actually experienced. Thus, possible cases are combinations of circumstances and outcomes that both appeared in actual memory, but not necessarily in the same case. Obviously, starting with memory of size n, there are at most n^2 such combinations. In the formal presentation we will deal with this special assumption, and the extension to an arbitrary finite set, R, is obvious. This approach further demands that any memory, generated by replication of such cases, be considered.

[3] For related derivations see Grodal [8], Gul [9], and Wakker [16, Theorem IV.2.7].

The first approach seems quite natural when the outcomes are monetary payoffs, as in the case of investment problems. Similarly, one may imagine the experience of consuming an arbitrary bundle in a convex set, as in the neo-classical economic model. There are, however, situations where it is hard to imagine outcomes that were not actually experienced. For instance, it is often argued that people who have not experienced extreme outcomes such as starvation or prolonged unemployment cannot imagine these experiences. Other, less dramatic examples, include the choice of a destination for a vacation trip. In such examples, the second approach taxes the decision maker's imagination to a lesser degree.

Sections 2 and 3 provide the two axiomatizations. All proofs and related analyses are relegated to the appendix.

2. THE OUTCOME TRADEOFF APPROACH

Consider an n-list E, to be interpreted as the circumstances (problem-act pairs) that have been encountered in the past, say $E = ((p_1, a_1), ..., (p_n, a_n))$. (Since the memory M is a list in $C = P \times A \times R$, E is the projection of M on the first two coordinates, i.e., $P \times A$.) The set of outcomes R is endowed with a binary relation $\succcurlyeq$, weakly ordering the outcomes from good to bad. R contains a *neutral outcome*, denoted θ. A *context* x is an n-list $(x_1, ..., x_n) \in R^n$ of outcomes. The interpretation is as follows. There are n cases in memory. Each case is characterized by a pair problem-act. The n-list $(x_1, ..., x_n)$ describes the context where for i $(i = 1, ..., n)$, x_i is the (hypothetical) outcome obtained when the problem p_i was encountered and the act a_i was chosen. For $x \in R^n$, $i \leqslant n$, and $\alpha \in R$, $\alpha_i x$ we denote the context x with x_i replaced by α. Thus $\alpha_i \beta_j x$ denotes the context x with x_i replaced by α and x_j replaced by β. We will also denote $\alpha_i \beta_j x$ by $\alpha\beta_{ij} x$.

Each context x generates a preference relation $\succcurlyeq_x$. It is important to emphasize that the axiomatizations in this paper (as well as the one in Gilboa and Schmeidler [5]) employ the so-called "context approach": preferences over a set A are assumed to depend on the context x. Much of the axioms and most of the mathematical work is done in the space of contexts. However, contexts are not the objects of choice and they are therefore not ranked. This stands in contrast to the bulk of the literature on axiomatic derivations in decision theory.

Because the problems and acts chosen are fixed, the preference relation induced by a context depends only on the preferences over the outcomes. We will thus assume, for outcomes α, β, that, if $\alpha \sim \beta$, then $\succcurlyeq_{\alpha_i x} = \succcurlyeq_{\beta_i x}$. A function V *represents* $\succcurlyeq_x$ if $V: A \to \mathbb{R}$ is such that $a \succcurlyeq_x b$ if and only if $V(a) \geqslant V(b)$. The binary relations $\sim$, $\succ$, $\preccurlyeq$, $\prec$, and the same symbols with subscripts, are defined as usual. For the constant-neutral context, i.e.,

$x=(\theta, \ldots, \theta)$, all acts are assumed to be indifferent. We also assume that there exist outcomes strictly preferred to the neutral outcome θ and others that θ is strictly preferred to.

This paper considers the following quantitative representations for the relations $\succcurlyeq_x$: There exists a function u: $R \to \mathbb{R}$ and for each act $a \in A$ and each problem i there exists a real number s_i^a, such that for every context x, the binary relation $\succcurlyeq_x$ is represented by

$$a \mapsto \sum_{i=1}^{n} s_i^a u(x_i), \tag{2}$$

where u is the *utility function* that will represent $\succcurlyeq$ in the main theorems, and s_i^a is the *similarity weight* of act a with respect to the act chosen in problem i. Clearly, Eq. (2) coincides with Eq. (1) when $s((p, a), (q, b)) = s_i^a$ and (q, b) varies over E. For $a, b \in A$, whether a is preferred to b is decided by the sign of

$$\sum_{i=1}^{n} (s_i^a - s_i^b)\, u(x_i). \tag{3}$$

Under a minimal nondegeneracy assumption, this representation requires that $u(\theta)=0$. Problem i is *favorable* for a vis-a-vis b if $\alpha \succ \beta$ and $a \succcurlyeq_{\beta_i x} b$ imply $a \succ_{\alpha_i x} b$. It is *neutral* to a vis-a-vis b if the preference between a and b does not change when the ith coordinate of the context is changed, and it is *unfavorable* for a vis-a-vis b if $\alpha \succ \beta$ and $a \preccurlyeq_{\beta_i x} b$ implies $a \prec_{\alpha_i x} b$. In the presence of certain richness conditions, these concepts will correspond to the sign of $(s_i^a - s_i^b)$ and hence are mutually exclusive and exhaustive. They entail an ordinal case-independence of the utilities of outcomes. If $(s_i^a - s_i^b)$ is positive, then improving x_i leads to additional evidence in favor of a.

We now comment on the empirical measurement of utility, which will naturally lead to a preference characterization. First, some additional notation is needed. Let I_{ab} denote the *indifference set* for acts a and b, i.e., $I_{ab} = \{x \in R^n \mid a \sim_x b\}$. Assume that $\alpha\sigma_{ik}x \in I_{ab}$ and $\beta\tau_{ik}x \in I_{ab}$. Then the extra evidence in favor of a, obtained when β replaces α, is exactly offset by the extra evidence obtained by replacing σ with τ. If a numerical representation as in (3) exists then

$$(s_i^a - s_i^b)(u(\alpha) - u(\beta)) = (s_k^a - s_k^b)(u(\tau) - u(\sigma)). \tag{4}$$

Assume also that $\gamma\sigma_{ik}\, y \in I_{ab}$ and $\delta\tau_{ik}\, y \in I_{ab}$. Then the extra evidence in favor of a, obtained by replacing γ with δ, offsets the same evidence (replacing σ by τ in problem k) as did the replacement of α by β. Substituting (3) yields

$$(s_i^a - s_i^b)(u(\gamma) - u(\delta)) = (s_k^a - s_k^b)(u(\tau) - u(\sigma)). \tag{5}$$

Because the right-hand sides of (4) and (5) are identical, their left-hand sides are equal. Assuming nonneutrality of problem i, which implies $s_i^a - s_i^b \neq 0$, it follows that

$$u(\alpha) - u(\beta) = u(\gamma) - u(\delta). \tag{6}$$

Hence, equalities of utility differences can be elicited by observations such as $\alpha\sigma_{ik}x, \beta\tau_{ik}x, \gamma\sigma_{ik}y, \delta\tau_{ik}y \in I_{ab}$. Such elicitations suffice to measure cardinal utility.

