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# EXPECTED UTILITY WITH PURELY SUBJECTIVE NON-ADDITIVE PROBABILITIES

Itzhak GILBOA*

*Tel-Aviv University, Ramat-Aviv, Tel-Aviv 69978, Israel*

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Acts are functions from the set of states of the world into the set of consequences. Savage proposed axioms regarding a binary relation on the set of acts which are necessary and sufficient for it to be representable by the functional $\int u(\cdot)\,dP$ for some real-valued (utility) function $u$ on the set of consequences and a (probability) measure $P$ on the set of states of the world. The Ellsberg paradox leads us to reject one of Savage's main axioms – the Sure Thing Principle – and develop a more general theory, in which the probability measure need not be additive.

## 1. Introduction

The problem of 'subjective' (or 'personalistic') probability, which is at the root of Bayesianism, has aroused interest since the early works of Bayes. The research on subjective probability attained new momentum with the works of F.P. Ramsey and B. De Finetti. However, the most convincing and well-known axiomatization of subjective probability was given by Savage (1954). He started with a preference relation over acts (i.e., functions from the states of the world into the consequences), in order to end up with a utility function and a probability measure, such that the individual's decisions are being made so as to maximize the expected utility.

However compelling Savage's axioms and results are, they are not immune to attacks. The following example is due to Ellsberg (1961): Suppose there are two urns, each one containing 100 balls. The balls may be either black or red. Urn A is known to contain 50 black balls and 50 red ones. There is no information whatsoever about the number of black (or red) balls in urn B. You are now asked to choose an urn and a color, and then to draw a ball from the urn you named. (Of course, you are not allowed to see the balls in the urn when choosing one of them.) If the ball you draw will be of the color

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you have chosen, you will gain \$10. Otherwise you will not gain anything. What will be your choice?

It has been claimed that, while most people are indifferent with respect to the color they bet on, they are not indifferent with respect to the urn they choose. For instance, a 'reasonable' preference relation would be to strictly prefer any bet (red or black) on the urn in which 'probabilities are known' over any bet on the other urn. (Note that no 'objective' or 'physical' probabilities are known in either case.) It is easy to see that such a decision-making pattern violates Savage's 'Sure Thing Principle' [axiom P2 in Savage (1954)]: The decision maker has four possible acts to be denoted by $AR$, $AB$, $BR$, $BB$: ($AR$ – betting on a red ball drawn from urn A; $AB$ – betting on a black one, etc.). Each act has two possible consequences (a gain of \$10 or \$0) and we have four states in the world: $s_{RR}$: a ball drawn from either urn will turn out to be red; $s_{RB}$: a ball drawn from urn A will be red, while a ball drawn from urn B will be black; and $s_{BR}$, $s_{BB}$ are defined symmetrically.

We thus have the following table which specifies the acts as functions from states of the world into the numerical prizes:

| | $s_{RR}$ | $s_{RB}$ | $s_{BR}$ | $s_{BB}$ |
|---|---|---|---|---|
| $AR$ | 10 | 10 | 0 | 0 |
| $AB$ | 0 | 0 | 10 | 10 |
| $BR$ | 10 | 0 | 10 | 0 |
| $BB$ | 0 | 10 | 0 | 10 |

Now, let $C$ be the event $\{s_{RR}, s_{BB}\}$. Then we have

$$AR(s) = BR(s) \quad \text{and} \quad AB(s) = BB(s) \quad \text{for} \quad s \in C$$

and

$$AR(s) = BB(s) \quad \text{and} \quad AB(s) = BR(s) \quad \text{for} \quad s \in C^c.$$

Savage's Sure Thing Principle implies that the preference relation over acts ($\geqq$) must satisfy

$$AR \geqq BR \Leftrightarrow BB \geqq AB$$

while for the preference relation discussed above we have

$$AR \sim AB > BB \sim BR$$

(where $>$ and $\sim$ are defined as the asymmetric and symmetric parts of $\geqq$, respectively).

One can also directly verify that no additive probability measure may explain these preferences. However, if we allow the measure $v$ to be non-additive, as suggested by Schmeidler (1982 and 1984) we may have

$$v(\{s_{RR}, s_{RB}\}) = v(\{s_{BR}, s_{BB}\}) = 1/2$$

but

$$v(\{s_{RR}, s_{BR}\}) = v(\{s_{RB}, s_{BB}\}) = 0,$$

such that the expected value of acts $AR$ and $AB$ is 5 while that of $BR$ and $BB$ is zero. (The way in which expectation is computed when the probability measure is not necessarily additive will be discussed in the sequel; for the time being we may consider the last statement as a requirement which the new definition of expectation will have to satisfy.)

Roughly speaking, in this paper we shall replace the Sure Thing Principle by weaker axioms, which will allow for non-additivity of the measure, thus explaining the Ellsberg Paradox.

Although this was not the primary motive for developing the non-additive expected utility theory, it turned out that Schmeidler's model may also explain some of the 'paradoxes' or counterexamples to the von Neumann–Morgenstern (1947) expected utility theory, which have already stimulated many studies of various generalizations of expected utility theory. Some of the latest of these studies (most of which are in a purely objectivistic context) such as Quiggin's (1982), Yaari's (1984) and others, lead to results that are special cases of the non-additive theory.

Schmeidler's works provide an axiomatization for expected utility maximization, where the probability measure is not necessarily additive, in the framework of Anscombe and Aumann (1963). Their model, as opposed to Savage's, involves both 'objective' ('physical') and 'subjective' probabilities, while only the latter are derived from a preference relation over acts ('Horse Lotteries'), and the former are primitives of the model. This model is mathematically simpler than that of Savage, but it has the drawback of using the controversial concept of 'objective' probabilities.

This paper is the non-objectivistic counterpart of that of Schmeidler: it axiomatizes expected utility maximization with a non-additive subjective probability in a Savageian spirit. That is to say, it does not presume the existence of 'physical' probabilities with respect to which one may 'mix' lotteries, satisfy the von Neumann–Morgenstern independence axiom and so forth. The connections among the models will be clarified by the following table:

| | Objective and subjective probabilities | Only subjective probabilities |
|---|---|---|
| Additive probabilities | Anscombe–Aumann (1963) | Savage (1954) |
| Non (necessarily) additive probabilities | Schmeidler (1982) | The paper you're now reading |

(The lower row models are generalizations of the respective upper row ones.)

Mathematically speaking, there is a great difference between the right and the left columns of the table, since the mathematical objects involved in them are quite different, whereas there is a considerable similarity in the nature of the mathematical work within each column. However, it should be pointed out that this paper, although constantly comparing itself to that of Savage, differs significantly from the latter. In fact, almost none of Savage's results were proved applicable, and even the fundamental von Neumann–Morgenstern expected utility theorem (1947), which is at the basis of all three existing theories, could not be used here.

The paper is organized as follows: Section 2 deals with some preliminaries, namely: the framework of the model and some useful definitions; The Choquet integral; Savage's theorem (for comparison purposes) and Statement and brief discussion of the axioms for the non-additive theory. Sections 3 and 4 contain the proof of the main representation theorem. In section 3 the probability measure is almost constructed, or rather, something that is almost a measure is constructed. Section 4 develops the utility theory, by defining a utility function and proving some representation theorems. One of the stages is, of course, the completion of the construction of the measure. However, the distinction between these sections, which is undeniably somewhat arbitrary, is based on their subject-matter: section 3 goes as far as the theory proceeds without mentioning the word 'utility', and there begins section 4.

*Note.* Additional results regarding the independence of the axioms and the continuity of the measure are to be found in Gilboa (1985).

