Frustrated spin Hamiltonians with binary input vectors

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Structures of composite spin operators are analyzed which appear in models of neural networks of the type which Amit *et al.* have recently investigated. A binary basis of size $N=2^M$ is introduced to study a problem of N quantum-mechanical spin operators. The $[Z(2)]^M$ group structure of the binary basis allows for many decompositions of the $[SU(2)]^N$ spin algebra. These become useful in studying and solving generalized frustrated Heisenberg as well as Ising models. Using these techniques for quantum-mechanical generalized spin operators, we derive an explicit representation of the partition function of classical statistical-mechanics models, in terms of a series summation over components of collective spin variables.

I. INTRODUCTION

Amit *et al.*[1] have recently investigated the properties of frustrated spin-glass models defined by the Hamiltonian

$$H=-\sum_{i,j} J_{ij}S^iS^j , \tag{1.1}$$

where S^i are classical spin variables at N sites i, and J_{ij} are couplings between any two of them chosen according to

$$J_{ij}=\frac{1}{N}\sum_{\mu=1}^{p}\xi_i^{\mu}\xi_j^{\mu} . \tag{1.2}$$

The elements of the p vectors ξ are chosen randomly to be 1 or -1. In the thermodynamic limit the ground state of the free energy at low temperature has $2p$ minima with spin expectation values that are proportional to one of these p vectors. The binary components of the vectors ξ can be viewed as representing some information which can be retrieved from the structure of the corresponding state with minimal free energy. In this sense, these vectors form an input memory of the system. The system defined by (1.1) is a generalization of a dynamical model for neural networks that has been proposed by Hopfield.[2]

We are going to investigate structures in which the variables S^i are replaced by quantum-mechanical spin operators obeying the commutation relations

$$[S_{\alpha}{}^i,S_{\beta}{}^j]=i\epsilon_{\alpha\beta\gamma}\delta^{ij}S_{\gamma}{}^i , \tag{1.3}$$

where $\epsilon_{\alpha\beta\gamma}$ is the completely antisymmetric rank-3 Levi-Cività tensor and δ^{ij} is the Kronecker delta function. We will limit ourselves to binary ξ vectors which have an equal number of positive and negative entries. In Sec. II we build a basis of such vectors and investigate its structure. This binary basis is used in Sec. III to construct collective spin operators. Using the properties of the binary basis, we obtain a simple group-theoretical spin structure which, for certain Hamiltonian systems, allows us to characterize completely the spectrum. Such a system is the frustrated Heisenberg model discussed in Sec. III. We show in Sec. IV how to use the insight gained by this approach in analyzing classical statistical-mechanics models like (1.1). We express the partition function in a closed form in terms of series of a finite number of variables and display the results of numerical evaluations. Section V is devoted to a discussion of our results.

II. THE BINARY BASIS

We are interested in constructing a basis for an N-dimensional vector space out of a set of the binary vectors ξ. All these vectors have $N/2$ entries of $+1$ and $N/2$ entries of -1 and should be orthogonal to each other. This can be achieved if we limit ourselves to the particular case in which the dimension of the system is a power of two:

$$N=2^M . \tag{2.1}$$

In this case one can construct a complete basis by performing M outer products of the two two-dimensional vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$. This basis will contain one vector with entries of $+1$ only, which we call the unit vector although its norm is N, and $N-1$ vectors which have an equal number of $+1$ and -1 entries and are orthogonal to one another.

As a particular example let us look at the case $N=8$. The N eigenvectors are given in Table I. Clearly there are, in general, $N!/(N/2)!^2$ vectors with an equal number of positive and negative entries of module 1. All of them can be viewed as permutations carried out on the entries of one of them, e.g., vector a in Table I. There are, therefore, many different bases one can construct, but they are all equivalent to the basis we chose.

The binary vectors have an intriguing algebraic structure. This becomes evident by defining a multiplication procedure in which the vectors multiply each other component by component,

$$a*b=c \Longleftrightarrow \xi_i{}^a\xi_i{}^b=\xi_i{}^c . \tag{2.2}$$

The fact that every two vectors in our binary basis produce, under this multiplication, a third vector which belongs to this basis follows from our construction: Every binary vector is an outer product of the two basic two-

TABLE I. The binary basis for $N=8$.

e	a	b	c	d	f	g	h
1	1	1	1	1	1	1	1
1	1	1	1	−1	−1	−1	−1
1	1	−1	−1	1	1	−1	−1
1	1	−1	−1	−1	−1	1	1
1	−1	1	−1	1	−1	1	−1
1	−1	1	−1	−1	1	−1	1
1	−1	−1	1	1	−1	−1	1
1	−1	−1	1	−1	1	1	−1

dimensional vectors, taken in a certain order. In multiplying two such outer products one has to get a third one since, under this multiplication, the elementary two-dimensional vectors reproduce themselves. Moreover, since the $M=1$ elementary system displays a $Z(2)$ symmetry under this product, it follows that our binary basis has a group structure of $[Z(2)]^M$.

