FINITE MATRIX MODELS WITH CONTINUOUS LOCAL GAUGE INVARIANCE ☆

D. HORN

Department of Physics and Astronomy, Tel Aviv University, Ramat Aviv, Israel

We construct a hamiltonian lattice gauge theory which possesses local SU(2) gauge invariance and yet is defined on a Hilbert space of 5-dimensional real vectors for every link. This construction does not allow for generalization to arbitrary SU(N), but a small variation of it can be generalized to an SU(N) × U(1) local gauge invariant model. The latter is solvable in simple gauge sectors leading to trivial spectra. We display these by studying a U(1) local gauge invariant model with similar characteristics.

Conventional SU(N) gauge theories are expressed in terms of variables which are continuous SU(N) group elements. In this letter we present an SU(9) gauge-invariant lattice model which uses finite numerical matrices as the basic link variables. Previous comparisons between theories of continuous and discrete variables were carried out for globally symmetric theories: the XY model which has a global U(1) symmetry [1] and all O(N) models in 1 + 1 dimensions [2]. The discrete theory can be regarded as a truncated version of the continuous model and both exhibit the same phase structure [2]. We follow the spirit of the construction in ref. [2] in building the SU(2) gauge invariant model. This model cannot be generalized to arbitrary SU(N). A slight variation of its structure leads to an SU(2) × U(1) local gauge invariant model, which can be generalized to SU(N) × U(1). However, the additional U(1) symmetry leads to a trivial physical structure as can be shown by the simple solution in the various gauge sectors.

Let us start by displaying the SU(2) gauge-invariant model. We construct a hamiltonian lattice gauge theory whose elements are 5 × 5 matrices associated with the links of a hypercubic lattice. These matrices operate on real five-dimensional link state vectors. The components 0 and 1 to 4 correspond to bases of (1, 1) and (2, 2) representations of SU(2) × SU(2). To introduce

the generators of these SU(2) let us define the following Hermitian matrices:

$$(L_m)_{\alpha\beta} = -\epsilon_{m\alpha\beta} \, ,$$

$$(N_m)_{\alpha\beta} = i(\delta_{\alpha,4}\delta_{\beta,m} - \delta_{\alpha,m}\delta_{\beta,4}) \, . \tag{1}$$

We choose Latin indices to take the values 1, 2, 3, whereas Greek indices run over 0 to 4. $\epsilon_{m\alpha\beta}$ is the normal 3-dimensional cyclic tensor and vanishes if any index takes the value 0 or 4. The vector operators L_i and N_i obey the algebra

$$[L_i, L_j] = i\epsilon_{ijk}L_k \, , \quad [L_i, N_j] = i\epsilon_{ijk}N_k \, ,$$

$$[N_i, N_j] = i\epsilon_{ijk}L_k \, , \tag{2}$$

from which it follows that

$$J_i = \tfrac{1}{2}(L_i + N_i) \, , \quad K_i = \tfrac{1}{2}(L_i - N_i) \, , \tag{3}$$

are the generators of the two independent SU(2) groups. It should be noted that by construction these generators are hermitian and antisymmetric and as such are also generators of the orthogonal group SO(4).

Let us introduce four step operators which connect the representation (1, 1) to all elements $\alpha = 1$ to 4 of (2, 2):

$$(M_k)_{\alpha\beta} = \delta_{\alpha,0}\delta_{\beta,k} + \delta_{\alpha,k}\delta_{\beta,0} \, ,$$

$$(M_4)_{\alpha\beta} = \delta_{\alpha,0}\delta_{\beta,4} + \delta_{\alpha,4}\delta_{\beta,0} \tag{4}$$

Under L_i and N_i they transform according to

$$[L_i, M_j] = i\epsilon_{ijk}M_k, \quad [L_i, M_4] = 0,$$

$$[N_i, M_j] = i\delta_{i,j}M_4, \quad [N_i, M_4] = -iM_i. \tag{5}$$

In particular note that M_k is a vector and M_4 a scalar under the generators L_i. Let us now construct a new kind of object:

$$V = M_4 + i\sigma_k M_k, \tag{6}$$

in terms of Pauli matrices σ_k which are outside the 5-dimensional space on which we defined all the operators up to now. It follows from eqs. (5) and (6) that

$$[J_i, V] = -\tfrac{1}{2}\sigma_i V, \quad [K_i, V] = \tfrac{1}{2}V\sigma_i. \tag{7}$$