## 2. Preliminaries

### 2.1. Framework and definitions

Let $S$ be the set of *states of the world*, $X$ the set of *consequences*, and $F=\{f:S\to X\}$ the set of *acts*. Subsets of $S$ will be called *events*. For $f, g\in F$ and $A\subset S$ we will define $f/_A^g$ to be the element of $F$ satisfying:

$$f/_A^g(s)=f(s)\quad \forall s\in A^c, \qquad f/_A^g(s)=g(s)\quad \forall s\in A.$$

For $x \in X$ we will define $x \in F$ to be the constant act:

$$x(s) = x \quad \forall s \in S.$$

Since no confusion may result, we will not distinguish between the notations of the two entities (the consequence and the act). $\geqq$ will denote a binary relation over $F$: $\geqq \subset F \times F$, to be interpreted as the preference relation. $(>, \leqq, <, \sim)$ are defined in the usual way. An act assuming only finitely many values is said to be a *simple act* or a *step function.*

We will use the following:

*Notation.* For $x_1 > x_2 > \cdots > x_n (x_i \in X)$ and $\phi = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n = S$, $(x_1, A_1; x_2, A_2; \ldots; x_n, A_n)$ denotes the simple act $f$ satisfying

$$f(s) = x_i \quad \forall s \in A_i - A_{i-1}, \qquad 1 \leqq i \leqq n.$$

Using this notation will henceforth presuppose that $x_1 > x_2 > \cdots > x_n$ and $A_1 \subset A_2 \subset \cdots \subset A_n$. That is to say, any statement involving the act $(x_1, A_1; \ldots; x_n, A_n)$ should be read as follows: '$x_1 > \cdots > x_n$, $A_1 \subset \cdots \subset A_n$, and...'.

A set function $v: 2^S \to R$ will be called *a measure* iff it satisfies

(i) $\quad E \subset F \Rightarrow v(E) \leqq v(F)$,

(ii) $\quad v(\phi) = 0, \qquad v(S) = 1.$

If not explicitly stated, a 'measure' is not assumed to be additive. A measure $v$ is said to have a *convex range* if for any $B \subset A \subset S$ and any $\alpha \in [0, 1]$ there is an event $C$, $B \subset C \subset A$, such that

$$v(C) = \alpha v(B) + (1 - \alpha) v(A).$$

A real function over $X$ will be called a *utility.*

Two acts $f$, $g \in F$ are said to be *comonotonic* iff there are no $s$, $t \in S$ such that

$$f(s) > f(t) \quad \text{and} \quad g(s) < g(t).$$

An event $A$ will be said to be *f-convex* for an act $f$, iff the following condition holds:

For any $s$, $t \in A$, $r \in S$ such that $f(s) < f(r) < f(t)$, it is true that $r \in A$.

### *2.2. The Choquet integral*

The introduction of non-additive probabilities poses some difficulties. First of all, the integration w.r.t. (with respect to) such (probability) measures is not well-defined: Consider a constant function over $[a,b]\subset R$, and note that the partial Riemann sums (which are all supposed to equal the integral) depend upon the specific partition of the domain. Straightforward definitions of the integral (such as summation over maximal sets, on which the integrand is constant) are bound to face problems of non-monotonicity and/or discontinuity of the functional. It turns out that the natural integral for the non-necessarily additive measures is the Choquet integral, defined as follows: Let $S$ be the domain of the integrands, and $v$ – a measure on $S$. The integral of $w:S\to R$ w.r.t. $v$ (over $S$) is defined to be

$$\int w\,\mathrm{d}v=\int_0^\infty v(\{s\mid w(s)\geqq t\})\,\mathrm{d}t-\int_{-\infty}^0 [v(\{s\mid w(s)\geqq t\})-1]\,\mathrm{d}t. \qquad (*)$$

This integral was defined in Choquet (1955), and is used and discussed in Schmeidler (1986). In this paper the symbol $\int w\,\mathrm{d}v$ will always stand for this functional.

Note that, since the integrands in the two extended Riemann integrals in (*) are monotone functions, the Choquet integral always exists, which is a lot to ask of an integral.

A useful definition will be the following: the utility $u$ and the measure $v$ are said to constitute an Integral-Representation (IR) of $\geqq$ over $\bar{F}\subset F$, iff

$$f\geqq g\Leftrightarrow\int u(f)\,\mathrm{d}v\geqq\int u(g)\,\mathrm{d}v \quad \forall f,g\in\bar{F}.$$

### 2.3. Savage's theorem

To formulate Savage's theorem, one has to cite the axioms and definitions involved in it: [The symbol $\mathrm{P}n(1\leqq n\leqq 7)$ denotes an axiom.]

*P1.* $\geqq$ is complete and transitive.
*P2.* (Sure Thing Principle). For all $f$, $g$, $h_1$, $h_2\in F$ and any $A\subset S$,

$$f/^{h_1}_{A^c}\geqq g/^{h_1}_{A^c}\Leftrightarrow f/^{h_2}_{A^c}\geqq g/^{h_2}_{A^c}.$$

*Definition.* If $f/^h_{A^c}\geqq g/^h_{A^c}$ for some ($\Leftrightarrow$ all by P2) $h\in F$, we shall say that $f\geqq g$ *given A*.

*Definition.* If for all $f$, $g\in F$, $f\geqq g$ given $A$, $A$ will be said to be *null.*

*P3.* If $A\subset S$ is not null, then for all $f\in F$; $x,y\in X$,

$$f/^x_A\geqq f/^y_A\Leftrightarrow x\geqq y.$$

*P4.* For all $x_1, y_1, x_2, y_2, \in X$ and all $A, B \subset S$,

$$(x_1, A; y_1, S) \geqq (x_1, B; y_1, S) \quad \text{iff} \quad (x_2, A; y_2, S) \geqq (x_2, B; y_2, S).$$

(Recall that the above notation presupposes that $x_1 > y_1$, $x_2 > y_2$.)

*Definition.* If for some ($\Leftrightarrow$ all by P4) $x, y \in X$, $(x, A; y, S) \geqq (x, B; y, S)$ then $A \geqq \cdot B$.

*P5.* There are $x^*, x_* \in X$ such that $x^* > x_*$.

*P6.* For any $f, g, h \in F$ such that $f > g$, there is a finite partition of $S$ $(B_1, \ldots, B_n)$ such that

$$f/_{B_i}^{h} > g \quad \text{and} \quad f > g/_{B_i}^{h} \quad \forall i.$$

*P7.* If $f \leqq (\geqq) g(s)$ given $A$ for all $s \in A$, then $f \leqq (\geqq) g$ given $A$.

*Savage's Theorem. Suppose $\geqq$ satisfies P1–P7. Then there are a unique (finitely) additive probability measure P on S with a convex range, and a bounded utility u, unique up to a positive linear transformation (p.l.t.), such that $\geqq$ is integral-represented by $(u, P)$ over all F.*

[This is a slight rephrasing of the original (Savage's) theorem. The axioms are basically the original, rewritten with some new notations, whereas the conclusion is based on that appearing in Fishburn (1970).]

### 2.4. Axioms for a non-additive theory

The main difference between the additive and non-additive theories is the Sure Thing Principle, accepted by Savage, but rejected by the non(necessarily) additive theory. This means that we cannot accept Savage's P2, and consequently have to replace it by a weaker version.