It is interesting to note that under this multiplication procedure one obtains abelian cycles of order 3 or less. Thus one finds that the unit vector e, which obviously obeys $e*e=e$ and $e*\mu=\mu$ is also given by the square of each element $\mu*\mu=e$. e and any μ therefore constitute a 2-cycle. Every two other elements are members of a 3-cycle:

$$a*b=c,\quad b*c=a,\quad c*a=b\ . \tag{2.3}$$

In the example of Table I we find seven such 3-cycles: abc, ahg, adf, bdg, bfh, chd, cgf. For general N the number of 3-cycles in a given basis is $(N-1)(N-2)/6$.

It follows from the multiplication procedure that we can construct a basis by starting with M binary vectors which are orthogonal to one another as well as to all their products. Let us call such a set of binary vectors a prime set. In the example of Table I, one can use the vectors a, b, and d as the prime set. The prime set together with all its products forms the basis. Using any prime set of binary vectors, one can construct the same basis or an equivalent one.

Given a basis of vectors ξ^μ of order N one can construct a basis of vectors λ^μ of order $2N$ by a simple $Z(2)$ outer product:

$$\begin{aligned}&\lambda_i{}^\mu=\xi_i{}^\mu,\quad \lambda_{i+N}{}^\mu=-\xi_i{}^\mu,\quad i=1,2,\ldots,N\\ &\lambda_i{}^{\mu+N}=\xi_i{}^\mu,\quad \lambda_{i+N}{}^{\mu+N}=\xi_i{}^\mu,\quad \mu=1,2,\ldots,N\ .\end{aligned} \tag{2.4}$$

Conversely one may identify in any basis B of vectors ξ^μ of order N subsets of vectors ζ^μ of order $N/2$ which allow for such a construction for some order of the indices. This leads to a division of the basis B into two halves, B^+ and B^-, which have the properties of even and odd parity, respectively, in the sense that if the vector a belongs to the sector with parity π_a then $a*b$ belongs to the sector with parity $\pi_a\pi_b$. Examples of such divisions in the case $N=8$ of Table I are

$$B^+=(e,c,f,g),\quad B^-=(a,b,d,h)\ ,$$

$$B^+=(e,b,d,g),\quad B^-=(a,c,f,h)\ .$$

There are seven such divisions that one can construct since B^+ has to be a basis of order four and, therefore, contain one of the seven possible 3-cycles. Prime sets can never belong to a half-basis with positive parity. In fact, every prime set determines a well specified division of the basis by assigning all odd multiples of the prime set to B^- and all even multiples to B^+.

In summary let us note that we encounter three different scales as N becomes large. One is $M=\log_2 N$, the order of the relevant group for our binary basis and the size of a prime set. The next is N, The size of our basis. Finally, we find that the number of possible binary vectors ξ is still much larger than that, it is of order $2^N/\sqrt{N}$. This leaves a large choice for the binary basis of the vector space of spin sites.

III. GENERALIZED SPIN OPERATORS AND COLLECTIVE SPIN VARIABLES

Using a Hamiltonian with a structure of the type of Eq. (1.1) we have to study operators which are specific combinations of the spins on the different sites

$$S^\mu=\sum_i S^i\xi_i{}^\mu\ . \tag{3.1}$$

We will refer to the operators S^μ as generalized spin operators. Clearly they cannot be regular spins since they obey a different algebra. Their commutation relations are

$$[S_\alpha{}^a,S_\beta{}^b]=i\epsilon_{\alpha\beta\gamma}S_\gamma{}^{a*b}\ . \tag{3.2}$$

This result follows by combining the commutation relations of the N independent spin variables, Eq. (1.3), with the multiplication property (2.2) of the vectors of the binary basis. The lower index displays the familiar SU(2) algebra while the upper index reflects the group multiplication property of the vectors ξ.

