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BOSONIZATION OF THE SU(N) THIRRING MODELS*

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Bosonization is applied to the SU(N) Thirring models, and interesting relations between various two-dimensional field theories arise. In particular, we show that the SU(2) model is equivalent to a version of the Sine-Gordon equation plus a free massless field.

1. Introduction

The kinematical constraints on field theories in two space-time dimensions lead to many unusual effects which have no analogue in higher dimensions. Perhaps the strangest of these is the equivalence between large classes of Bose and Fermi field theories. This correspondence, which has been dubbed bosonization, provides a transparent unified method for solving two dimensional Fermi theories [1**, 2–4].

In this paper we will apply bosonization to the non-abelian SU(N) Thirring models [5,6,7+]. Although we will not solve these models, we will find several amusing relations between them and other two-dimensional theories. Our most striking result is for the case $N = 2$: we show that the SU(2) Thirring model is equivalent to the theory of a free massless scalar field and a Sine-Gordon field. The bare coupling constant of the Sine-Gordon theory is fixed at $\beta^2 = 8\pi$, the value which makes the model exactly renormalizable. According to the results of Coleman [2], this implies a correspondence between the SU(2) model and another fermion model: a free massless Fermi field and a massive abelian Thirring model.

This work is divided into three parts. In sect. 2 we set up a precise operator scheme which implements the bosonization of free massless Fermi fields. Our formulas are very similar to those proposed by Dell'Antonio et al. [1] and by Mandelstam [3]. In sect. 3 we apply our formalism to the SU(N) models in the interaction picture and give a heuristic derivation of the equivalent boson Lagrangians for these models. Sect. 4 is devoted to a proof of the equivalence for the SU(2) case

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** We use the γ-matrix conventions of Klaiber which imply $\psi_{L,R} = \psi_{1,2}$.
+ This author has arrived at some of the results of the present paper, in particular the Lagrangian (26).

based on the method and results of Coleman [2]. Finally we discuss the rather unusual picture of renormalization that emerges from our results and point out areas in which further study is necessary.

2. Bosonization of massless free fermion fields

We are going to write an explicit expression for a massless free fermion field as a non-local function of a free massless pseudo-scalar field. As is well-known, the latter needs infrared regularization in two dimensions; we will supply this regularization by working in a spatial box. We have found that we cannot write the fermion in terms of the boson field alone. We will have to introduce two discrete Fermi degrees of freedom. This is in accord with the results of Schroer [1].

The left- and right-handed components of the massless pseudo-scalar field are defined in a box of length L with periodic boundary conditions in the following explicit fashion:

$$\varphi_{\mathrm{L,R}}(x \pm t) = \frac{1}{\sqrt{L}} \sum_{n \neq 0} [a_n e^{ik_n(x \pm t)} + \text{h.c.}] \frac{\theta(\mp n)}{\sqrt{2|k_n|}} \; ; \; k_n = 2\pi n/L \,, \quad n = \pm \text{ integer} . \tag{1}$$

This field φ does not have the zero momentum mode. The latter is treated separately in terms of the charge (q) and axial charge ($\tilde{q}$) operators and their conjugate momenta (p and $\tilde{p}$):

$$\hat{\varphi}_{\mathrm{L,R}}(x \pm t) = \frac{1}{2} \left\{ \frac{1}{\sqrt{\pi}} (\tilde{p} \pm p) + \sqrt{\pi} (q \mp \tilde{q}) \frac{x \pm t}{L} \right\},$$

$$[q,p] = [\tilde{q}, \tilde{p}] = i \,, \qquad [q,\tilde{q}] = [q,\tilde{p}] = [\tilde{q},p] = [\tilde{p},p] = 0 \,. \tag{2}$$

The operators p and $\tilde{p}$ are assumed to be angle variables so that q and $\tilde{q}$ have integer eigenvalues. We will designate henceforth the Hilbert space on which q and $\tilde{q}$ operate by $H_{q\tilde{q}}$. This is distinct from the Fock space H_{B} on which the field φ operates. We will also use the pseudo-scalar field

$$\Phi = \varphi_{\mathrm{L}} + \varphi_{\mathrm{R}} + \hat{\varphi}_{\mathrm{L}} + \hat{\varphi}_{\mathrm{R}} \,. \tag{3}$$

It obeys the massless Klein-Gordon equation and operates on the Hilbert space $H_{\mathrm{B}} \otimes H_{q\tilde{q}}$.