However, it turns out that there are some technical differences between the two theories, which call for modifying or replacing other axioms as well:

- P3 as phrased, turns out to be too strong an axiom, excluding some of the measures we have no reason to object to;
- P4 is implied by P2's substitute;
- P5 is too weak, since the minimal number of $\geqq$-distinguishable consequences needed for the uniqueness of the measure is 3 in the non-additive theory (rather than 2);
- P6 is simply insufficient for any kind of continuity in a non-additive context. Here it will be replaced by two axioms, of non-atomicity and archimedianity;

P7 is used in the sequel in a slightly different version than Savage's, but the difference stems mainly from terminological reasons.

In order to facilitate comparison, we will name the axioms after those of Savage. An asterisk will indicate that the axiom differs from its Savageian counterpart. When more than one axiom is used to replace a single axiom in the original model, the number of asterisks will increase monotonically.

The axioms we will need are the following:

(1) P1. – as Savage's.

(2) P2.* – For all $f_1, f_2, g_1, g_2 \in F$, all $A, B \subset S$, and all $x_1, x_2, y_1, y_2 \in X$ such that $y_1 > x_1$ and $y_2 > x_2$, *if*

(i) $f_1/^{x_1}_A$, $f_1/^{y_1}_A$, $g_1/^{x_2}_A$, $g_1/^{y_2}_A$ are pairwise comonotonic (p.c.), and so are $f_2/^{x_1}_B$, $f_2/^{y_1}_B$, $g_2/^{x_2}_B$, $g_2/^{y_2}_B$, and

(ii) $f_1/^{x_1}_A \sim f_2/^{x_1}_B$, $g_1/^{x_2}_A \sim g_2/^{x_2}_B$ and $f_1/^{y_1}_A \geqq f_2/^{y_1}_B$

*then* $g_1/^{y_2}_A \geqq g_2/^{y_2}_B$.

*2.4.1. Observation.* P2* implies Savage's P4.

*Proof.* Take $f_1 = f_2 = x_1$ and $g_1 = g_2 = x_2$. □

Since P4 justifies the definition of $\geqq\cdot$ (over $2^S$), we may use Savage's definition.

(3) P3*. For all $A \subset S$, $x, y \in X$, $f \in F$, if $x < y$ then $f/^y_A \geqq f/^x_A$.

*2.4.2. Observation.*[1] If, furthermore, $f(s) \leqq x < y$ for all $s \in S$, and $A >\cdot \phi$, then $f/^y_A > f/^x_A$.

*Proof.* In P2*, take $f_1 = f_2 = x$, $g_1 = g_2 = f/^x_A$, $B = \phi$, and $x_1 = x_2 = x$, $y_1 = y_2 = y$. □

(4) P5*. There are at least three consequences $x^*$, $x$, $x_*$ such that $x^* > x > x_*$.

(5) P6*. (Non-atomicity). Let $x, y \in X$, $f, g \in F$ and $A \subset S$ satisfy $f/^x_A > g > f/^y_A$, where $f/^x_A$ and $f/^y_A$ are comonotonic. Then there exists an event $B \subset A$ such that

$$(f/^{x}_{A-B})/^y_B \sim g.$$

(6) P6**. (Archimedianity). Let there be a sequence $\{f_n\}_{n \geqq 1} \subset F$, which for some $x, y \in X$, $x > y$, and $A \subset S$ satisfies the following two conditions:

(i) $\forall s \in S$, $\forall n \geqq 1$, $f_n(s) \leqq y$,

(ii) $f_n/^x_A \sim f_{n+1}$,

then $A \sim\cdot \phi$.

[1]This observation follows from P2*, but it is closely connected to P3*, since both mean monotonicity. In the sequel we shall refer to P3* and 2.4.2. together as 'P3*'.

(7) P7*. Let $A$ be an $f$-convex event for $f \in F$, and suppose that for some $g \in F$,

$$f/_A^{f(s)} \geqq (\leqq) g \quad \forall s \in A.$$

Then $f \geqq (\leqq) g$.

*The main theorem is:* P1–P7* hold iff there are a measure $v$ with a convex range and a bounded utility $u$, such that $(u, v)$ are an IR of $\geqq$ over $F$. Furthermore, $v$ is unique and so is $u$, up to a p.l.t. (The proof is given in sections 2 and 3.) The difference between this theorem and that of Savage is that $v$ is not necessarily additive, and, consequently, the integration operation refers to the Choquet integral.

*A discussion of the axioms.* Considering the axioms, one should distinguish between conceptually-essential axioms, such as P1, and technical ones, such as P6*. The 'essential' axioms are those that are easily defendable on philosophical grounds, and it usually turns out to be the case that they are also easily defended on mathematical grounds, since one can construct simple examples of preference orders, satisfying all the axioms but the one under discussion, but having no IR.

The axioms we will consider to be 'essential' are P1, P2* and P3*, and they will be discussed first.

P1 is identical to that of Savage, and we will not expatiate on it.

P3* is a weaker version of P3, and is easily justifiable since it means monotonicity. A technical point should, however, be clarified: Under Savage's P3, if $A > \cdot \phi$ and $A \cap B = \phi$, then $A \cup B > \cdot B$. (This fact is also implied by P2.) This is not necessarily true in the non-additive case, so that P3 must be modified in order to include probability measures not satisfying the above condition.

P2* is a new axiom, and deserves some deliberation. First, suppose that none of the eight acts involved in it are required to be comonotonic. The axiom simply states that there is a preference order between events: suppose $f_1/_A^{x_1} \sim f_2/_B^{x_1}$ and $f_1/_A^{y_1} \geqq f_2/_B^{y_1}$ where $y_1 > x_1$. This means that the improvement on $A$ is more weighty than the same improvement on $B$, so that in some sense $A$ is preferred to (or considered more likely than) $B$. This statement would be reversed if there were $g_1/_A^{x_2} \sim g_2/_B^{x_2}$ such that $g_1/_A^{y_2} < g_2/_B^{y_2}$ with $y_2 > x_2$. The axiom basically state that this reversal is impossible. (For simple acts it is equivalent to Savage's P2 and P4.) However, this condition is restricted to the case where $f_1/_A^{x_1}$, $f_1/_A^{y_1}$, $g_1/_A^{x_2}$, $g_1/_A^{y_2}$ are p.c., and so are

$$f_2/_B^{x_1}, \quad f_2/_B^{y_1}, \quad g_2/_B^{x_2}, \quad g_2/_B^{y_2}.$$

The meaning of comonotonicity is that each event ($A$ or $B$ above) is indeed conceived in the same way in each of the above acts in which it appears. What the Ellsberg paradox shows, in these terms, is that event-assessment is not context-free, i.e., the same event may have different weights when the 'better' or 'worse' events are different.

In Schmeidler (1984) comonotonicity plays a similar role: restricting one of the essential axioms (Independence) to comonotonic acts allows the probability measure to be non-additive.

Next we turn to the technical axioms, namely, P5*, P6*, P6**, P7*.

P5*, to start with, is the most innocuous of all. It merely states that $|X/\sim|\geqq 3$, and should it not hold, one cannot expect to have a unique measure.

P6* is a non-atomicity axiom. It is supposed to sound reasonable. The same can be said about P6**, which is an archimedian axiom: it asserts that an act cannot be indefinitely improved if all 'improvements' are equally weighty.

Both P6* and P6** have very similar counterparts in Luce and Krantz (1971), which is one of the few existing models in the Savageian spirit. However, the justification of these axioms is mainly pragmatic: without each of them, IR of $\geqq$ is not guaranteed. This is proved by counterexamples in Gilboa (1985, sect. 4).

Finally consider P7*. Basically it is similar to P7, only that the latter is phrased in terms of '$\geqq$ given an event', which, in the absence of P2, is not well-defined.