Since the generalized spin operators S^μ are just linear superpositions of the S^i it is clear that this new algebraic structure closes on the $[\mathrm{SU}(2)]^N$ algebra of the original N independent spin operators S^i. This observation, coupled with the simple cyclic multiplication of the upper index, allows us to draw interesting conclusions. It enables us to locate many subalgebras of the form $[\mathrm{SU}(2)]^n$, which can be decomposed into combinations of collective spin variables S^I. The latter are just straightforward sums of subsets of the original spins S^i. Thus, if we are given a problem which is defined in terms of spin operators built on p vectors, which belong to a binary basis, then there exists

some $m \leqq p$ such that the p vectors close on a $[Z(2)]^m$ group. Correspondingly, the p generalized spin operators S^μ close on an algebra of $[SU(2)]^n$ where $n=2^m$, and the whole problem can be solved in terms of n independent collective spins S^I.

Let us go through several examples by starting with the trivial observation that S^e (where e is the unit element of the binary basis) is an SU(2) generator. In fact this operator is just the total spin of the N-dimensional system. If the original spins S^i carry spin $\frac{1}{2}$, S^e will have a spectrum of all spins ranging from 0 to $N/2$. Next note that S^e together with any other generalized spin operator S^a close on the algebra of $SU(2)\times SU(2)$. The two independent collective spin operators are given by

$$S^+=\frac{S^e+S^a}{2},\quad S^-=\frac{S^e-S^a}{2}\ . \tag{3.3}$$

Their spectra will range from 0 to $N/4$. Starting with any two elements a and b it is clear they close on an algebra by adding their product $c=a*b$ and the unit element e. This way one obtains the algebra of $[SU(2)]^4$ with each collective spin ranging from 0 to $N/8$,

$$\begin{aligned} S^{++}&=\frac{S^e+S^a+S^b+S^c}{4},\quad S^{+-}=\frac{S^e+S^a-S^b-S^c}{4}\ ,\\ S^{-+}&=\frac{S^e-S^a+S^b-S^c}{4},\quad S^{--}=\frac{S^e-S^a-S^b+S^c}{4}\ . \end{aligned} \tag{3.4}$$

These collective spins are guaranteed to obey the regular spin algebra and they are, therefore, simple sums of $N/4$ elementary spins S^i. The generalization to larger systems is self-evident. It follows the same lines of the $Z(2)$ structure that we used for constructing the binary basis. Repeating this procedure by doubling the size of the system each time through the addition of generalized spin operators, one ends up inevitably with the original N spin-$\frac{1}{2}$ operators.

As an application of this reasoning let us analyze the structure of a frustrated Heisenberg model defined by the Hamiltonian

$$H_{\mathrm{FH}}=c\sum_{i,j}\sum_{\alpha}J_{ij}S_\alpha{}^iS_\alpha{}^j \tag{3.5}$$

with a coupling matrix defined by a set of p binary vectors according to Eq. (1.2). Starting with the simple case of a single binary vector ξ^a, i.e., $p=1$, we find

$$H=\frac{c}{N}\sum_{\alpha}S_\alpha{}^aS_\alpha{}^a=\frac{c}{N}(S^+-S^-)^2\ . \tag{3.6}$$

Thus we end up with the problem of coupling two collective spin structures of our system. The result has a spectrum which is simple to construct. In the ferromagnetic case, where c is negative, the ground state is unique. It corresponds to both spins having their highest value (i.e., $N/4$), while being coupled together to form a total spin-0 state. This follows from rewriting H as

$$\begin{aligned} H&=\frac{c}{N}[2(S^+)^2+2(S^-)^2-(S^++S^-)^2]\\ &=\frac{c}{N}[2(S^+)^2+2(S^-)^2-S_{\mathrm{total}}{}^2]\ . \end{aligned} \tag{3.7}$$

In the antiferromagnetic case of positive c we find many possible ground states, corresponding to all possible ways of constructing $S^+=S^-=0$ out of the $N/2$ spin-$\frac{1}{2}$ operators that each one contains.

The structure obtained in Eq. (3.7) is characteristic of the case $p=n-1$, i.e., when one uses all the binary vectors which form (together with the unit vector) a basis of size n. This follows from the identity

$$\sum_{\mu=1}^{n}S^\mu S^\mu=n\sum_{I=1}^{n}S^IS^I\ . \tag{3.8}$$

The sum on the left includes all the generalized spin operators corresponding to the basis. By moving the unit element (i.e., S_{total}) to the right, one finds a structure as seen in Eq. (3.7).