We now construct the massless spinor field

$$\psi_{\mathrm{L,R}} = \frac{1}{\sqrt{L}} : e^{\mp 2i\sqrt{\pi}\varphi_{\mathrm{L,R}}(x \pm t)} : e^{\mp 2i\sqrt{\pi}\hat{\varphi}_{\mathrm{L,R}}(x \pm t)} \chi_{\mathrm{L,R}} \,. \tag{4}$$

The normal ordering is used on the fields $\varphi_{\mathrm{L,R}}$. Here we introduced two additional fermionic degrees of freedom

$$\chi_\alpha = A_\alpha + A_\alpha^+ , \qquad \{A_\alpha, A_\beta^+\} = \delta_{\alpha,\beta}, \qquad \{A_\alpha, A_\beta\} = 0, \qquad \alpha, \beta = \mathrm{L,R} , \tag{5}$$

which operate on the Hilbert space H_χ spanned by the operation of A_α^+ and $A_\mathrm{L}^+ A_\mathrm{R}^+$ on the vacuum. Altogether ψ is defined on $H_\mathrm{B} \otimes H_{q\tilde{q}} \otimes H_\chi$. It is a straightforward exercise to show that ψ satisfies the Dirac equation and the correct Fermi anticommutation relations. We construct explicitly the Wightman functions of these ψ fields:

$$\langle 0|\psi_1(x_1) \ldots \psi_1(x_n)\psi_2(x_{n+1}) \ldots \psi_2(x_{n+m})\psi_1^+(y_1) \ldots \psi_1^+(y_n)\psi_2^+(y_{n+1}) \ldots \psi_2^+(y_{n+m})|0\rangle$$

$$= \frac{(-1)^{m(n+1)}}{(2\pi i)^{n+m}} \frac{\prod_{1 \leqslant j<k \leqslant n} (L/\pi) \sin [(\pi/L)(u_j - u_k)] \, (L/\pi) \sin [(\pi/L)(U_j - U_k)]}{\prod_{j,k=1}^{n} \{(L/\pi) \sin [(\pi/L)(u_j - U_k)] - i\epsilon\}}$$

$$\times \frac{\prod_{1 \leqslant i \leqslant l \leqslant m} (L/\pi) \sin[(\pi/L)(v_i - v_l)] \, (L/\pi) \sin[(\pi/L)(V_i - V_l)]}{\prod_{i,l=1}^{m} \{(L/\pi) \sin[(\pi/L)(v_i - V_l)] + i\epsilon\}} ,$$

$$u_k = x_k^0 + x_k^1 , \qquad U_k = y_k^0 + y_k^1 , \qquad k = 1, \ldots n ,$$

$$v_{k-n} = x_k^1 - x_k^0, \quad V_{k-n} = y_k^1 - y_k^0, \qquad k = n+1, \ldots n+m . \tag{6}$$

In the limit $L \to \infty$ these expressions converge to the correct forms for a free massless spinor field in two dimensions (see, e.g. Klaiber, ref. [1]).

By introducing fermion normal ordering *via* point splitting, one can prove that the fermion number current is given by

$$j^\mu = N_\mathrm{F}(\bar{\psi}\gamma^\mu \psi) = -\frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \Phi . \tag{7}$$

The appropriate current field algebra is now easily obtained. With some appropriate modifications, one can view eq. (4) as a construction of the fermion field in terms of the currents in this theory, as advocated by Dell-Antonio et al. [1].

Using the bosonization formulas one is able to easily solve a large class of fermion models by writing everything in the interaction picture. We have checked the validity of this procedure in the following soluble cases:

(a) The Schwinger model in the Coulomb gauge [4];

(b) The derivative coupling model [8];

(c) The abelian SU(N) Thirring models [1,7].

The solutions of these models obtained by bosonization are completely consistent with known solutions.

In the following we will apply this method to field theoretical models that have not been solved yet. Throughout this paper we will use an interaction picture approach which is based on the free field construction described in this section. Our approach is therefore different from that of Mandelstam [3], although the formulas may look similar.