P7*, as phrased, seems to be the natural way of stating the axiom in our model. However, Savage's example of a preference order, which satisfied P1–P6 but is not integral-representable, may also serve as a justification of P7*, since that preference relation also satisfies P1–P6**.

## 3. Defining something like a measure

We begin with a preliminary lemma which will be used extensively hereafter.

*3.1. Lemma.* *Let a and b be two simple acts such that*

$$a=(z_1,C_1;z_2,C_2;\ldots;z_n,C_n), \qquad b=(z_1,D_1;\ldots;z_n,D_n),$$

*with* $C_i\sim\cdot D_i\,\forall\, i\leqq n$. *Then* $a\sim b$.

*Proof.* Use P2* inductively. □

Throughout the rest of this section and subsection 4.1, we shall assume $X$ to be the triple $T=\{x_*,x,x^*\}$ satisfying $x^*>x>x_*$. Since the least-preferred consequence is always (=until subsection 4.2) $x_*$, we can write any act $f$ as

$(x^*, A; x, B)$, meaning

$$A = \{s \mid f(s) = x^*\} \quad \text{and} \quad B = \{s \mid f(s) = x\} \cup A.$$

All definitions made in this context should be understood as dependent upon the triple $T$. However, for convenience of notation, the subscript $T$ will be omitted.

We shall need some lemmas:

*3.2. Lemma. Let $E \subset E'$, $F \subset F'$ be events, and $a$, $a'$, $b$, $b'$, $c$, $c'$, $d$, $d'$ be acts satisfying one of the following three sets of conditions:*

(*i*) $a = (x^*, E; x, A) \sim b = (x^*, F; x, B)$
$a' = (x^*, E'; x, A); b' = (x^*, F'; x, B)$
$c' = (x^*, C; x, E) \sim d = (x^*, D; x, F)$
$c' = (x^*, C; x, E'); d' = (x^*, D; x, F')$

(*ii*) $a = (x^*, E; x, A) \sim b = (x^*, F; x, B)$
$a' = (x^*, E'; x, A); b' = (x^*, F'; x, B)$
$c = (x^*, E; x, C) \sim d = (x^*, F; x, D)$
$c' = (x^*, E'; x, C);\ d' = (x^*, F'; x, D)$

*or*

(*iii*) $a = (x^*, A; x, E) \sim b = (x^*, B; x, F)$
$a' = (x^*, A; x, E');\ b' = (x^*, B; x, F')$
$c = (x^*, C; x, E) \sim d = (x^*, D; x, F)$
$c' = (x^*, C; x, E');\ d' = (x^*, D; x, F')$

*then $a' \geqq b$ iff $c' \geqq d'$.*

*Proof.* Follows from P2*, Lemma 3.1, P3* and P6*. □

We shall now define a partial binary operation on $2^S/\sim\cdot$, which is to be thought of as an addition. It will, eventually, be equivalent to summation of the measure. In order to simplify notations and facilitate the discussion, we will not define the operation on equivalence classes of events formally, but rather use the following:

*Notation.* If there are events $H_0, H_1, H'_0, H'_1$ such that $H_0 \sim\cdot H'_0, H_1 \sim\cdot H'_1$ and

$$(x^*, B_1; x, H_0) \sim (x^*, \phi; x, H_1),$$

$$(x^*, B; x, H'_0) \sim (x^*, B_2; x, H'_1)$$

then we shall write $B \sim\cdot B_1 \oplus B_2$ (henceforth read '$B$ is the circle-sum of $B_1$ and $B_2$').

Note that, as defined, circle-addition need not be commutative, nor should it be defined for all pairs of events $(B_1, B_2)$.

Next, let us observe the following facts:

*3.3. Lemma*

(*i*) *If* $B \sim \cdot B_1 \oplus B_2$, *then* $B \geqq \cdot B_1$ *and* $B \geqq \cdot B_2$;

(*ii*) *If* $B' \sim \cdot B$, $B'_1 \sim \cdot B_1$, $B'_2 \sim \cdot B_2$ *and* $B \sim \cdot B_1 \oplus B_2$, *then* $B' \sim \cdot B'_1 \oplus B'_2$;

(*iii*) *Let* $B \sim \cdot B_1 \oplus B_2$, *and suppose that* $F_0 \sim \cdot F'_0$, $F_1 \sim \cdot F'_1$, *where* $B_1 \subset F_0$, $B \subset F'_0$ *and* $B_2 \subset F'_1$. *If* $(x^*, B_1;\ x, F_0) \sim (x^*, \phi;\ x, F_1)$ *then* $(x^*, B;\ x, F'_0) \sim (x^*, B_2; x, F'_1)$.

*Proof.* Using 2.1, P3* and P6* shows (i) and (ii) to be trivial, whereas (iii) becomes a direct application of Lemma 3.2. □

Lemma 3.3.(iii) means, in fact, that the circle-sum of two events $B_1$, $B_2$ does not depend upon other events.

Having the circle-addition operation, we wish to construct a measure which is additive w.r.t. (with respect to) it. Constructing the measure is based on the familiar principle of measuring each event with an ever-increasing precision, for which one should have an ever-decreasing measurement unit. We are now about to construct these units.

Since $(x^*, S) > (x, S) > (x_*, S)$, there is an event $A_1$ such that $(x^*, A_1) \sim (x, S)$, whence $\phi \cdot < A_1 \cdot < S$. Similarly, there is an event $A_2 \subset A_1$ such that $(x^*, A_2) \sim (x, A_1)$, and, arguing inductively, we have a sequence $\{A_k\}_{k \geqq 1}$ for which the following conditions hold:

(i) $A_k \supset A_{k+1}$,

(ii) $A_k > \cdot A_{k+1}$ (which also implies $A_k > \cdot \phi$).

(The notation $A_k$ will be reserved for members of this sequence even beyond sub-section 4.1, only that there the subscript $T$ will be added to it.) We would like to know that this sequence is indeed fine enough to construct a measure. This is guaranteed by

*3.4. Lemma. Suppose* $H > \cdot \phi$. *Then there exists an integer* $k$ *such that* $A_k \cdot < H$.

*Proof.* Use the archimedian axiom (P6**). □

Another notation will be proved useful: for $B, C \subset S$ and $n \in N$, we will say that $B \sim \cdot nC$ if there are $C = C_1, \ldots, C_n = B$ such that $C_i \sim \cdot C_{i-1} \oplus C$ for $2 \leqq i \leqq n$. We shall refer to the symbol $nC$ as an event, meaning 'any $B$ such that $B \sim \cdot nC$'. If there is no $B$ such that $B \sim \cdot nC$, we will write '$nC > \cdot B$' for all $B$. Now we can formulate:

*3.5. Lemma. If* $C > \cdot \phi$ *and* $B \subset S$, *there is an integer* $n$ *such that* $nC \geqq \cdot B$.

*Proof.* This is a straightforward application of P6**. □

The preceding lemma allows us the following:

*Definition.* For $B \subset S$ such that $\phi \cdot < B \cdot \leqq A_1$ and $k \geqq 1$, $n_k^B$ is the unique integer satisfying

$$n_k^B A_k \cdot \leqq B \cdot < (n_k^B + 1) A_k.$$

(The existence is implied by the lemma, whereas the uniqueness follows from the fact that $A_k > \cdot \phi$.)