For the representation in (2) to be valid, it is necessary that different inferences not lead to inconsistencies. *Evidence tradeoff consistency*, or *tradeoff consistency* for short, holds if

$$\begin{matrix} \alpha\sigma_{ik}x,\ \beta\tau_{ik}x,\ \gamma\sigma_{ik}y,\ \delta\tau_{ik}y \in I_{ab} \text{ and} \\ \alpha\mu_{jl}v,\ \beta v_{jl}v,\ \gamma\mu_{jl}w,\ \delta' v_{jl}w \in I_{cd} \end{matrix} \quad \text{imply} \quad \delta' \sim \delta \tag{7}$$

whenever i is not neutral with respect to a, b and j is not neutral with respect to c and d. In other words, if the first quadruple suggests that $u(\alpha) - u(\beta) = u(\gamma) - u(\delta)$ and the second that $u(\alpha) - u(\beta) = u(\gamma) - u(\delta')$ then the outcomes δ and δ' must be equally preferred. Tradeoff consistency is obviously necessary for the representation (2). In the presence of natural preference conditions plus continuity, tradeoff consistency also turns out to be sufficient, i.e., it characterizes (2). The condition entails separability because the relative attitude toward given outcomes in given cases are unaffected by what outcomes occurred in other (possibly similar) cases.

Preference continuity holds if the $\succcurlyeq$ order topology on R is connected and for each pair of acts a, b, the set $\{x \in R^n \mid a \succcurlyeq_x b\}$ is closed in R^n, where the latter is endowed with the product topology of the order topology on R.[4]

We first present the characterization result for two available acts; two acts suffice to determine the case-based model and its primitives. The extension to three or more available acts is more complex and is presented in Proposition 2.2.

THEOREM 2.1. *Assume that there are two acts a and b. The following two statements are equivalent for $\{\succcurlyeq_x \mid x \in R^n\}$:*

[4] The $\succcurlyeq$ order topology on R is generated by the sets $\{\alpha \in R \mid \alpha \succ \beta\}$ and $\{\alpha \in R \mid \alpha \prec \beta\}$, where $\beta \in R$.

(i) (C1) *(Completeness) For every context x, $a \succcurlyeq_x b$ or $b \succcurlyeq_x a$.*

(C2) *(Preference-based evidence) If $x_i \sim y_i$ for all i then $\succcurlyeq_x = \succcurlyeq_y$.*

(C3) *(Monotonicity) Every problem is either favorable or neutral or unfavorable for a vis-a-vis b.*

(C4) *(Continuity) Preference continuity holds.*

(C5) *(Richness)*

— *There are at least three nonneutral problems.*

— *a and b are equivalent for the context $(\theta, ..., \theta)$.*

— *$\alpha \succ \theta \succ \beta$ for some outcomes α, β.*

— *For all problems i that are nonneutral with respect to a, b, and all contexts x, there exists an outcome α such that $a \sim_{\alpha_i x} b$ (solvability).*

(C6) *Tradeoff consistency holds.*

(ii) *There exists a function u: $R \rightarrow \mathbb{R}$ and for each act $a \in A$ and each problem i there exists a real number s_i^a, such that the following conditions hold:*

(P1) *For every context x, $a \mapsto \sum_{i=1}^n s_i^a u(x_i)$ and $b \mapsto \sum_{i=1}^n s_i^b u(x_i)$ represent the preference relation between a and b.*

(P2) *u represents $\succcurlyeq$ on R.*

(P3) *u is continuous and its range is $\mathbb{R}$; $u(\theta) = 0$.*

(P4) *For at least three problems i, $(s_i^a - s_i^b)$ is nonzero.*

Furthermore, the uniqueness of the similarity weights in (ii) *is as in Gilboa and Schmeidler [5] or the auxiliary theorem in the Appendix below and u is unique up to a positive scale factor.*

The mathematical novelty in this theorem, as compared to Koebberling and Wakker [10] and other works, is twofold. First, we employ the context approach. This implies that the objects of choice are different than in the standard approach. Second, this representation theorem is not derived assuming a complete product space but a subset thereof. This subset is different in nature from the subsets studied in the rank-dependent theories.

Next we present the extension to more than two acts. Without further conditions, the similarity weights s_i^a in the preceding theorem can depend on the other available act b, and intransitive preferences can result. Necessary and sufficient conditions to prevent such intransitivities are as yet unknown. Some sufficient conditions have been introduced by Gilboa and Schmeidler [7] and are incorporated next. Conditions (C5') and (A4') hereafter are the plausible but restrictive conditions that remain to be relaxed.

PROPOSITION 2.2. *The following two statements are equivalent for* $\{ \succcurlyeq_x \mid x \in R^n \}$:

(i) *Statement* (i) *of Theorem* 2.1 *holds (with conditions* (C1), (C3), (C4), *and* (C5) *imposed on all act pairs a, b). Further,*

(C1') *Every* $\succcurlyeq_x$ *is transitive (Weak ordering).*

(C5') *For all distinct acts a, b, c, d, there exists a context x such that* $a \succ_x b \succ_x c \succ_x d$ *(diversity).*

(ii) *Statement* (ii) *of Theorem* 2.1 *holds (with* (P1) *imposed for all distinct acts a, b).*[5] *Also,*

(P4') *For all distinct acts a, b, c, d, the vectors* $(s_1^a - s_1^b, \ldots, s_n^a - s_n^b)$, $(s_1^b - s_1^c, \ldots, s_n^b - s_n^c)$, $(s_1^c - s_1^d, \ldots, s_n^c - s_n^d)$ *are linearly independent.*

Furthermore, uniqueness holds as in Theorem 2.1.

3. THE CASE REPETITION APPROACH

The decision maker facing a choice from a set of available (feasible) acts A has one actual memory $M_0 = \{(q_i, b_i, r_i) \mid 1 \leqslant i \leqslant n\}$. However, we assume that preferences between acts are given not only for M_0, but also for other memories, which may differ from M_0 in two respects. First, we also consider cases (q_i, b_i, r_j) for $j \neq i$. That is, we require that the decision maker express preferences also if certain past cases yielded outcomes that differ from the outcomes they yielded in actuality. Yet, we do not require the decision maker to imagine outcomes that he or she has never experienced. We only consider recombinations of circumstances that have actually appeared with outcomes that have actually been experienced.[6] Second, we assume that the decision maker can imagine memories in which these (recombined) cases appear any number of times, including zero. Thus, for example, President Clinton might have objected to the U.S. intervention in Bosnia had the militarily successful experiences (from the United States' point of view) of the Gulf war and the Noriega capture never occurred, or had episodes like the Marines' losses in Lebanon had occurred many times.

[5] Condition (P1) now means that $a \mapsto \sum_{i=1}^n s_i^a u(x_i)$ represents $\succcurlyeq_x$ for each x.

[6] The techniques and some of the results of this section can be extended to any arbitrary, finite set of outcomes. Our goal here is to obtain a representation while using as few outcomes as possible.

Notational convention for this section. The lower case letter c, with or without superscripts or subscripts, denotes cases. The lower case letters a, b, d, e and f, again with or without superscripts or subscripts, denote acts.