Halpern [7] has also used an interaction picture formalism to discuss bosonization of the SU(N) models. He works with T products and enforces covariance via the Dirac-Schwinger commutation relations for the energy density. Instead, we will use T* products and Mathews' theorem, a procedure which simplifies the derivations.

3. Bosonization of the SU(N) Thirring model in the interaction picture

We begin the bosonization of the SU(N) Thirring model by defining the N two-component spinors:

$$\psi^a_{\mathrm{L,R}}(x,t) = \frac{1}{\sqrt{L}} : \mathrm{e}^{\mp 2i\sqrt{\pi}\Phi^a_{\mathrm{L,R}}(x,t)} : \chi^a_{\mathrm{L,R}} , \quad a = 1, ..., N . \tag{8}$$

Here Bose normal ordering should be understood to act only on the H_{B} part of the field Φ. The U(N) currents are defined by

$$J^\mu = \sum_a \bar{\psi}^a \gamma^\mu \psi^a , \qquad J^{(i)\mu} = \sum_{a,b} \bar{\psi}^a \gamma^\mu \, \tfrac{1}{2} \lambda^{(i)}_{ab} \, \psi^b , \tag{9}$$

where the $N \times N$ matrices $\lambda^{(i)}$, $i = 1, ..., N$ form the regular representation of SU(N). Eq. (9) has to be understood as a point-split definition so that the currents are regular. In terms of the boson field they turn out to be

$$J^0 = \frac{1}{\sqrt{\pi}} \sum_{a=1}^{N} \partial_x \Phi^a , \quad J^1 = -\frac{1}{\sqrt{\pi}} \sum_{a=1}^{N} \partial_t \Phi^a ,$$

$$J^{(i)0} = \sum_{a \neq b}^{N} \frac{\lambda^{(i)}_{ab}}{2L} \{\chi^a_1 \chi^b_1 : \mathrm{e}^{2i\sqrt{\pi}(\Phi^a_{\mathrm{L}} - \Phi^b_{\mathrm{L}})} : + \chi^a_2 \chi^b_2 : \mathrm{e}^{-2i\sqrt{\pi}(\Phi^a_{\mathrm{R}} - \Phi^b_{\mathrm{R}})} :\}$$
$$+ \frac{1}{\sqrt{\pi}} \sum_{a=1}^{N} \tfrac{1}{2} \lambda^{(i)}_{aa} \, \partial_x \Phi^a , \qquad i = 1, ..., N^2 - 1 ;$$

$$J^{(i)1} = \sum_{a \neq b}^{N} \frac{\lambda^{(i)}_{ab}}{2L} \{-\chi^a_1 \chi^b_1 : \mathrm{e}^{2i\sqrt{\pi}(\Phi^a_{\mathrm{L}} - \Phi^b_{L})} : + \chi^a_2 \chi^b_2 : \mathrm{e}^{-2i\sqrt{\pi}(\Phi^a_{\mathrm{R}} - \Phi^b_{\mathrm{R}})} :\}$$
$$- \frac{1}{\sqrt{\pi}} \sum_{a=1}^{N} \tfrac{1}{2} \lambda^{(i)}_{aa} \, \partial_t \Phi^a , \qquad i = 1, ..., N^2 - 1 . \tag{10}$$

Notice that the diagonal currents are simple functions of the boson fields. It is a straightforward exercise to check that these expressions satisfy the non-abelian current algebra as given by Dashen and Frishman [5].

In order to perform the bosonization of the model it is necessary to use a point split definition of Bose normal ordering for arbitrary solutions of the Klein-Gordon equation. In ref. [9] it was shown that this is the way to get the right quantum expression for the energy-momentum tensor in the Sugawara form. The prescription is:

$$J_1(x)\,J_2(y) \to \lim_{y\to x}\ \{\tfrac{1}{2}(J_1(x)\,J_2(y) + J_2(y)\,J_1(x)) - \text{V.E.V.}\}\,. \tag{11}$$

We now start from the Lagrangian

$$\mathcal{L} = i\bar\psi\partial\!\!\!/\psi - \tfrac{1}{2}g_\mathrm{B}\,J^\mu J_\mu - \sum_i \tfrac{1}{2}g_\mathrm{V}\,J^{\mu(i)}J^{(i)}_\mu\,, \tag{12}$$

and use the Gell-Mann-Low formula to express time ordered products of Heisenberg operators in terms of interaction-picture fields.