The various matrix operators L, N and M are defined separately for every link and commute if they belong to different links. The σ-matrices are common to all links and are introduced in order to enable us to construct in a simple fashion a gauge-invariant interaction term. The latter is a Wilson loop of four link operators associated with one plaquette p in which we perform a trace over the σ-matrices only:

$$W(p) = \text{tr}_\sigma [V(1)V(2)V^\dagger(3)V^\dagger(4)]. \tag{8}$$

This remains a finite matrix operating on the 5^4 real vector space associated with the four links in question. Eq. (7) guarantees its gauge invariance.

On our hypercubic lattice we define a set of axes. Links no. 1 and 4 in eq. (8) are chosen to lie along positive directions of the basis vectors at the vertex adjoining these two links. $W(p)$ is then invariant under local SU(2) transformations associated with the vertex v and generated by

$$G_i(v) = \sum_{l_+} J_i(l) + \sum_{l_-} K_i(l), \tag{9}$$

where l_+ (l_-) are the positive (negative) links which meet at vertex v. Using eq. (7) it is easy to verify that

$$[G_i(v), W(p)] = 0, \tag{10}$$

for any plaquette p and vertex v, thus guaranteeing gauge invariance.

Finally we construct the hamiltonian of our model in the following way [+1]

[+1] W is real in the SU(2) case. The combination $W + W^\dagger$ in eq. (11) will be important in the other cases to which we apply the same general form of H.

$$H = \sum_l J^2(l) - x \sum_p \{W(p) + W^\dagger(p)\}, \tag{11}$$

which is quite similar to the Kogut–Susskind hamiltonian [3]. The similarity goes beyond the statement of local gauge invariance: also here $J^2 = K^2$ and the SU(2) rotations of J and those of K cover the same manifold of real state vectors. This property is very important: since the SU(2) gauge group of one end of a link is presented by J while that of the other end acts via K this means that both cover the same vector space on the link. The main difference between this and the conventional model is that V is not unitary. Nonetheless it does have the same gauge group, it exhibits confinement in the strong coupling limit ($x = 0$), and looks like a truncated version of the usual theory limited just to two representations of SU(2) $\times$ SU(2) on every link. Following the experience of ref. [2] one may expect this model to have the same physical features as the one based on continuous SU(2) group elements.

Let us consider now a small variation of the above construction in which we replace M of eq. (4) by the asymmetric matrix

$$(M_k)_{\alpha\beta} = \delta_{\alpha,0}\delta_{\beta,k}, \quad (M_4)_{\alpha\beta} = \delta_{\alpha,0}\delta_{\beta,4}. \tag{12}$$

Eq. (5) will still hold and, therefore, the whole construction of H still goes through. It is this variation of the model which allows the generalization to SU(N): classify link vectors as bases of the $(1, 1)$ and $(\bar{N}, N)$ representation of SU(N) $\times$ SU(N) and define

$$(F_k)_{\alpha\beta} = -if_{k\alpha\beta}, \tag{13}$$

$$(D_k)_{\alpha\beta} = d_{k\alpha\beta} + (2/N)^{1/2}(\delta_{\alpha,k}\delta_{\beta,N^2} + \delta_{\alpha,N^2}\delta_{\beta,k}).$$

Latin indices run now over 1 to $N^2 - 1$ while Greek indices take all values from 0 to N^2. f_{ijk} and d_{ijk} are the conventional SU(N) symbols and vanish if any index takes on the value 0 or N^2. The sum and difference

$$J_k = \tfrac{1}{2}(F_k + D_k), \quad K_k = \tfrac{1}{2}(F_k - D_k), \tag{14}$$

define two independent SU(N) algebras.