Formally, let M be the set of all possible recombinations: $M = \{c = (q_i, b_i, r_j) \mid 1 \leqslant i, j \leqslant n\}$. We refer to M as an $n \times n$ matrix. The set of repetitions of cases is $\mathbb{J} = \mathbb{Z}_+^M = \{I \mid I\colon M \to \mathbb{Z}_+\}$, where $\mathbb{Z}_+$ denotes the non-negative integers. For simplicity, we will refer to elements of $\mathbb{J}$ as *memories*. We assume that for every $I \in \mathbb{J}$ the decision maker has a binary relation *at least as desirable as* $\succcurlyeq_I$ on the set A. The latter is assumed to be non-empty and finite. Thus, the relation $\succcurlyeq_I$ represents the decision maker's ranking of acts in the problem p given the memory I. As usual, we define $\succ_I$ and $\sim_I$ to be the asymmetric and symmetric parts of $\succcurlyeq_I$, respectively. Algebraic operations on $\mathbb{J}$ are performed pointwise.

Comparing the case repetitions framework to the outcomes tradeoffs framework from a technical point of view, we have two different spaces representing the decisions contexts. In the previous section it was the n-dimensional product of a connected topological space, and in this section we have an n^2-dimensional vector space where the coordinates are non-negative integers i.e., $\mathbb{Z}_+^M$. In the previous model there must be infinitely many outcomes that constitute a connected space in the order topology. In the present model only finitely many outcomes are allowed, and the interpretation that each outcome has been actually encountered at least once is made possible.

We would like to obtain a representation of the relation $\succcurlyeq_I$ for every $I \in \mathbb{J}$. For this purpose we define, for all $I \in \mathbb{J}$ and $a \in A$:

$$(\ddagger) \quad U_I(a) = \sum_{(q_i, b_i, r_j) \in M} I(q_i, b_i, r_j)\, s^a(q_i, b_i)\, u(r_j),$$

and we wish to prove that $a \succcurlyeq_I b$ iff $U_I(a) \geqslant U_I(b)$. This constitutes the extension of Eq. (1) to all $I \in \mathbb{J}$, where $s^a(q_i, b_i)$ stands for $s((p, a), (q_i, b_i))$.

In view of ($\ddagger$), if for all $I \in \mathbb{J}$, $U_I(\cdot)\colon A \to \mathbb{R}$ represents $\succcurlyeq_I$, then clearly the following three axioms holds:

A1: *Order*. For every $I \in \mathbb{J}$, the relation $\succcurlyeq_I$, is complete and transitive on A.

A2: *Combination*. For every $I, J \in \mathbb{J}$ and every $a, b \in A$, if $a \succcurlyeq_I b$ ($a \succ_I b$) and $a \succcurlyeq_J b$, then $a \succcurlyeq_{I+J} b$ ($a \succ_{I+J} b$).

A3: *Archimedean axiom*. For every $I, J \in \mathbb{J}$ and every $a, b \in A$, if $a \succ_I b$, then there exists $k \in \mathbb{N}$ such that, $a \succ_{kI+J} b$.

Are the axioms plausible, independently of their implications? Axiom 1 requires that, given any conceivable memory, the decision maker's preference relation over acts is a weak order. Axiom 2 states that if act a is preferred to act b given two disjoint memories, a should also be preferred to b given the combination of these memories. If one data set leads to the conclusion that a is preferred to b, and the same conclusion follows given another data set, disjoint from the first, combining the data sets should not lead to another conclusion. In our setup, combination (or concatenation) of memories takes the form of adding the number of repetitions of each case in the two memories. Thus, Axiom 2 requires that preferences obey additivity with respect to the number of repetitions, i.e., additivity in cases. One can imagine nonadditivity in cases similarly to nonadditivity in probabilities.

Axiom 3 is a continuity axiom. It states that if, given memory I, the decision maker expresses strict preference for act a over b, then, no matter what his or her preferences are for another memory, J, there is a number of repetitions of I that is large enough to overwhelm the preferences induced by J. The meaning of this axiom may be clearer if we consider a violation thereof. Assume that a decision maker has to choose among candidates for a public office. Memory I contains cases in which candidate a performed well. Given this memory, a is preferred to b. Memory J contains but a single case in which candidate a was convicted of embezzlement. It is quite reasonable for a voter to prefer candidate b to a given memory $kI+J$ for every k. This would reflect lexicographic preferences for integrity and competence. Axiom A3 rules out such preferences. It requires that the most hideous crime be forgivable provided many instances of (even minor) good deeds.

Until now, in A1–A3, we did not require the structural assumption that cases are decomposable into problems, act, and results. These three axioms are consistent with the representation formula (‡), where for all $a \in A$ and $c=(q_i, b_i, r_j) \in M$, one substitutes $s^a(q_i, b_i)\, u(r_j)$ with the less informative $v^a(c) = v^a(q_i, b_i, r_j)$. In the next two axioms we will use the special structure of $v^a(c)$. To state it we need some further notation.

For a vector of non-negative integers $L=(l_1, ..., l_n) \in \mathbb{Z}_+^n$ and an outcome $r \in R = \{r_1, ..., r_n\}$, let $L(r) \in \mathbb{J}$ be defined by $L(r)(q_i, b_i, r) = l_i$ and $L(r)(q_i, b_i, t) = 0$ for $t \neq r$. An outcome s is *neutral*[7] if for every $L \in \mathbb{Z}_+^n$ and every $a, b \in A$, $a \sim_{L(s)} b$. Intuitively, neutral outcomes correspond to a utility value of zero. If only such outcomes have ever been experienced, no matter under which circumstances or how many times, there is no reason to distinguish between any two available acts. We assume that there are non-neutral outcomes. We can now formulate

A4: *Diversity*. For every list (a, b, d, e) of distinct elements of A and for every nonneutral $r \in R$, there exists $L \in \mathbb{Z}_+^n$ such that $a \succ_{L(r)} b \succ_{L(r)} d \succ_{L(r)} e$. If $|A| < 4$, then the same holds for any list of length $|A|$.

[7] Not to be confused with θ of the previous section. Here neutrality is defined and not assumed.

The diversity axiom is not necessary for the functional form we would like to derive. While the theorem we present is an equivalence theorem, it characterizes a more restricted class of preferences than the decision rule discussed in the introduction, namely those preferences satisfying Axiom 4 as well. The axiom implies that for any four alternatives, there is a memory that would distinguish among all four of them. It rules out dominated acts: an act b cannot belong to A if for some act $a \in A$ and for all $I \in \mathbb{J}$, $a \succ_I b$. Thus, we assume that the decision maker cannot know that one act dominates another by some logical deduction. Any preference between a and b should be based on past experience, and the set of circumstances that have been encountered is assumed rich enough to induce (via repetitions) preference for a over b but also vice versa. Similarly, any ranking of four alternatives is assumed to be possible given an appropriately chosen vector of repetitions of the circumstances.