$$_\mathrm{H}\langle 0|\mathrm{T}\prod_{i=1}^{n} O_\mathrm{H}(x_i)\ |0\rangle_\mathrm{H} = \frac{_\mathrm{I}\langle 0|\mathrm{T}^*[\prod_{i=1}^{n} O_\mathrm{I}(x_i)\exp(i\int \mathrm{d}^2y\ \ \mathcal{L}_\mathrm{I}\,(\psi_\mathrm{I}(y)))]\,|0\rangle_\mathrm{I}}{_\mathrm{I}\langle 0|\mathrm{T}^*[\exp(i\int \mathrm{d}^2y\ \mathcal{L}_\mathrm{I}\,(\psi_\mathrm{I}(y)))]\,|0\rangle_\mathrm{I}}\,. \tag{13}$$

We must use a T* product in eq. (13) because of the Schwinger terms in the current algebra.

Using eqs. (10) and (11) we can rewrite (12) in boson language. The effective boson interaction Lagrangian is

$$\mathcal{L}_\mathrm{I} = \frac{g_\mathrm{B}N}{2\pi}\ :\partial_\mu\Phi'^1\partial^\mu\Phi'^1: + \frac{g_\mathrm{V}}{4\pi}\sum_{a\geqslant 2}\ :\partial_\mu\Phi'^a\partial^\mu\Phi'^a:$$

$$-\frac{g_\mathrm{V}}{L^2}\sum_{a\neq b}^{N}\chi_1^a\chi_2^a\,\chi_1^b\chi_2^b\ :\cos[2\sqrt{\pi}\sum_{c\geqslant 2}(C^{ca}-C^{cb})\,\Phi'^c]: \tag{14}$$

C is an orthogonal real $N\times N$ matrix that satisfies

$$C^{1a} = \frac{1}{\sqrt{N}}\qquad \text{for } a = 1,\dots,N;\quad \sum_{bd} C^{ab}C^{cd} = N\delta^{a1}\delta^{c1}\,;\ \sum_{b=1}^{N} C^{ab}C^{cb} = \delta^{ac}\,, \tag{15}$$

and the Φ' are related to the original fields Φ by

$$\Phi'^a = \sum_b C^{ab}\Phi^b\,. \tag{16}$$

We will now assume that eq. (13) can be implemented in the boson language by the use of Mathews' theorem [10], i.e. we use the standard recipe of regarding T* as a T product which commutes with time derivatives. The fermion Lagrangian, on the other hand, contains no derivative interactions. Formal evaluation of a T product of Fermi interaction Lagrangians *via* Wick's theorem leads to a covariant but divergent expression. When we define this expression by means of a covariant regularization and subtraction procedure we effectively convert it into a T* product. The crucial assumption of this section is that these two definitions coincide. This conjecture is actually justified in all of the soluble models mentioned above, and we will assume that it remains valid here. We are then led to a theory which can be represented by the Lagrangian

$$\mathcal{L} = \frac{\pi + g_B N}{2\pi} \partial_\mu \Phi'^1 \partial^\mu \Phi'^1 + \frac{\pi + \frac{1}{2} g_V}{2\pi} \sum_{a=2}^{N} \partial_\mu \Phi'^a \partial^\mu \Phi'^a$$
$$- \frac{g_V}{L^2} \sum_{a \neq b} \chi_1^a \chi_2^a \chi_1^b \chi_2^b : \cos[2\sqrt{\pi} \sum_{c=2}^{N} (C^{ca} - C^{cb}) \Phi'^c] : . \tag{17}$$

Let us perform a finite wave function renormalization on the fields Φ':

$$\theta^1 = \sqrt{1 + \frac{g_B N}{2\pi}} \Phi'^1 , \qquad \theta^a = \sqrt{1 + \frac{g_V}{2\pi}} \Phi'^a , \qquad a = 2,\ldots,N . \tag{18}$$