We shall also need

*3.6. Lemma. Suppose $A_1 \geqq \cdot B \geqq \cdot C$. Then there exists an event H such that $B \sim \cdot C \oplus H$. If, furthermore $B > \cdot C$, then $H > \cdot \phi$.*

*Proof.* The first part is proved by P6*, whereas the 'furthermore' clause is a consequence of P3*. □

Now we can prove

*3.7. Lemma. If $B \subset S$ is such that $\phi \cdot < B \cdot < A_1$, then $n_k^B \to_{k \to \infty} \infty$.*

*Proof.* The sequence is obviously non-decreasing. Lemmas 3.4 and 3.6 imply that it cannot be bounded. □

Now we are in a position to define a set-function for all events $B$ such that $B \cdot \leqq A_1$, which will be the measure of these events, up to a scaling factor: for each $k \geqq 1$ define $\varepsilon_k$ to be $(n_k^{A_1})^{-1}$. Note that by 3.7, $\varepsilon_k \to_{k \to \infty} 0$. Now let there be given an event $B \cdot \leqq A_1$. Define $\tilde{v}(B) = \limsup_{k \to \infty} \varepsilon_k n_k^B$. To see that this set-function is indeed 'almost' a measure, which is monotonic w.r.t. $\geqq \cdot$, we have

*3.8. Lemma*

*(i) If $C \cdot \leqq B \cdot \leqq A_1$, then $\tilde{v}(C) \leqq \tilde{v}(B)$,*
*(ii) $\tilde{v}(A_1) = 1$,*
*(iii) $\tilde{v}(\phi) = 0$.*

*Proof.* Trivial. □

The main property of the function $\tilde{v}$ is its circle-additivity:

*3.9. Theorem. Let $B_1$, $B_2$, $B \cdot \leqq A_1$ satisfy $B \sim \cdot B_1 \oplus B_2$. Then $\tilde{v}(B) = \tilde{v}(B_1) + \tilde{v}(B_2)$.*

*Proof.* First we note that for any $k \geqq 1$

$$n_k^{B_1} A_k \cdot \leqq B_1 \cdot \leqq (n_k^{B_1}+1) A_k,$$

$$n_k^{B_2} A_k \cdot \leqq B_2 \cdot \leqq (n_k^{B_2}+1) A_k,$$

whence

$$(n_k^{B_1}+n_k^{B_2}) A_k \cdot \leqq B \cdot \leqq (n_k^{B_1}+n_k^{B_2}+2) A_k.$$

The left-hand side inequivalence implies

$$n_k^{B_1}+n_k^{B_2} \leqq n_k^B,$$

while the right-hand-side one implies

$$(n_k^{B_1}+n_k^{B_2}+2) \geqq n_k^B+1,$$

so that we may write

$$n_k^{B_1}+n_k^{B_2} \leqq n_k^B \leqq n_k^{B_1}+n_k^{B_2}+1.$$

Now suppose that $\{k_i\}_{i \geqq 1}$ is a sub-sequence of $N$ such that $\exists \lim_{i \to \infty} \varepsilon_{k_i} n_{k_i}^B = \tilde{v}(B)$. (Such a sub-sequence exists because of the definition of $\tilde{v}$ as limsup.) Obviously, $\lim_{i \to \infty}(\varepsilon_{k_i} n_{k_i}^{B_1} + \varepsilon_{k_i} n_{k_i}^{B_2}) = \tilde{v}(B)$. But, considering the definition of $\varepsilon_k$, one may easily see that $\varepsilon_{k_i} n_{k_i}^B \in [0,1]$ for all $k_i$, i.e., the sequence is bounded. Hence $\{k_i\}_i$ has a sub-sequence $\{k_{i_j}\}_j$ for which $\{\varepsilon_{k_{i_j}} n_{k_{i_j}}^{B_1}\}_j$ converges. Since $\{\varepsilon_{k_{i_j}} n_{k_{i_j}}^B\}_j$ also converges (to $\tilde{v}(B)$), we deduce that

$$\exists \lim_{j \to \infty} \varepsilon_{k_{i_j}} n_{k_{i_j}}^{B_2} = \tilde{v}(B) - \lim_{j \to \infty} \varepsilon_{k_{i_j}} n_{k_{i_j}}^{B_1}.$$

Since $\{k_{i_j}\}_j$ is a converging sub-sequence for both events $B_1$ and $B_2$, we may write

$$\tilde{v}(B_1) \geqq \lim_{j \to \infty} \varepsilon_{k_{i_j}} n_{k_{i_j}}^{B_1}, \qquad \tilde{v}(B_2) \geqq \lim_{j \to \infty} \varepsilon_{k_{i_j}} n_{k_{i_j}}^{B_2},$$

and, as a conclusion, $\tilde{v}(B_1)+\tilde{v}(B_2) \geqq \tilde{v}(B)$.

Now we wish to prove that the converse inequality holds as well. For this we shall need the following:

*3.9.1. Lemma. Let there be $I=\{k_i\}_{i \geqq 1}$ and $J=\{k_j\}_{j \geqq 1}$, two indices sequences, such that*

$$\exists \lim_{i \to \infty} \varepsilon_{k_i} n_{k_i}^{B_1} \equiv v_1^I \quad \textit{and} \quad \exists \lim_{j \to \infty} \varepsilon_{k_j} n_{k_j}^{B_2} \equiv v_2^J.$$

*Furthermore, assume that*

$$\exists \lim_{i\to\infty} \varepsilon_{k_i} n_{k_i}^{B_2} \equiv v_2^I \quad \textit{and} \quad \exists \lim_{j\to\infty} \varepsilon_{k_j} n_{k_j}^{B_1} \equiv v_1^J.$$

*Then it is impossible that* $v_1^I > v_1^J$ *and* $v_2^I < v_2^J$.

*Proof.* Assume the contrary, i.e., that indeed $v_1^I > v_1^J$ and $v_2^I < v_2^J$. W.l.o.g. assume $B_1 \geqq \cdot B_2$, whence, by Lemma 3.6, there is an event $B_3$ such that $B_1 \sim \cdot B_2 \oplus B_3$. We already know that

$$n_k^{B_2} + n_k^{B_3} \leqq n_k^{B_1} \leqq n_k^{B_2} + n_k^{B_3} + 1,$$

and therefore

$$\exists \lim_{i\to\infty} \varepsilon_{k_i} n_{k_i}^{B_3} \equiv v_3^I, \qquad \exists \lim_{j\to\infty} \varepsilon_{k_j} n_{k_j}^{B_3} \equiv v_3^J,$$

which satisfy

$$v_1^I = v_2^I + v_3^I, \qquad v_1^J = v_3^J + v_3^J.$$

Subtraction will yield

$$v_1^I - v_1^J = (v_2^I - v_2^J) + (v_3^I - v_3^J),$$

or

$$v_3^I - v_3^J = (v_1^I - v_1^J) + (v_2^J - v_2^I) > v_1^I - v_1^J,$$

and, in particular,

$$v_3^I > v_3^J.$$

Now we have $B_2$ and $B_3$, and we may proceed in this way to construct a sequence $\{B_n\}_{n\geqq 1}$ such that $(v_{n-1}^I - v_{n-1}^J)(v_n^I - v_n^J) < 0$. It is important to note that $v_n^I, v_n^J > 0$ for all $n$. [To see this, note that if $B_n \sim \cdot B_{n+1}$, $v_n^I = v_{n+1}^I$ and $v_n^J = v_{n+1}^J$, contrary to the induction assumption. If, for instance, $B_n > \cdot B_{n+1}$ (the case $B_n \cdot < B_{n+1}$ is identical), both $v_n^I > v_{n+1}^I$ and $v_n^J > v_{n+1}^J$, and consequently $v_{n+2}^I, v_{n+2}^J > 0$.] This sequence satisfies

$$\max(v_n^{I(J)}, v_{n+1}^{I(J)}) - \min(v_n^{I(J)}, v_{n+1}^{I(J)}) = v_{n+2}^{I(J)},$$

that is, any number in $\{v_n^I\}_{n\geqq 3}$ (or in $\{v_n^J\}_{n\geqq 3}$) is equal to the absolute difference between its two consecutive predecessors. This implies