Each of the memories used in A4 has only one, nonneutral, outcome. Intuitively, if for such a memory, say $L(r)$, the decision maker prefers a to b, it may be because of one of two reasons: either r is a desired outcome and circumstance (p, a) is more like those in the memory $L(r)$ than is (p, b), or r is an undesired outcome and (p, b) is more like the circumstances in $L(r)$ than is (p, a). The same reasoning applies to Axiom 5.

A5: *Case independence of desirability.* Assume that two outcomes r and s are not neutral. Then either

(i) for every $L \in \mathbb{Z}_+^n$ and every $a, b \in A$, $a \succcurlyeq_{L(r)} b \Leftrightarrow a \succcurlyeq_{L(s)} b$; or

(ii) for every $L \in \mathbb{Z}_+^n$ and every $a, b \in A$, $a \succcurlyeq_{L(r)} b \Leftrightarrow a \preccurlyeq_{L(s)} b$.

It is quite obvious that A5 is necessary for the desired representation. This axiom is reminiscent of the state independence axiom in Anscombe and Aumann's [1] expected utility model, as presented in Fishburn [3]. Now the main result of this section can be stated.

THEOREM 3. *Let there be given M_0, M and $\{\succcurlyeq_I\}_{I \in \mathbb{J}}$ with $\mathbb{J} = \mathbb{Z}_+^M$, as above. Then the following two statements are equivalent if $|A| \geqslant 4$:*

(i) $\{\succcurlyeq_I\}_{I \in \mathbb{J}}$ *satisfy* A1–A5;

(ii) *There are similarity vectors $\{s^a \in \mathbb{R}^n\}_{a \in A}$ and a utility vector $(u(r_j))_{j \leqslant n}$ such that:*

$$(*) \qquad \begin{cases} \text{for every } I \in \mathbb{J} \text{ and every } a, b \in A; \\ a \succcurlyeq_I b \quad \text{iff} \quad U_I(a) \geqslant U_I(b), \end{cases}$$

where $U_I(a) = \sum_{(q_i, b_i, r_j) \in M} I((q_i, b_i, r_j))\, s^a(q_i, b_i)\, u(r_j)$.

Also, for every list, (a, b, d, e), of distinct elements of A, the convex hull of the vectors $(s^a - s^b)$, $(s^b - s^d)$, and $(s^d - s^e)$ does not intersect $\mathbb{R}^n_-$.

Furthermore, in this case, the vectors $\{s^a\}_{a \in A}$ and the vector $(u(r_j))_{j \leqslant n}$ are unique in the following sense: if $(\{s^a\}_{a \in A}, (u(r_j))_{j \leqslant n})$ and $(\{\hat{s}^a\}_{a \in A}, (\hat{u}(r_j))_{j \leqslant n})$ both satisfy (*), then there are scalars α, β with $\alpha\beta > 0$, and a matrix $w \in \mathbb{R}^M$ such that for all $a \in A$, $\hat{s}^a = \alpha s^a + w$, and for all $j \leqslant n$, $\hat{u}(r_j) = \beta u(r_j)$.

Finally, in the case $|A| < 4$, the numerical representation result (as in (*)) holds and uniqueness as above is guaranteed.

The implications of A1–A5 include, in addition to the required representation, the convex hull condition. This latter condition implies, together with the representation (**), the diversity axiom, A4, which, as pointed out earlier, is not necessary for the representation itself.

Axioms A1–A3 are not new and appeared in Gilboa and Schmeidler [6, 7]. Similar axioms appeared in Gilboa and Schmeidler [5]. Axiom 4 introduced here is slightly more restrictive than similar axioms in the above references. For comparison and use in the proof, the weaker version is stated here:

A4*: *Diversity.* For every list (a, b, d, e) of distinct elements of A there exists $I \in \mathbb{J}$ such that $a \succ_I b \succ_I c \succ_I d$. If $|A| < 4$, then for any strict ordering of the elements of A there exists $I \in \mathbb{J}$ such that $\succ_I$ is that ordering.

Extensive discussions of these axioms appear in Gilboa and Schmeidler [6, 7]. There also appears a basic result which is used in our proof of Theorem 3.

APPENDIX: PROOFS AND RELATED ANALYSIS

Proof of Theorem 2.1. We first assume (ii) and derive (i). Completeness is immediate. (C2) follows because equivalent outcomes have the same u value. For monotonicity note that, because u represents $\succcurlyeq$, $s_i^a - s_i^b > 0$ implies that problem i is favorable for a vis-a-vis b, $s_i^a - s_i^b = 0$ implies that problem i is neutral, and $s_i^a - s_i^b < 0$ implies that problem i is unfavorable. Note that favorableness, neutrality, and unfavorableness of problem i are mutually exclusive because of the presence of a neutral context $(\theta, ..., \theta)$ and the existence of outcomes $\alpha \succ \theta \succ \beta$.

For continuity, the set of $\sim$ equivalence classes of R is homeomorphic to the range of u, hence is connected; therefore, so is R. Continuity of u implies continuity in x of $\sum_{i=1}^n (s_i^a - s_i^b)\, u(x_i)$, hence the inverse of the closed subset $[0, \rightarrow)$ of $\mathbb{R}$ is also closed. This inverse is $\{x \in R^n \mid a \succcurlyeq_x b\}$.

Next we consider richness. Given that a and b are equivalent under context $(\theta, \ldots, \theta)$ and $u(\alpha) > u(\theta)$, $\alpha_i(\theta, \ldots, \theta)$ has a and b not indifferent whenever $(s_i^a - s_i^b) \neq 0$. For these problems, replacing the outcome θ by the outcome α has affected the preference between a and b and hence the belonging problems are nonneutral. Because of (A4), at least three problems are nonneutral. $u(\theta) = 0$ and $u(\alpha) > u(\theta) > u(\beta)$ for some α and β imply that all acts are equivalent for context $(\theta, \ldots, \theta)$ and that $\alpha \succ \theta \succ \beta$. Solvability follows because $(s_i^a - s_i^b) \neq 0$ and $u(R) = \mathbb{R}$.

Tradeoff consistency was derived in Section 2. $u(\delta) = u(\delta')$ indeed implies $\delta \sim \delta'$ because u represents $\succcurlyeq$. This completes the derivation of the implication (ii) $\Rightarrow$ (i).

Henceforth, we assume (i) and derive (ii) and the uniqueness results. We first modify the model so as to have $\succcurlyeq$ antisymmetric, i.e., if $\alpha \sim \beta$ then $\alpha = \beta$. To this effect, define, for any outcomes α, $[\alpha]$ as its $\sim$-indifference class, i.e., $[\alpha] = \{\beta \mid \beta \sim \alpha\}$. For every context x, write $[x]$ for $([x_1], \ldots, [x_n])$. We can define preference relations $\succcurlyeq_{[x]}$ by choosing any element $y \in [x]$ and defining $\succcurlyeq_{[x]} = \succcurlyeq_y$; because evidence is preference-driven (C2), the definition is independent of the particular choice of y. Let $[R]$ denote the set of indifference classes in R. We can replace R by $[R]$ while preserving all conditions (C1)–(C6) for R. In particular, the order topology on $[R]$ is connected and preference continuity holds, and so does tradeoff consistency. From now on, we write x instead of $[x]$ etc., that is, we assume that every indifference class in R contains exactly one element. Proving the result for this modified structure implies the same result for the original structure.