The Lagrangian (17) then becomes

$$\mathcal{L} = \tfrac{1}{2} \sum_{a=1}^{N} \partial_\mu \theta^a \partial^\mu \theta^a - \frac{g_V}{L^2} f(L^2\Lambda^2) \sum_{a \neq b} \chi_1^a \chi_2^a \chi_1^b \chi_2^b : \cos[2\sqrt{\pi} \left(1 + \frac{g_V}{2\pi}\right)^{-1/2}$$
$$\times \sum_{c=2}^{N} (C^{ca} - C^{cb}) \theta^c] : ; \qquad f(L^2\Lambda^2) = \left(\frac{L^2\Lambda^2}{4\pi^2}\right)^{g_V/(2\pi + g_V)} , \tag{19}$$

where Λ^2 is an ultra-violet cutoff. The cutoff-dependent factor in front of the cosine comes from re-normal ordering the cosine in terms of the fields θ.

One now sees immediately how to solve the problem for $g_V = 0$. The ground state of the Heisenberg Hamiltonian is the Fock vacuum of the fields θ. We can get a Fermi field which has finite Green functions in this ground state by re-normal ordering our field (8) with respect to the θ's. The finite wave function renormalization between Φ' and θ induces an infinite wave function renormalization of ψ and we pick up the correct anomalous dimension for the Fermi field.

$$\psi_r^{(a)} = \left(\frac{L\Lambda}{2\pi}\right)^{N g_B^2 / 4\pi(\pi + N g_B)} \psi^{(a)} . \tag{20}$$

The full $g_V \neq 0$ Lagrangian (19) becomes particularly simple for the case of SU(2).

$$\mathcal{L} = \tfrac{1}{2} \partial_\mu \theta^1 \partial^\mu \theta^1 + \tfrac{1}{2} \partial_\mu \theta^2 \partial^\mu \theta^2 - \frac{2 g_V}{L^2} f(L^2\Lambda^2) \chi_1^1 \chi_2^1 \chi_1^2 \chi_2^2 : \cos\left[\sqrt{8\pi} \left(1 + \frac{g_V}{2\pi}\right)^{-1/2} \theta^2\right] \tag{21}$$

Thus, the SU(2) Thirring model is equivalent to a massless free Bose field plus a Sine-Gordon field. The correspondence between couplings is (we use Coleman's notation) [2]

$$\frac{\alpha}{\beta^2} \leftrightarrow \frac{2g_V}{L^2} f(L^2\Lambda^2)\,, \qquad \beta \leftrightarrow \left(1+\frac{g_V}{2\pi}\right)^{-\frac{1}{2}} \sqrt{8\pi}\,. \tag{22}$$

Let us summarise this section. By using an interaction picture formalism based on eq. (8) we wrote the interaction Lagrangian in terms of the scalar fields. It separated automatically into two parts which can be simply expressed in terms of ϕ'^a which are linear combinations of ϕ^a. This field theory of bosons can be represented by the Lagrangian (17). Performing a finite wave function renormalization on ϕ', one is led to a new Lagrangian which is equivalent (in the SU(2) case) to one free Bose field plus one Sine-Gordon field with a fixed relation between the two Sine-Gordon couplings α and β. This is a rather surprising result and the skeptical reader may be suspicious of our cavalier treatment of T* products. We therefore check the result using a different method in sect. 4.

4. Proof of equivalence between the Bose and Fermi Lagrangians for the case of SU(2)

In this section we will use Coleman's results to prove the correspondences suggested in sect. 3. We will therefore give up our box normalization in favour of Coleman's infrared regularization. We will also set $g_B = \frac{1}{4}g_V$. This simplifies many formulas and implies no essential loss of generality because we already know that bosonization works for the Abelian part of the model.

After applying a Fierz transformation, we can write the fermion Lagrangian as

$$\begin{aligned}\mathcal{L} &= i\bar{\psi}\partial\!\!\!/\psi - \tfrac{1}{4}g_V\,(\bar{\psi}^1\gamma^\mu\psi^1)^2 - \tfrac{1}{4}g_V(\bar{\psi}^2\gamma^\mu\psi^2)^2 \\ &\quad + g_V\,[(\psi_1^{1+}\psi_2^1)(\psi_2^{2+}\psi_1^2) + (\psi_1^{2+}\psi_2^2)(\psi_2^{1+}\psi_1^1)]\,.\end{aligned} \tag{23}$$