$v_n^I, v_n^J \to_{n\to\infty} 0$. (For instance: for any $n \geqq 1$ there is a finite $M$ such that $v_{n+M}^{I(J)} \leqq 1/2 v_n^{I(J)}$.) But this means that $|v_n^I - v_n^J| \to_{n\to\infty} 0$, while we have shown that

$$|v_n^I - v_n^J| > \min\{(v_1^I - v_1^J), (v_2^J - v_2^I)\}.$$

This contradicts our assumption and thereby proves the Lemma. □

We return now to the proof of the theorem: Let $I = \{k_i\}_{i \geqq 1}$, and $J = \{k_j\}_{j \geqq 1}$ be subsequences such that

$$\exists \lim_{i\to\infty} \varepsilon_{k_i} n_{k_i}^{B_1} = \tilde{v}(B_1) \quad \text{and} \quad \exists \lim_{j\to\infty} \varepsilon_{k_j} n_{k_j}^{B_2} = \tilde{v}(B_2).$$

Since $\varepsilon_k n_k^B \in [0,1]$ (for all $k \geqq 1$, $B \cdot\leqq A_1$), one can choose subsequences of $I$ and $J$, to be denoted by $\bar{I} = \{k_{i_s}\}_{s \geqq 1}$ and $\bar{J} = \{k_{j_r}\}_{r \geqq 1}$ respectively, such that

$$\exists \lim_{s\to\infty} \varepsilon_{k_{i_s}} n_{k_{i_s}}^{B_2} \equiv v_2^{\bar{I}} \quad \text{and} \quad \exists \lim_{r\to\infty} \varepsilon_{k_{j_r}} n_{k_{j_r}}^{B_1} \equiv v_1^{\bar{J}}.$$

If $v_2^{\bar{I}} = v_2^J$, then $\bar{I}$ is a subsequence attaining $\tilde{v}(B_1)$ and $\tilde{v}(B_2)$ simultaneously, and this implies

$$\exists \lim_{s\to\infty} \varepsilon_{k_{i_s}} n_{k_{i_s}}^{B} = \tilde{v}(B_2) + \tilde{v}(B_1),$$

whence $\tilde{v}(B) \geqq \tilde{v}(B_2) + \tilde{v}(B_1)$. Therefore we may assume $v_2^{\bar{I}} < v_2^{\bar{J}} = v_2^J$. But according to the lemma, this is possible only if $v_1^{\bar{I}} \leqq v_1^{\bar{J}}$ Since $v_1^{\bar{I}} = v_1^I = \limsup_{k\to\infty} \varepsilon_k n_k^{B_1}$, we have $v_1^{\bar{J}} = \tilde{v}(B_1)$, while we already know that $v_2^{\bar{J}} = \tilde{v}(B_2)$. In that case again $\tilde{v}(B) \geqq \tilde{v}(B_1) + \tilde{v}(B_2)$. Combining the two inequalities we have

$$\tilde{v}(B) = \tilde{v}(B_1) + \tilde{v}(B_2),$$

which completes the proof. □

Another important property of the function $\tilde{v}$ is that it agrees with $\geqq\cdot$:

*3.10. Theorem.* *Let $B, C \cdot\leqq A_1$. Then $B \cdot\leqq C$ iff $\tilde{v}(B) \leqq \tilde{v}(C)$.*

*Proof.* By 3.6 and 3.9, it suffices to show that $\tilde{v}(H) > 0$ for $H > \cdot\phi$. This is proved by P6** and 3.9. □

We will also be interested in the range of $\tilde{v}$. First we prove

*3.11. Lemma. Suppose that H satisfies* $A_1 \geqq \cdot H > \cdot \phi$; *then there is an* $H^1 > \cdot \phi$ *such that*

$$1/2\tilde{v}(H) \geqq \tilde{v}(H^1) > 0.$$

*Proof.* Use 3.4 and 3.6. □

This last lemma proves useful in

*3.12. Theorem.* $\tilde{v}$ *has a convex range. (This property was originally defined for a measure, and* $\tilde{v}$ *fails to be one, but the definition is extended in an obvious manner.)*

*Proof.* In view of 3.9 and 3.11, the proof is straightforward. □

*3.13. Conclusion.* $\{\tilde{v}(B)\}_{B \cdot < A_1} = [0, 1]$.

So far we have defined $\tilde{v}(B)$ for $B \cdot \leqq A_1$. Defining a measure for all $2^S$ should be postponed until after we have said something about integral representation of $\geqq$, which will be done in the next section.

## 4. Integral representation of the preference order

This section is divided into three subsections: Subsection 4.1 constructs an IR of $\geqq$, retaining section 3's assumption of $X = \{x^*, x, x_*\}$. This requires, of course, a definition of a measure for all $2^S$.

Subsection 4.2 removes the restriction on $X$, but constructs an IR of $\geqq$ only for step functions. This step includes, however, the comparison of the measures and utilities constructed in 4.1 for any triple of consequences.

Subsection 4.3 proves that the utility and the measure that were constructed in 4.2 constitute and IR of $\geqq$ over all acts, and not only over simple ones (='step functions').

### *4.1. IR for a three-consequence world*

The steps in constructing the IR of $\geqq$ for a specific triple of consequences are:

(a) IR for $T^S \cap \underline{F}_{(x, A_1)}$, where $T$ is the triple of consequences, and $\underline{F}_f = \{g \in F \mid g \leqq f\}$ (for $f \in F$).
(b) Extending $\tilde{v}$ and normalizing it to construct a measure for $2^S$ [this is done in view of ($a$)].
(c) IR for all $T^S$.

It should be noted that we do not have a measure until step (*b*), so that the term 'IR' is not well-defined. However, the way we will define it will not be surprising:

Since only three consequences are involved, one may safely assume that any utility $u: T \to R$ satisfies $u(x^*) = 1$ and $u(x_*) = 0$. Hence for $f = (x^*, B; x, C)$ with $B, C \cdot\leqq A_1$, we may define

$$\int u(f)\,\mathrm{d}\tilde{v} = [1 - u(x)]\tilde{v}(B) + u(x)\tilde{v}(C).$$

Bearing this definition in mind until we have a 'real' measure, step (*a*) is no more than

*4.1.1. Theorem.* *For $T = \{x^*, x, x_*\}$ with $x^* > x > x_*$ and the function $\tilde{v}$ attached to it, there exists a $u: T \to R$ such that $(u, v)$ is an IR of $\geqq$ (in the sense of the above definition) over $T^S \cap \underline{F}_{(x, A_1)}$.*