The following lemma follows from preference continuity by, first, interchanging the role of a and b and, second, by taking the intersection $\{x \in R^n \mid a \succcurlyeq_x b\} \cap \{x \in R^n \mid a \preccurlyeq_x b\}$.

LEMMA 1. *$\{x \in R^n \mid a \preccurlyeq_x b\}$ and I_{ab} are closed.*

We first derive the representation for the contexts in I_{ab}, i.e., we obtain a representation as in the theorem that is 0 on I_{ab}. There are at least three nonneutral problems. Assume that problem 1 is nonneutral and assume that in fact it is favorable. The case of unfavorable problem 1 is analogous. We will consider R^{n-1} for some time; the generic notation for its elements is $x' = (x_2, \ldots, x_n)$. Because of solvability, there exists α for each x' such that $(\alpha, x') \in I_{ab}$. Here (α, x') denotes $(\alpha, x_2, \ldots, x_n)$. By favorableness of problem 1 and antisymmetry, the mentioned α is unique. We define the map $V: R^{n-1} \to R$ by assigning to each x' the mentioned α.

$\beta_i x'$ denotes the $n-1$ list x' with x_i replaced by β. $(\sigma, \beta_i x')$ denotes the n-list $(\sigma, x_2, \ldots, x_{i-1}, \beta, x_{i+1}, \ldots, x_n)$. V generates a binary relation, denoted $\mathscr{R}$, on R^{n-1} defined by $x' \mathscr{R} y'$ if and only if $V(x') \preccurlyeq V(y')$. Thus higher-preferred values of V correspond to lower-$\mathscr{R}$-preferred elements of R^{n-1}.

We say that V *anti-represents* $\mathscr{R}$ and denote by $\mathscr{I}$ and $\mathscr{P}$ the symmetric and asymmetric parts of $\mathscr{R}$, respectively. We first show that $\mathscr{R}$ can be represented by a form $\sum_{j=2}^{n} \lambda_j u(x_j)$. It is obvious that $\mathscr{R}$ inherits weak ordering from $\succcurlyeq$ on R

LEMMA 2. *$\mathscr{R}$ satisfies monotonicity, i.e., for all $i \geqslant 2$:*

— *If i is favorable for a vis-a-vis b, then $\alpha_i x' \mathscr{P} \beta_i x'$ whenever $\alpha \succ \beta$.*
— *If i is neutral for a vis-a-vis b, then always $\alpha_i x' \mathscr{I} \beta_i x'$.*
— *If i is unfavorable for a vis-a-vis b, then $\beta_i x' \mathscr{P} \alpha_i x'$ whenever $\alpha \succ \beta$.*

Proof. Assume that $(\sigma, \alpha_i x')$ and $(\tau, \beta_i x')$ belong to I_{ab}, where $\alpha \succ \beta$. If problem i is favorable then, because so is problem 1, $\sigma \succcurlyeq \tau$ and $(\tau, \beta_i x') \in I_{ab}$ would imply $a \succ_{\sigma, \alpha_i x'} b$, contradicting $(\sigma, \alpha_i x') \in I_{ab}$. Hence $\sigma \prec \tau$ must hold, thus $V(\alpha_i x') \prec V(\beta_i x')$, and $\alpha_i x' \mathscr{P} \beta_i x'$ follows.

If problem i is neutral then $(\sigma, \alpha_i x') \in I_{ab}$ implies $(\sigma, \beta_i x') \in I_{ab}$, i.e., $V(\alpha_i x') = V(\beta_i x')$, and $\alpha_i x' \mathscr{I} \beta_i x'$ follows.

The case of unfavorable i is similar to favorable i. Q.E.D.

LEMMA 3. *V is continuous.*

Proof. Let $\alpha \in R$. The set $V^{-1}\{\beta \in R \mid \beta \succcurlyeq \alpha\}$ is the projection on R^{n-1} of the closed set $I_{ab} \cap \{x \in R^n \mid x_1 \succcurlyeq \alpha\}$, hence is closed again. (I_{ab} is closed by Lemma 1.) $V^{-1}\{\beta \in R \mid \beta \preccurlyeq \alpha\}$ is similarly closed. Their complements are open. Because the sets $\{\beta \in R \mid \beta \prec \alpha\}$ and $\{\beta \in R \mid \beta \succ \alpha\}$ generate the topology on R, V is continuous. Q.E.D.

COROLLARY 4. *$\mathscr{R}$ is continuous.*

$\mathscr{R}$ satisfies *equivalence-tradeoff consistency* if, for any nonneutral problems i, j, $\alpha_i x' \mathscr{I} \gamma_i y'$, $\beta_i x' \mathscr{I} \delta_i y'$, and $\alpha_j v' \mathscr{I} \gamma_j w'$ imply $\beta_j v' \mathscr{I} \delta_j w'$.

LEMMA 5. *$\mathscr{R}$ satisfies equivalence-tradeoff consistency.*

Proof. We use solvability throughout. The three $\mathscr{I}$ relations in the premise imply that there exist σ, τ, μ such that $(\sigma, \alpha_i x')$, $(\sigma, \gamma_i y')$, $(\tau, \beta_i x')$, $(\tau, \delta_i y')$, $(\mu, \alpha_j v')$, $(\mu, \gamma_j w') \in I_{ab}$. Take v and δ' such that also $(v, \beta_j v')$, $(v, \delta'_j w') \in L_{ab}$. By evidence-tradeoff consistency, $\delta' \sim \delta$ (so $\delta' = \delta$). Hence $(v, \delta_j w') \in I_{ab}$. This and $(v, \beta_j v') \in I_{ab}$ imply $\beta_j v' \mathscr{I} \delta_j w'$. Q.E.D.

The *generalized Reidemeister condition* is the special case of equivalence-tradeoff consistency with $i = j$. It implies the representation in the following lemma. Functions V_j, $j = 2, \ldots, n$, are *joint ratio scales* if they can be replaced by functions W_j, $j = 2, \ldots, n$ if and only if there exists a positive σ such that $W_j = \sigma V_j$ for all j.

LEMMA 6. *There exist continuous functions V_j: $R \to \mathbb{R}$, $j = 2, \ldots, n$, such that $\mathscr{R}$ is represented by $\sum_{j=2}^{n} V_j(x_j)$ and $V_j(\theta) = 0$ for all j. The V_j-s are joint ratio scales. For each favorable i, V_i represents $\succcurlyeq$, for each unfavorable i, $-V_i$ represents $\succcurlyeq$.*

Proof. Because at least three problems are nonneutral, and by Lemma 2, at least two of the problems 2, ..., n are essential (j is *essential* if $\alpha_j x \mathscr{P} \beta_j x$ for some x, α, β). The lemma can now be derived from Wakker [16, Theorem III.6.6]. Unfortunately, the result needed here is not stated exactly in the mentioned reference and can only be obtained by combining several remarks and lemmas, as follows: By Wakker's Remark III.7.1, topological separability of R is not needed. Monotonicity (C3) implies Wakker's weak separability. For the case where exactly two of the problems 2, ..., n are nonneutral (so essential), weak separability is equivalent to Wakker's CI (see his Remark III.7.4), and the generalized Reidemeister condition is equivalent to Wakker's Reidemeister condition. The statement at the end of Wakker's Theorem III.6.6 now gives the desired result. For the case of three or more nonneutral problems among 2, ..., n, by Wakker's Remark III.7.3, his condition CI is only needed with equivalences instead of preferences and that condition is derived from the generalized Reidemeister condition exactly as Wakker's CI derived from his generalized triple cancellation in Lemma III.6.5.