Defining

$$\mathcal{L}^{\,0}(\psi) = \sum_{a=1}^{2} \,[i\bar{\psi}^a\partial\!\!\!/\psi^a - \tfrac{1}{4}g_V(\bar{\psi}^a\gamma^\mu\psi^a)^2]\,,$$

$$\sigma_\pm^{1,2}(x) \propto \lim_{(x-y)^2\to 0} (x-y)^{2\delta}\,\psi_{1,2}^{+1,2}(x)\,\psi_{2,1}^{1,2}(y)\,; \quad \delta = \frac{g_V}{8\pi}\,\frac{4\pi+g_V}{2\pi+g_V}\,, \tag{24}$$

we get (following Coleman [2])

$$\langle 0|\mathrm{T}\prod_{i=1}^{I} \sigma_+^1(\xi_i) \prod_{k=1}^{K} \sigma_-^1(\eta_k) \prod_{j=1}^{J} \sigma_+^2(\tau_j) \prod_{l=1}^{L} \sigma_-^2(\lambda_l) \prod_{\alpha=1}^{n} \sigma_+^1(x_\alpha)\,\sigma_-^2(x_\alpha)$$

$$\times \prod_{\beta=1}^{m} \sigma_+^2(y_\beta)\,\sigma_-^1(y_\beta)|0\rangle \propto (\tfrac{1}{2})^{4(I+J+n+m)} \left\{ \prod_{i>j}^{I} (\xi_i - \xi_j)^2 \right. \tag{25}$$

$$\times \prod_{\alpha>\beta}^{n} (x_\alpha - x_\beta)^2 \prod_{i=1}^{I} \prod_{\alpha=1}^{n} (x_\alpha - \xi_i)^2 \prod_{i>j}^{K} (\eta_i - \eta_j)^2 \prod_{\alpha>\beta}^{m} (y_\alpha - y_\beta)^2$$

$$\times \prod_{i=1}^{K} \prod_{\beta=1}^{m} (\eta_i - y_\beta)^2 \left[\prod_{i=1}^{I} \prod_{k=1}^{K} \prod_{\alpha=1}^{m} \prod_{\beta=1}^{n} (\xi_i - \eta_k)^2 (\xi_i - y_\alpha)^2 (x_\beta - \eta_k)^2 \right.$$

$$\left. \times (x_\beta - y_\alpha)^2 \right]^{-1} \times \text{[same expression with } I \to J; K \to L; \xi \to \tau; \eta \to \lambda;$$

$$\left. \times\, n \leftrightarrow m; x \leftrightarrow y] \right\}^{1/(1+g_V/2\pi)} .$$

if $I + n = K + m$ and $J + m = L + n$ and zero otherwise.

We write the boson Lagrangian as [7]

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\theta^1\partial^\mu\theta^1 + \tfrac{1}{2}\partial_\mu\theta^2\partial^\mu\theta^2 - \tfrac{1}{2}\mu^2[(\theta^1)^2 + (\theta^2)^2]$$

$$- g(x)\,Gm^2N_m \cos[\gamma\sqrt{8\pi}\,\theta^2] , \tag{26}$$

$$\gamma = \left(1 + \frac{g_V}{2\pi}\right)^{-\frac{1}{2}} .$$

The function g has compact support in space-time and is supposed to be taken to 1 everywhere after the perturbation series is summed. μ^2 is a regulator mass which allows us to make calculations with massless scalar fields in two dimensions. It should be taken to zero order by order in the perturbation theory. Note that the factors χ_i which appear in eq. (21) are absent from eq. (26). The reason is that they do not play any role in the matrix elements of eq. (25) since they will always appear squared and therefore lead to no observable factor in the boson sector of the theory. Therefore, although we will be able to show the equivalence using Coleman's method, we will not get the explicit representation of the spinor from which we started.