Defining the step operators

$$(M_k)_{\alpha\beta} = \delta_{\alpha,0}\delta_{\beta,k}, \quad (M_{N^2})_{\alpha\beta} = \delta_{\alpha,0}\delta_{\beta,N^2}, \tag{15}$$

we can construct

$$V = (2/N)^{1/2}M_{N^2} + \lambda_i M_i \tag{16}$$

in terms of external λ-matrices which are common to all links and obey

$$\lambda_i \lambda_j = (2/N)\delta_{ij} + (d_{ijk} + i f_{ijk})\lambda_k . \tag{17}$$

It is then straightforward to check that

$$[J_k, V] = -\tfrac{1}{2}\lambda_k V , \qquad [K_k, V] = \tfrac{1}{2} V\lambda_k . \tag{18}$$

This is the property which guarantees that a hamiltonian of the kind of eq. (11) will be locally gauge invariant under the groups generated by eq. (9).

In the absence of d_{ijk}, i.e. for SU(2), it was possible to define the phases somewhat differently so that the whole algebra was realized by SO(4) on a purely real basis. This is no longer true for SU(N) which needs a complex basis. This is also why we needed the non-hermitian form for the M-matrices in eq. (15) — a hermitian form would have led to a purely real interaction matrix which is allowed only for SU(2). But this form of the interaction hides a higher symmetry: an additional local U(1) gauge symmetry. If we define diagonal matrices τ which take on the values $+1$ for the $(1, 1)$ and -1 for the $(\bar{N},N)$ bases then the non-hermitian M-matrices obey

$$[\tau, M] = 2M , \qquad e^{i\alpha\tau} V e^{-i\alpha\tau} = e^{2i\alpha} V. \tag{19}$$

We can now define the local gauge generator

$$g(v) = \sum_{l_+} \tau(l) - \sum_{l_-} \tau(l) , \tag{20}$$

under which $W(p)$ and H are invariant. Hence all the SU(N) models constructed with the M-matrices of eq. (15) are invariant under SU(N) $\times$ U(1) local gauge transformations.

This additional U(1) gauge symmetry has far reaching consequences. It turns the ground-state calculation of every sector into a finite problem. To analyze it let us discuss the simpler case of a local U(1) gauge-invariant theory defined by Pauli matrices as follows

$$H = \sum_l \{1 - \tau_3(l)\}$$
$$- x \sum_p (\tau_+(1)\tau_+(2)\tau_-(3)\tau_-(4) + \text{h.c.}) , \tag{21}$$

where the gauge generator is given by

$$g(v) = \sum_{l_+} \tau_3(l) - \sum_{l_-} \tau_3(l) . \tag{22}$$

The gauge invariant sector is specified by $\tau_3(l) = 1$ for every link, it has energy $E = 0$ and no finite excitation. Other sectors have charges (i.e. g-values) which are integer multiples of ± 2. The wave function for a charge -2 at point A and $+2$ at point B is given by a string of $\tau_3 = -1$ links, on otherwise $\tau_3 = +1$ background, which runs from A to B along the positive directions only. For any given distribution of a finite number of charges there is always only a finite number of such strings on which H can be represented as a finite matrix. This problem was recently solved for two charges on the corners of an arbitrary rectangle [4]. If A and B lie along a straight line on the lattice, separated by L links, then there is only one such wave function with energy $2L$ on which the interaction term of eq. (21) is inoperative as it was in the gauge-invariant sector. The tension is therefore 2 independently of x.

The generalization to the SU(N) $\times$ U(1) models is straightforward. The interaction term cannot act on either the $x = 0$ vacuum, which is given by the $(1, 1)$ state on all links, or the straight string. Therefore the tension is constant. The trivial structure of the spectrum is caused by the large symmetry of the model: each vertex is associated with a conservation law whose implication leads to strong restrictions on the wave functions in our finite Hilbert space. One needs symmetric step matrices in order to obtain non-trivial dynamics. In the set of models which we considered this was possible for SU(2) only. It is gratifying to realize that in this case one obtains a truncated version of the conventional model which possesses the continuous local gauge symmetry.

References

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