The uniqueness result follows from Wakker [16, Observation III.6.6'] and the additional requirement of $V_j(\theta) = 0$. The results on representation of $\succcurlyeq$ follow from Lemma 2 and antisymmetry. Q.E.D.

LEMMA 7. *There exist continuous functions V_j: $R \to \mathbb{R}$, $j = 1, \ldots, n$, such that $[x \in I_{ab} \Rightarrow \sum_{j=1}^{n} V_j(x_j) = 0]$ and $V_j(\theta) = 0$ for all j. For each favorable i, V_i represents $\succcurlyeq$, for each unfavorable i, $-V_i$ represents $\succcurlyeq$. Given all these restrictions, the V_j-s are joint ratio scales.*

Proof. Take the V_j, $j = 2, \ldots, n$ of Lemma 6. Because of solvability, for each $\alpha \in R$ there exists an x' such that $(\alpha, x') \in I_{ab}$. For another y' such that $(\alpha, y') \in I_{ab}$ also, $y' \mathscr{I} x'$ and hence $\sum_{j=2}^{n} V_j(y_j) = \sum_{j=2}^{n} V_j(x_j)$. We can therefore define $V_1(\alpha) = -\sum_{j=2}^{n} V_j(x_j)$. By continuity and connectedness, the range of $-\sum_{j=2}^{n} V_j(x_j)$ is connected. Because of solvability, for each x' there exists an α such that $V_1(\alpha) = -\sum_{j=2}^{n} V_j(x_j)$, i.e., V_1 has the same range as $-\sum_{j=2}^{n} V_i(x_i)$ which is therefore also connected. Assume that $\alpha \succ \beta$. We show that $V_1(\alpha) > V_1(\beta)$. Because of solvability we can find x' such that $(\beta, x') \in I_{ab}$. There is an $i > 1$ nonneutral, say it is unfavorable (the case of i favorable is similar). Because of solvability, $(\alpha, \sigma_i x') \in I_{ab}$ for some σ. Because $\alpha \succ \beta$ and because of unfavorableness of i, $\sigma \preccurlyeq x_i$ would imply $a \succ_{\alpha, \sigma_i x'} b$ contradicting $(\alpha, \sigma_i x') \in I_{ab}$. Hence $\sigma \succ x_i$. Because $-V_i$ represents $\succcurlyeq$, $V_i(\sigma) < V_i(x_i)$. This, $V_1(\beta) + \sum_{j=2}^{n} V_j(x_j) = 0$, and $V_1(\alpha) + \sum_{j=2}^{i-1} V_j(x_j) + V_i(\sigma) + \sum_{j=i+1}^{n} V_j(x_j) = 0$, imply $V_1(\alpha) > V_1(\beta)$. So V_1 represents $\succcurlyeq$.

V_1 representing $\succcurlyeq$ and having a connected range, and $\succcurlyeq$'s order topology being connected, imply that V_1 is continuous.

To invoke the uniqueness result of Lemma 6, we have to demonstrate that $\sum_{j=2}^{n} V_j(x_j)$ as in Lemma 7 necessarily represents $\mathscr{R}$, as was the case in Lemma 6. This is derived from V_1 representing $\succcurlyeq$, as follows. Consider x' and y'. Then $x' \mathscr{R} y'$ holds if and only if $V(x') \preccurlyeq V(y')$, which holds if and only if $V_1(V(x')) \leqslant V_1(V(y'))$. Because $V_1(V(x')) + \sum_{j=2}^{n} V_j(x_j) = 0 = V_1(V(y')) + \sum_{j=2}^{n} V_j(y_j)$, the inequality of the preceding sentence is equivalent to $\sum_{j=2}^{n} V_j(x_j) \geqslant \sum_{j=2}^{n} V_j(y_j)$. Hence, indeed, $\sum_{j=2}^{n} V_j(x_j)$ represents $\mathscr{R}$.

$V_2, \ldots, V_n$ uniquely determine V_1 and the uniqueness result of Lemma 6 now straightforwardly implies that the V_js, including V_1, are ratio scales.

The requirement that V_1 (or, by symmetry, any other V_i for some nonneutral problem i) represents $\succcurlyeq$ was used in the proof to relate the functions in Lemma 7 to those in Lemma 6.

LEMMA 8. *Assume that V_j: $R \to \mathbb{R}$, $j = 1, \ldots, n$, are as in Lemma 7. Then every nonconstant V_i is proportional to every other nonconstant V_j.*

Proof. The proof of this lemma is identical to that of Lemma 11 with the following substitutions: b for c, a for d, $W_j = -V_j$. Q.E.D.

COROLLARY 9. *There exists a continuous u: $R \to \mathbb{R}$ that represents $\succcurlyeq$, satisfies $u(0) = 0$, and there exist real numbers $\lambda_1^{ab}, \ldots, \lambda_n^{ab}$ such that $a \sim_x b \Rightarrow \sum_{j=1}^{n} \lambda_j^{ab} u(x_j) = 0$. u is a ratio scale. λ_j^{ab} is positive for each favorable problem, zero for each neutral problem, and negative for each unfavorable problem.* Q.E.D.

LEMMA 10. *For the parameters of Corollary 9, the preference between a and b corresponds with the sign of $\sum_{j=1}^{n} \lambda_j^{ab} u(x_j)$ ($\succ$ if positive, $\sim$ if zero, $\prec$ if negative).*

Proof. Say $a \succ_x b$. Take a nonneutral i, say i is favorable. By solvability there exists α such that $a \sim_{\alpha_i x} b$. By favorableness of i, $\alpha \prec x_i$. We have $\sum_{j=1}^{n} \lambda_j^{ab} u(x_j) > \sum_{j=1}^{i-1} \lambda_j^{ab} u(x_j) + \lambda_i^{ab} u(\alpha) + \sum_{j=i+1}^{n} \lambda_j^{ab} u(x_j) = 0$. Similarly, $a \prec_x b$ implies $\sum_{j=1}^{n} \lambda_j^{ab} u(x_j) < 0$. These implications and Corollary 9 prove the lemma. Q.E.D.

Solvability readily implies that u is unbounded from both sides. (If u were bounded, say from above, then we could take, for instance, a (say) favorable, i for a vis-a-vis b, and α with $u(\alpha)$ very close to the supremum, and then x with $x_i = \alpha$ and $\sum_{j=1}^{n} \lambda_j^{ab} u(x_j)$ so negative (possible due to solvability), that no β would exist such that $a \sim_{\beta_i x} b$.)