Now, consider the following expression:

$$\lim_{\mu^2\to 0} \langle 0,\mu| \mathrm{T} \prod_{i=1}^{I} N_m \mathrm{e}^{-i\gamma\sqrt{2\pi}(\theta^1+\theta^2)(\xi_i)} \prod_{k=1}^{K} \mathrm{e}^{i\gamma\sqrt{2\pi}(\theta^1+\theta^2)(\eta_k)}$$

$$\times \prod_{j=1}^{J} N_m \mathrm{e}^{-i\gamma\sqrt{2\pi}(\theta^1-\theta^2)(\tau_j)} \prod_{l=1}^{L} N_m\, \mathrm{e}^{i\gamma\sqrt{2\pi}(\theta^1-\theta^2)(\lambda_l)} \tag{27}$$

$$\times \prod_{\alpha=1}^{n} N_m \mathrm{e}^{-i\gamma\sqrt{2\pi}\theta^2(x_\alpha)} \prod_{\beta=1}^{m} N_m \mathrm{e}^{i\gamma\sqrt{8\pi}\theta^2(y_\beta)}|0,\mu\rangle .$$

The dependence on μ^2 is

$$(\mu^2)^{\frac{1}{4}[(-L+K+J-I+2m-2n)^2+(L+K-I-J)^2]} , \tag{28}$$

so a contribution is non-zero as $\mu^2 \to 0$ if $J + m = L + n$ and $K + m = I + n$. When these conditions are satisfied it is easy to see (using Coleman's formulae again) that the result is proportional to eq. (25).

So, we are led to deduce the correspondences

$$\sigma^1_\pm \propto mN_m \, e^{\mp i\gamma\sqrt{2\pi}(\theta^1+\theta^2)} , \tag{29}$$

$$\sigma^2_\pm \propto mN_m e^{\mp i\gamma\sqrt{2\pi}(\theta^1-\theta^2)} .$$

These are exactly the consequences that one would anticipate on the basis of the discussion in sect. 3. Thus we have proven the equivalence conjectured in the previous section, at least for a class of matrix elements. We have not checked the equivalence of other matrix elements of the two theories but we are confident that they are, in fact, equal.

5. Discussion

Let us discuss the renormalization procedure for the Lagrangian (21) of section 3 keeping the box length L fixed. We therefore ask what has to be done to the coupling constant g_V and the field normalization in order to make all Green functions Λ independent as $\Lambda \to \infty$. For $\Lambda < \infty$ the terms in the Lagrangian (and all of their powers) are finite operators on Fock space (we always work in a finite volume). The coefficient of the cosine term blows up as $\Lambda \to \infty$ so we must obviously let $g_V \to 0$ to obtain anything finite in this limit. The crucial point now is that as we let $g_V \to 0$ the coefficient inside the cosine goes to $\sqrt{8\pi}$. This is the very edge of the positivity domain which Coleman [2] has obtained for the Sine-Gordon equation. Moreover, for our purpose it is important to note that $\beta = \sqrt{8\pi}$ is exactly the value for which $\cos \beta\phi$ becomes an operator of scale dimension 2. Thus, the cosine interaction, which for $\beta^2 < 8\pi$ was superrenormalizable, becomes exactly renormalizable in this limit. It is easy to verify that new divergences appear in the Sine-Gordon perturbation series for this value of β^2.

The upshot of this is that if we take $g_V \to 0$ in such a way that

$$g_V(\Lambda) \left(\frac{L^2\Lambda^2}{4\pi^2}\right)^{g_V(\Lambda)/(2\pi+g_V(\Lambda))} \underset{\Lambda\to\infty}{\to} g_V^0 = \text{finite} , \tag{29}$$

then the higher powers of the resulting Hamiltonian will not be well defined. In more mathematical terms, the Hamiltonian will not converge to an essentially selfadjoint operator on Fock space. We know that if we take g_V to zero fast enough as $\Lambda \to \infty$, then we get a well-defined operator. For example, if we simply set $g_V = 0$ for some finite Λ, then we get back the abelian Thirring model. Since we have shown that the massive Thirring model is equivalent to the SU(2) model (which we know can be renormalized), it is now reasonable to conjecture that there is a way to choose the bare couplings

$$g_V \to 0\,, \qquad \beta = \sqrt{8\pi}\left(1 + \frac{g_V}{2\pi}\right)^{-1/2}, \tag{30}$$

in the Sine-Gordon equation and obtain finite Green functions for the theory which are essentially different from the $g_V(\Lambda) \equiv 0$ case.