*Proof.* For any $f = (x^*, B; x, C) \in \underline{F}_{(x, A_1)}$ there is (by P6*) an event $D \cdot\leqq A_1$ such that $f \sim (x, D)$. Therefore it suffices to show that there is an $\alpha \in (0, 1)$, $(\alpha = u(x))$ such that

$$(1 - \alpha)\tilde{v}(B) + \alpha\tilde{v}(C) = \alpha\tilde{v}(D)$$

for all $(x^*, B; x, C) \sim (x, D)$ with $D \cdot\leqq A_1$. First we observe that, since $\tilde{v}$ agrees with $\geqq\cdot$, $\tilde{v}(D)$ depends on $B$ and $C$ only through $\tilde{v}(B)$ and $\tilde{v}(C)$, respectively. That is, if $B'$ and $C'$ are such that $\tilde{v}(B) = \tilde{v}(B')$ and $\tilde{v}(C) = \tilde{v}(C')$, and $B' \subset C'$, then, by Theorem 3.10 and 3.1, $(x^*, B'; x, C') \sim (x, D)$. Denoting by $\tilde{V}_1$ the set $\{\tilde{v}(B), \tilde{v}(C))\}_{(x^*, B; x, C) \in \underline{F}_{(x, A_1)}}$ we have proved the existence of a function $\psi_1: \tilde{V}_1 \to [0, 1]$ such that

$$(x^*, B; x, C) \sim (x, D) \Leftrightarrow \tilde{v}(D) = \psi_1(\tilde{v}(B), \tilde{v}(C))$$

$$\text{for all } (x^*, B; x, C) \in \underline{F}(x, A_1) \quad \text{and} \quad D \cdot\leqq A_1.$$

Since $D \geqq\cdot C$, $\tilde{v}(D) \geqq \tilde{v}(C)$ and we may write

$$\psi_1(\tilde{v}(B), \tilde{v}(C)) = \tilde{v}(C) + \psi_2(\tilde{v}(B), \tilde{v}(C))$$

with $\psi_2: \tilde{V}_1 \to [0, 1]$. We now note that

*4.1.1.1. Lemma.* *$\psi_2$ is independent of its second argument.*

*Proof.* Implied by 3.9. □

Consequently there is a $\psi_3 : \tilde{V}_3 \to [0,1]$, with $\tilde{V}_3$ being the projection of $\tilde{V}_1$ onto its first coordinate, such that

$$f = (x^*, B; x, C) \sim (x, D) \Leftrightarrow \tilde{v}(D) = \tilde{v}(C) + \psi_3(\tilde{v}(B))$$

$$\text{for} \quad f, (x, D) \in \underline{F}_{(x, A_1)}.$$

It is obvious that $\psi_3$ is a non-negative, monotonically increasing function, with $\psi_3(0) = 0$. Another important fact about $\psi_3$ is

*4.1.1.2. Lemma.* *$\psi_3$ is additive.*

*Proof.* Follows from the definition of $\oplus$ and from 3.9. □

In the light of Conclusion 3.13, $\tilde{V}_3$ is no more than an interval (either closed or half-closed), so that monotonicity and additivity imply the linearity of $\psi_3$: There exists a $\lambda > 0$ for which

$$\psi_3(v) = \lambda v \quad \forall v \in \tilde{V}_3.$$

Taking $\alpha = (\lambda + 1)^{-1}$, one completes the proof of Theorem 4.1.1. □

Now we may turn to step (*b*), i.e., finally define the measure $v$ for a given triple $T$, using the set function $\tilde{v}$ and the number $\alpha$ defined above: For $B \cdot\leqq A_1$, let $v(B) = \alpha\tilde{v}(B)$. For $B > \cdot A_1$, let $C \cdot\leqq A_1$ be such that $(x^*, C; x, A_1) \sim (x, B)$, and define $v(B) = v(A_1) + ((1-\alpha)/\alpha)v(C)$. [$v(B)$ is well defined in this case, since it does not depend upon the choice of $C$.] Note that $v(A_1) = \alpha$ and therefore $v(S) = 1$.

Having $v$ defined, we may proceed to the third step, namely, to construct an IR of $\geqq$ over all $T^S$. First we extend Theorem 4.1.1 in the following way:

*4.1.2. Lemma. If $(x^*, B; x, C) \sim (x, D)$, then $(1-\alpha)v(B) + \alpha v(C) = \alpha v(D)$. [Note that this means integral representation for all $f, g \leqq (x, S)$.]*

*Proof.* Use the definition of $v$ and 4.1.1. □

Now we wish to extend the circle-additivity of $v$ over $[0, \alpha]$ to $[0, 1]$:

*4.1.3. Lemma.* $v$ is circle-additive.

*Proof.* The lemma is a conclusion of the definition of $v$ and the previous two results. □

The time is ripe to prove

*4.1.4. Theorem.* *For $T=\{x^*, x, x_*)$ with $x^* > x > x_*$, the measure $v$ and the utility $u$ defined above constitute an IR $\geqq$ over $T^S$.*

*Proof.* Trivial in view of the preceding lemmas. □

*4.1.5. Corollary.* *$v$ has a convex range. (Use Theorem 3.12.)*

### 4.2. IR for step functions

It is the time to remind ourselves that the utility and the measure we have proved to constitute an IR of $\geqq$ over $T^S$ for a given triple $T$, are dependent upon this triple, and should be denoted by $u_T$ and $v_T$ respectively. We now come to the comparison among different triples:

*4.2.1. Lemma.* *Let $T_1$ and $T_2$ be non-trivial triples of consequences. (i.e., $|T_i/\sim|=3$, $i=1,2$). Then $v_{T_1}=v_{T_2}$.*

*Proof.* By P2*, $\oplus_T$ does not depend upon the triple $T$. 4.1.3–4.1.5 complete the proof. □

Henceforth we shall refer to the measure $v$ (without a subscript), since it does not depend on the defining triple.

We now turn to the comparison among $\{u_T\}_T$. We shall need some new definitions:

$$X_{x_*}^{x^*}=\{x\in X \mid x_* \leqq x \leqq x^*\} \quad \text{for } x^* \geqq x_*,$$

$$F^n=\{f\in F \mid \#\{x\in X \mid \exists s\in S, f(s)=x\} \leqq n\} \quad \text{for } n \geqq 1,$$

$$F^*=\bigcup_{n\geqq 1} F^n,$$

$$F_{x_*}^{x^*}=\{f\in F \mid f(s)\in X_{x_*}^{x^*}\ \forall s\in S\} \quad \text{for } x^*\geqq x_*, \quad \text{and}$$

$I(f)=\int u(f)\,\mathrm{d}v$ (any subscripts, superscripts, apostrophies, and other symbols attached to $u$ will be understood to define their corresponding $I$'s).

by which it is easier to formulate:

*4.2.2. Lemma.* *For any $x^*>x_*$ there exists a $u_{x^*,x_*}: X_{x_*}^{x^*}\to R$, such that for any $T=\{x_1^*, x_1, x_*\}$ with $x^*\geqq x_1^* > x_1 > x_*$, $u_{x^*,x_*}$ and $v$ are an IR of $\geqq$ over $T^S$.*

*Proof.* Let there be given an $x \in X_{x_*}^{x^*}$. There is an event $B_x$ satisfying $(x^*, B_x; x_*, S) \sim (x, S)$. Define $u_{x^*, x}(x) = v(B_x)$. □

Next we shall need

*4.2.3. Lemma. Let $a, b, c, d \in F^*$ be p.c., and for some $H \subset S$,*

$$a(s) = b(s) \quad \text{and} \quad c(s) = d(s) \quad \text{for} \quad s \in H,$$
$$a(s) = c(s) \quad \text{and} \quad b(s) = d(s) \quad \text{for} \quad s \in H^c.$$

*Suppose, furthermore that $a \sim c$. Then $b \sim d$.*

*Proof.* Let $k$ be the minimal number of disjoint events $\{H_i\}_{i=1}^k$ such that $\bigcup_i H_i = S$ and on each $H_i$ $a$, $b$, $c$ and $d$ are all constant. Now use induction on $k$, each step using P2*. □

Now we have

*4.2.4. Theorem. For any $x^* > x_*$ the utility $u_{x^*, x_*}$ and the measure $v$ form an IR of $\geqq$ over $F_{x_*}^{x^*} \cap F^*$. (i.e., over all step functions which are bounded by $x^*$ and $x_*$ from above and below, respectively.)*

*Proof.* Prove that $u_{x^*, x_*}$ and $v$ are an IR of $\geqq$ over $F_{x_*}^{x^*} \cap F^n$, for all $n \geqq 1$, by an inductive use of:

*4.2.4.1. Lemma. For any $f \in F^n \cap F_{x_*}^{x^*}$ $(n \geqq 3)$ there is an $f' \in F^{n-1} \cap F_{x_*}^{x^*}$ satisfying:*

*(i)* $f' \sim f$
*(ii)* $I_{x^*, x_*}(f') = I_{x^*, x_*}(f)$.