We can now define s_i^a arbitrary and $s_i^b = s_i^a - \lambda_j^{ab}$, for all j and the proof of Theorem 2.1 is complete. Q.E.D.

Proof of Proposition 2.2. For the implication (i) $\Rightarrow$ (ii), diversity (C5') follows mainly from (P4') as in Gilboa and Schmeidler [5]; the rest follows as in Theorem 2.1. Next we assume (ii) and derive (i). Again, the proof closely follows the proof of Theorem 2.1. Lemmas and Corollaries 1–7 now hold true for each distinct pair of acts a, b. To prove that the utility u does not depend on the acts a, b, we generalize Lemma 8 as follows.

LEMMA 11. *Assume that* $V_j\colon R \to \mathbb{R}$, $j = 1, \ldots, n$, *are as in Lemma* 7. *Assume that* $W_j\colon R \to \mathbb{R}$, $j = 1, \ldots, n$ *are similar functions, only for acts* $c \neq d$ *that are possibly different from* a, b. *Then every nonconstant* V_i *is proportional to every nonconstant* W_j.

Proof. Assume that problem i is nonneutral for a vis-a-vis b and problem j is nonneutral for c vis-a-vis d. Take any $\alpha, \beta, \gamma, \delta$ such that $V_i(\alpha) - V_i(\beta) = V_i(\gamma) - V_i(\delta)$. Because there is a nonneutral problem k, $k \neq i$, for a vis-a-vis b, and because of solvability, $\alpha\sigma_{ik}x \in I_{ab}$ for some x, σ. Because k is nonneutral, we can also find τ such that $\beta\tau_{ik}x \in I_{ab}$. It follows that $V_i(\alpha) - V_i(\beta) = V_k(\tau) - V_k(\sigma)$. Because there is a nonneutral problem, other than i or k, y can be found such that $\gamma\sigma_{ik}y \in I_{ab}$. Substituting $V_i(\gamma) - V_i(\delta) = V_i(\alpha) - V_i(\beta) = V_k(\tau) - V_k(\sigma)$ implies that also $\delta\tau_{ik}y \in I_{ab}$.

Because there is a nonneutral problem l, $l \neq j$, for c vis-a-vis d, and because of solvability, $\alpha\mu_{jl}v \in I_{cd}$ for some v, μ. Because l is nonneutral, we can also find ν such that $\beta\nu_{jl}v \in I_{cd}$. It follows that $W_j(\alpha) - W_j(\beta) = W_l(\nu) - W_l(\mu)$. Because there is a nonneutral problem, other than j or l, w can be found such that $\gamma\mu_{jl}w \in I_{cd}$. We can also find δ' such that $\delta'\nu_{jl}w \in I_{cd}$. Evidence tradeoff consistency now implies $\delta \sim \delta'$ (hence, by antisymmetry, $\delta = \delta'$) implying $\delta\nu_{jl}w \in I_{cd}$. This and $\gamma\mu_{jl}w \in I_{cd}$ implies $W_l(\nu) - W_l(\mu) = W_j(\gamma) - W_j(\delta)$ which is therefore equal to $W_j(\alpha) - W_j(\beta)$.

We have demonstrated that $V_i(\alpha) - V_i(\beta) = V_i(\gamma) - V_i(\delta)$ implies $W_j(\alpha) - W_j(\beta) = W_j(\gamma) - W_j(\delta)$. These functions being continuous on a connected domain implies that they are related by an affine transformation. They are all 0 at θ, hence they are related by a positive or negative scale factor. Q.E.D.

Corollary 9 and Lemma 10 remain valid for all a, b where u can be taken independent of a and b due to Lemma 11.

At this stage, we can map the structure into a structure that satisfies all conditions of Gilboa and Schmeidler [5, the Theorem] by replacing all outcomes by their u values. Then all axioms of their theorem are satisfied and the rest of Proposition 2.2 follows from their theorem. Q.E.D.

Proof of Theorem 3. We first quote Theorem 3.1 from Gilboa and Schmeidler [6] or the main theorem from Gilboa and Schmeidler [7] which is used here as an

AUXILIARY THEOREM. *Let there be given a finite, nonempty set K and for every $I \in \mathbb{J} = \mathbb{Z}_+^K$ let $\succcurlyeq \subset A \times A$. Then the following two statements are equivalent if $|A| \geqslant 4$:*

(i) $\{\succcurlyeq_I\}_{I \in \mathbb{J}}$ *satisfy* A1–A3 *and* A4*;

(ii) *There are vectors $\{v^a \in \mathbb{R}^K\}_{a \in A}$ such that:*

$$\text{for every } I \in \mathbb{J} \text{ and every } a, b \in A,$$

(**)
$$a \succcurlyeq_I b \quad \text{iff} \quad \sum_{c \in K} I(c)\, v^a(c) \geqslant \sum_{c \in K} I(c)\, v^b(c),$$

and, for every list (a, b, d, e) of distinct elements of A, the convex hull of the vectors $(v^a - v^b)$, $(v^b - v^d)$ and $(v^d - v^e)$ does not intersect $\mathbb{R}_-^K$.

Furthermore, in this case the vectors $\{v^a\}_{a \in A}$ are unique in the following sense: $\{v^a\}_{a \in A}$ and $\{\hat{v}^a\}_{a \in A}$ both satisfy (**), *iff there are a scalar $\alpha > 0$ and a vector $w \in \mathbb{R}^K$ such that for all $a \in A$, $\hat{v}^a = \alpha v^a + w$.*

Finally, in the case $|A| < 4$, the numerical representation result (as in (**)) *holds and uniqueness as above is guaranteed.*

We start with (i) implies (ii). By the same implication in the auxiliary theorem with $K = M$, one has for every $a \in A$ an array $v^a \in \mathbb{R}^M$ such that (**) holds. Using A5 we will show that (*) of Theorem 3 holds too. That is, we have to show that there is an n-lists $(u(r_j))_{j \leqslant n}$ and for every $a \in A$ there is an n-list, $(s^a(q_i, b_i))_{i \leqslant n}$, such that for all $c = (q_i, b_i, r_j) \in M$ and $a \in A: v^a((q_i, b_i, r_j)) = s^a(q_i, b_i)\, u(r_j)$.

For concreteness, choose an act $e \in A$ and set, without loss of generality, $v^e = 0$. Given this convention, $\{v^a\}_{a \in A}$ are unique up to multiplication by a positive constant.

For every neutral outcome $r \in R = \{r_1, ..., r_n\}$, **define** $u(r) = 0$.

Let $\bar{r} \in R$ be an arbitrary, but fixed from now on, nonneutral outcome. **Define** $u(\bar{r}) = 1$, and for every $a \in A$ and every $i \leqslant n$, **define** $s^a(q_i, b_i) = v^a(q_i, b_i, \bar{r})$. Suppose that there is another nonneutral outcome, say t, such that condition (i) of A5 holds for $\bar{r}$ and t. Then for every $L \in \mathbb{Z}_+^n$, $\succcurlyeq_{L(r)} = \succcurlyeq_{L(t)}$. We will show

LEMMA 12. *There is* $\lambda > 0$ *such that for every* $a \in A$ *and every* $i \leqslant n$, $v^a((q_i, b_i, t)) = \lambda v^a((q_i, b_i, \bar{r}))$.