In other words, the limit (27) leads to an equivalence between the SU(2) model and the Sine-Gordon model with parameters

$$\frac{\alpha}{\beta} = \frac{2g_V^0}{L^2}\,, \qquad \beta = \sqrt{8\pi}\,. \tag{31}$$

For finite g_V^0 this theory is not yet renormalized. However, our knowledge that the SU(2) model is renormalizable by coupling constant and Fermi wave function renormalization [6]* leads us to expect that this Sine-Gordon field theory is also renormalizable. We may expect that an additional finite renormalization of the boson field has to be carried out. This is because the anomalous dimension of the Fermi field in the non-abelian SU(2) model is not the same as that in the abelian model [5,11]. (The Bose renormalization must be finite because the curl of the Bose field is an SU(2) current which we know is not renormalized.) Thus, we expect that a finite version of the Sine-Gordon equation can be constructed even when $\beta^2 = 8\pi$. For larger values of β Coleman has shown [2] that the model does not exist.

The conjecture presented above is a bit disquieting when we think of it in terms of renormalization group trajectories. The standard way of computing renormalization constants by looking at the divergent parts of integrals leads us to believe that for $\beta^2 < 8\pi$ the trajectories for the Sine-Gordon equation are all β = constant, i.e. straight lines in the α, β plane. We seem to have found a new trajectory which moves in to the point ($\alpha = 0, \beta^2 = 8\pi$) like a square root. We believe that the apparent contradiction lies in the fact that the relation between the bare and renormalized β is a completely finite one and is missed by the usual method of only taking infinite renormalizations into account. This point deserves further study.

We would like to emphasize that our analysis does not reveal any hint of the second scale invariant solution of the SU(N) model found by Dashen and Frishman.

* These authors discuss a slightly more general SU(N) model than ours. One can use their results to show that the U(1) chiral symmetry may be consistently implemented. This implies that the only renormalizations needed in our model are the wave function and coupling constant renormalizations.

The existence of the $g_V = 4\pi/(N+1)$ Dashen Frishman solution is a consequence of anomalies which make the SU(N) axial currents conserved for this value of the coupling. In previous applications of bosonization, fermion anomalies always appeared as canonical results in the boson language, and one might naively expect the same to happen for the SU(N) models. The difference, of course, is that in all previous models the non-quadratic part of the equivalent boson Lagrangian was super-renormalizable, while for the SU(N) models it is renormalizable. Thus the boson interaction will have anomalies of its own and we will not see anomalies arise as canonical results.

Nonetheless, there is a way that we should see the Dashen-Frishman solution in our formalism. Since the diagonal SU(N) axial currents are conserved in the Dashen-Frishman model, our Bose fields (from which we get the axial currents by taking derivatives) should be free. The only way we can see to make this happen is to take the bare coupling constant g_V to zero so fast that the interaction vanishes even in the infinite cutoff limit. In this case the Fermi fields (as defined by us) would also be free. The Dashen-Frishman Fermi fields for $g_V = 4\pi/(N+1)$ can also be written in terms of free boson fields, but we see no limit in which our fields can be identified with theirs. It may be that the Dashen-Frishman model can be obtained by other limiting procedures and not by a quasi-perturbative scheme such as our own.

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References

[1] R. Klaiber, Lectures in theoretical physics ed. A. Barut and W. Britten (Gordon and Breach, 1968);
G.F. Dell'Antonio. Y. Frishman and D. Zwanziger, Phys. Rev. D6 (1972) 988;
B. Schroer, Freie Universität Berlin preprint (1975).
[2] S. Coleman, Phys. Rev. D11 (1975) 2088.
[3] S. Mandelstam, Phys. Rev. D11 (1975) 3026.
[4] J. Kogut and L. Susskind, Phys. Rev. D11 (1975) 3594.
[5] R. Dashen and Y. Frishman, Phys. Rev. D11 (1975) 2781.
[6] P.K. Mitter and P.H. Weisz, Phys. Rev. D8 (1973) 4410.
[7] M. Halpern, Berkeley preprint (1975).
[8] K.D. Rothe and I.O. Stamatescu, Heidelberg preprint (1975).
[9] S. Coleman, R.W. Jackiw and D.J. Gross, Phys. Rev. 180 (1969) 1359.
[10] P.T. Mathews, Phys. Rev. 76 (1949) 684L (erratum: 76 (1949) 1489 (L)).
[11] H. Romer, S. Yankielowicz and H.R. Rubinstein, Nucl. Phys. B79 (1974) 285.