*The proof of which is trivial.* □

We are now approaching the conclusion of this subsection. At long last we turn to define the utility $u$:

Choose any $x^* > x_*$, and for $x \in X_{x_*}^{x^*}$, let $u(x) = u_{x^*, x_*}(x)$. [So that $u(x^*) = 1$, $u(x_*) = 0$.] Now let $x \in X$ satisfy $x > x^*$. Consider the triple $T = \{x, x^*, x_*\}$, for which there exists a utility $u_T$. Define $u(x) = u_T(x^*)^{-1}$, so that $(u(x), u(x^*), u(x_*))$ is a scalar multiplication (and hence a p.l.t.) of $(u_T(x), u_T(x^*), u_T(x_*))$. Similarly, for $x$ satisfying $x < x_*$, take the triple $T = \{x^*, x_*, x\}$ and the utility $u_T$ attached to it, and define $u(x) = -u_T(x_*)/(1 - u_T(x_*))$, again preserving the equality

$$\frac{u_T(x_*) - u_T(x)}{u_T(x^*) - u_T(x)} = \frac{u(x_*) - u(x)}{u(x^*) - u(x)}.$$

*4.2.5. Theorem.* *The utility u defined above satisfies*

$$f \geqq g \Leftrightarrow I(f) \geqq I(g) \quad \forall f, g \in F^*.$$

*Proof.* Trivial. □

The results obtained so far may be summarized in

*4.2.6. Theorem.* *The following two statements are equivalent:*

*(i)* $\geqq$ *satisfies axioms P1, P2*, P3*, P5*, P6* and P6** for all step functions.*
*(ii) There is a utility u, which is unique up to p.l.t., and a unique measure v with a convex range, which constitute an IR of* $\geqq$ *over F*.*

*Proof.* (i)⇒(ii) is the conclusion of sections 3, 4.1, 4.2. (ii)⇒(i) is easy to check. □

### *4.3. IR for all functions*

We now turn to the general case, in which the acts under comparison need not be simple acts. In this section, P7* is assumed to hold, unless otherwise stated.

To begin with, we need

*4.3.1. Theorem.* *u is bounded.*

*Proof.* The proof is very similar to that of theorem 14.5 in Fishburn[2] (1970, pp. 206–207) and we shall not repeat it here. □

This theorem allows us to assume henceforth, w.l.o.g., that $\inf_{x \in X} u(x) = 0$ and $\sup_{x \in X} u(x) = 1$. A crucial property of a preference relation satisfying P7* is

*4.3.2. Lemma.* *Let* $\phi = B_0 \subset B_1 \subset \cdots \subset B_n = S$ *be events such that* $B_i - B_{i-1}$ *is f-convex for* $i \leqq n$. *Suppose*

$$\underline{u}_i = \inf_{s \in B_i - B_{i-1}} \{u(f(s))\}, \qquad \bar{u}_i = \sup_{s \in B_i - B_{i-1}} \{u(f(s))\}.$$

[2]This proof is not to be found in Savage (1954). Fishburn notes, that although the theorem is mainly due to Savage, it was not known to him until they discovered it together several years after the publication of 'The Foundations of Statistics'.

*Let $\bar{f} \in F^*$ satisfy $\bar{f} \sim f$. Then*

$$\sum_{i=1}^{n} (\underline{u}_i - \underline{u}_{i+1}) v(B_i) \leqq I(\bar{f}) \leqq \sum_{i=1}^{n} (\bar{u}_i - \bar{u}_{i+1}) v(B_i),$$

*where $\bar{u}_{n+1} \equiv \underline{u}_{n+1} \equiv 0$.*

*Proof.* We shall prove only one of the two inequalities, say the left-hand side one, for the other one is proved symmetrically. Assume the contrary, i.e.:

$$\underline{u} \equiv \sum_{i} (\underline{u}_i - \underline{u}_{i+1}) v(B_i) > I(\bar{f}).$$

Take $\bar{g} \in F^*$ to be such that $\underline{u} \geqq I(\bar{g}) > I(\bar{f})$, whence $\bar{g} > \bar{f} \sim f$, (such a $\bar{g}$ exists because $v$ has a convex range). Now for any $(s_1, s_2, \ldots, s_n)$ such that $s_i \in B_i - B_{i-1}$ and any $k \leqq n$, define

$$f^{(s_1,\ldots,s_k)}(s) = f(s_i), \qquad s \in B_i - B_{i-1}, \qquad i \leqq k$$

$$= f(s) \quad \text{otherwise.}$$

Note that $f^{(s_1,\ldots,s_n)} \in F^*$ for any sequence $(s_1, \ldots, s_n)$, and $I(f^{(s_1,\ldots,s_n)}) \geqq \underline{u} \geqq I(\bar{g})$ *so that* $f^{(s_1,\ldots,s_n)} \geqq \bar{g}$. This can be written as

$$f^{(s_1,\ldots,s_{n-1})} \Big/ \frac{f(s_n)}{B_n - B_{n-1}} \geqq \bar{g} \quad \forall s_n \in B_n - B_{n-1},$$

whence, by P7*, $f^{(s_1,\ldots,s_{n-1})} \geqq \bar{g}$ for all $(s_1, \ldots, s_{n-1})$. Arguing inductively, $f^{(s_1,\ldots,s_k)} \geqq \bar{g}$ for all $k \leqq n$ and all $(s_1, \ldots, s_k)$, and, in particular, $f \geqq \bar{g}$, which is known to be impossible. □

A straightforward consequence is

*4.3.3. Lemma. Let $f \in F$, $\bar{f} \in F^*$ satisfy $f \sim \bar{f}$.*
*Then $I(f) = I(\bar{f})$.*

*Proof.* Trivial. □

Now the time has come to phrase:

*4.3.4. Theorem. Let P1, P2*, P3*, P5*, P6* and P6** hold. Then P7* holds iff*

$$f \geqq g \Leftrightarrow I(f) \geqq I(g) \quad \textit{for all } f, g \in F.$$

*Proof.* First assume that P7* holds. Denote

$$\bar{F} = \{f \in F \mid \exists\, \bar{x}, \underline{x} \in X, \bar{x} \gtrsim f \gtrsim \underline{x}\}$$

Note that

(i) $f \in \bar{F}$ iff there is an act $\bar{f} \in F^*$ such that $f \sim \bar{f}$.

(ii) If $f \in \bar{F}$, then either $f > x$ for all $x \in X$, in which case $f \gtrsim g$ for all $g \in F$ and $I(f) = \sup\{u(x) \mid x \in X\}$, or $f < x$ for all $x \in X$, in which case $f \lesssim g$ for all $g \in F$ and $I(f) = \inf\{u(x) \mid x \in X\}$.

These observations, together with the previous results, complete the first half of the proof. However, the second half is trivial. □

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