Proof. Once again we apply the auxiliary theorem, this time for $K = \{(q_1, b_1, \bar{r}), (q_2, b_2, \bar{r}), ..., (q_n, b_n, \bar{r})\}$. For simplicity we write $\mathbb{Z}^n_+$ for $\mathbb{Z}^K_+$. Then for every $L \in \mathbb{Z}^n_+$ define $\succcurlyeq_L = \succcurlyeq_{L(\bar{r})}$.

For any $r \in \{r_1, ..., r_n\} = R$ denote $\mathbb{J}(r) = \{I \in \mathbb{J} \mid I(q_i, b_i, r') = 0$ for $r' \neq r\}$.

Since A1–A3 of part (i) of Theorem 3 hold when restricted to $\mathbb{J}(\bar{r})$, and A4 was stated so that A4* holds for $\mathbb{J}(\bar{r})$, part (ii) of the auxiliary theorem as well as the uniqueness hold. Note that, for all $L \in \mathbb{Z}^n_+$, the matrices $v^a \in \mathbb{R}^M$, $(a \in A)$ represent $\succcurlyeq_{L(\bar{r})}$. Hence, the same matrices truncated to their $\bar{r}$'s columns, $(v^a(q_i, b_i, \bar{r}))_{i \leqslant n} \in \mathbb{R}^n$, $(a \in A)$ represent $\succcurlyeq_L$ $(= \succcurlyeq_{L(\bar{r})})$.

Substituting t for $\bar{r}$ in the paragraph above results in the symmetric conclusion, i.e., the vectors $(v^a(q_i, b_i, t))_{i \leqslant n} \in \mathbb{R}^n$, $(a \in A)$ represent $\succcurlyeq_L$ $(= \succcurlyeq_{L(t)})$. Uniqueness and the normalization, $v^e = 0$, imply the assertion in the lemma. Q.E.D.

We now can define $u(s) = \lambda$. Similarly, applying Lemma 12 we define $u(r)$ for every other nonneutral $r \in R$ such that condition (i) of A5 holds for $\bar{r}$ and r. If for some nonneutral $t \in R$ condition (ii) of A5 holds for $\bar{r}$ and t, then for every $L \in \mathbb{Z}^n_+$, $\preccurlyeq_{L(t)} = \succcurlyeq_{L(\bar{r})}$. Thus $(-v^a(q_i, b_i, t))_{i \leqslant n} \in \mathbb{R}^n$, $(a \in A)$ represent $\succcurlyeq_{L(\bar{r})}$, and again Lemma 12 guarantees existence of some $\theta > 0$ such that $-v^a((q_i, b_i, t)) = \theta v^a((q_i, b_i, \bar{r}))$ for every $a \in A$ and every $i \leqslant n$. Defining $u(t) = -\theta$ and continuing in this way for any other nonneutral outcome will complete the construction of the required vector of utilities, $(u(r_j))_{j \leqslant n}$, as in (ii) of Theorem 3.

To complete the proof of (ii) we have to show that for every list, (a, b, d, e), of distinct elements of A, the convex hull of the vectors $(s^a - s^b)$, $(s^b - s^d)$ and $(s^d - s^e)$ does not intersect R^n_-. But since $s^a(q_i, b_i) = v^a(q_i, b_i, \bar{r})$ for all i and a, it has been shown in the proof of Lemma 12.

The opposite direction, that (ii) implies (i) is immediate. Note that $r \in R$ is neutral iff $u(r) = 0$, condition (i) of A5 holds iff $u(r) > 0$, and condition (ii) of A5 holds iff $u(r) < 0$. Furthermore, to prove that (ii) implies A4, the auxiliary theorem has to be used with $K = \{(q_1, b_1, r), (q_2, b_2, r), ..., (q_n, b_n, r)\}$, for every nonneutral $r \in R$. The remainder of the claims in Theorem 3 are proved in the same way. Q.E.D.

REFERENCES

1. F. J. Anscombe and R. J. Aumann, A definition of subjective probability, *Ann. Math. Stat.* **34** (1963), 199–205.

2. B. de Finetti, La Prévision: Ses Lois Logiques, ses Sources Subjectives, *Ann. Inst. H. Poincaré* **7** (1937), 1–68. [Translated into English by H. E. Kyburg, Foresight: Its logical laws, its subjective sources, *in* "Studies in Subjective Probability" (H. E. Kyburg and H. E. Smokler, Eds.), Wiley, New York, 1964.]
3. P. C. Fishburn, "Utility Theory for Decision Making," Wiley, New York, 1970.
4. I. Gilboa and D. Schmeidler, Case-based decision theory, *Quart. J. Econ.* **110** (1995), 605–639.
5. I. Gilboa and D. Schmeidler, Act-similarity in case-based decision theory, *Econ. Theory* **9** (1997), 47–61.
6. I. Gilboa and D. Schmeidler, "A Theory of Case-Based Decision," Cambridge Univ. Press, Cambridge, UK, 2001.
7. I. Gilboa and D. Schmeidler, "Inductive Inference: An Axiomatic Approach," Foerder Institute for Economic Research Working Paper 29–99, 1999. [In the revised version (2001), see the auxiliary theorem.]
8. B. Grodal, "Some Further Results on Integral Representation of Utility Functions," Institute of Economics, University of Copenhagen, Copenhagen, 1978.
9. F. Gul, Savage's theorem with a finite number of states, *J. Econ. Theory* **57** (1992), 99–110. [Erratum: *J. Econ. Theory* **61** (1993), 184.]
10. V. Koebberling and P. P. Wakker, "A Tool for Qualitatively Testing, Quantitatively Measuring, and Normatively Justifying Expected Utility," CentER, Tilburg University, Tilburg, The Netherlands, 1999.
11. H. E. Kyburg and H. E. Smokler (Eds.), "Studies in Subjective Probability," pp. 53–118, Wiley, New York, 1964; 2nd ed., 1980, Krieger, New York.
12. R. B. Myerson, Axiomatic derivation of scoring rules without the ordering assumption, *Soc. Choice Welfare* **12** (1995), 59–74.
13. F. P. Ramsey, Truth and probability, *in* "The Foundations of Mathematics and Other Logical Essays," pp. 156–198, Routledge and Kegan Paul, London, 1931. [Reprinted in "Studies in Subjective Probability" (H. E. Kyburg and H. E. Smokler, Eds.), Wiley, New York, 1964.
14. L. J. Savage, "The Foundations of Statistics," Wiley, New York, 1954. [Second edition, 1972, Dover, New York.]
15. J. H. Smith, Aggregation of preferences with variable electorate, *Econometrica* **41** (1973), 1027–1041.
16. P. P. Wakker, "Additive Representations of Preferences, A New Foundation of Decision Analysis," Kluwer Academic, Dordrecht, 1989.
17. P. H. Young, Social choice scoring functions, *SIAM J. Appl. Math.* **28** (1975), 824–838.