It is clear how to proceed with the analysis for arbitrary values of p. For any given p the Hamiltonian is a function of scalar products of collective spin operators whose number is $n=2^m$ where $m\leqq p$. The equality sign is obtained if the binary vectors form a prime set. The spectrum of this model can be completely classified within an algebra of $[SU(2)]^n$. This will hold also for more complicated Heisenberg models in which the coupling matrix is given by

$$J_{ij}=\frac{1}{N}\sum_{\mu=1}^{p}\omega_\mu\xi_i{}^\mu\xi_j{}^\mu\ . \tag{3.9}$$

Such a model depends again on n collective spin variables, but the different weights ω_μ introduce a structure in the spectrum which prefers certain alignments of the spins over others which were equally probable in the more symmetric case of equal weights. A simple example is the case $p=2$. With general weights we find

$$\begin{aligned} NH&=\omega_1S^aS^a+\omega_2S^bS^b\\ &=(\omega_1+\omega_2)[(S^{++}-S^{--})^2+(S^{+-}-S^{-+})^2]\\ &\quad+2(\omega_1-\omega_2)(S^{++}-S^{--})(S^{+-}-S^{-+})\ . \end{aligned} \tag{3.10}$$

Clearly, this system has the most symmetric structure when the weights are equal. In fact, in this case, the Hamiltonian reduces to a direct sum of two Hamiltonians which have the structure of the $p=1$ problem, Eq. (3.6).

Let us close this section with an elaboration of the last remark, by asking: Under what conditions does the Hamiltonian reduce into a sum of $p=1$ Hamiltonians? It is easy to see that this does not recur for $p=3$ but can occur for $p=4$, depending on which set of binary vectors is used. In fact, it occurs if one uses half the $n=8$ basis with a negative parity (e.g., the elements a, b, d, and h of Table I). To prove that this is a general result let us note that for a half-basis with negative parity one can choose

$$\xi_i=\zeta_i,\quad \xi_{i+n/2}=-\zeta_i,\quad i=1,2,\ldots,n/2 \tag{3.11}$$

and, therefore, one can express the corresponding generalized spin operators as

$$S^\mu=\sum_{i=1}^{n/2}\zeta_i{}^\mu(L^i-K^i)\ , \tag{3.12}$$

where L and K are two sets of $n/2$ independent spins. It follows then from the identity (3.8) that a Hamiltonian which depends on the sum of all these $S^{\mu}S^{\mu}$ will be decomposable into $p=1$ parts, each corresponding to a single combination of $(L-K)^2$.

IV. THE CLASSICAL PARTITION FUNCTION

We will make use of the collective spin operators that were developed in Sec. III to gain insight into the structure of the partition function of the classical model defined by Eq. (1.1). For this purpose let us write the partition function as a quantum-mechanical expectation value

$$Z=2^N\bar{Z},\quad \bar{Z}=\langle x\,|\,\exp(-\beta H)\,|\,x\rangle\ , \tag{4.1}$$

where we choose the Hamiltonian

$$H=-\sum_{i,j}J_{ij}S_z{}^iS_z{}^j \tag{4.2}$$

and the state

$$|\,x\rangle=\prod_i\,|\,S_x{}^i=\tfrac{1}{2}\rangle\ . \tag{4.3}$$

Since an eigenstate of S_x has equal contributions from both components of S_z it follows that Eq. (4.1) is indeed equal to the partition function of the statistical mechanics defined by Eq. (1.1).

Regarding the transfer from classical to quantum mechanics as the replacement of S^i by $S_z{}^i$, one may wonder whether the formalism of Sec. III is relevant at all to this problem since all the S_z operators commute with one another anyway. Since, however, the state (4.3) is defined in terms of S_x, the formalism becomes relevant and the power of the spin algebra comes into play. We know that the problem can be rewritten in terms of $n=2^m$ collective spin variables. Let us denote these variables by S^I. Since the state (4.3) has all $S_x{}^i$ aligned it follows that the same state, when described in terms of the S^I, will turn out to be composed of states with maximal spins which are maximally polarized along the x direction:

$$|\,x\rangle=\prod_I\,|\,S_x{}^I=Q,\ S^I=Q\rangle,\quad Q=\frac{N}{2^{m+1}}\ . \tag{4.4}$$

Hence the partition function depends only on collective spins which attain their maximal values. To rewrite this function as an algebraic series we have to project the states of Eq. (4.4) onto their polarizations along the z axis,

$$|\,S_x=Q,S=Q\rangle=\sum_{q=-Q}^{Q}c_q\,|\,S_z=q,S=Q\rangle\ . \tag{4.5}$$

The coefficients c_q are obtained from well-known representations of the rotation group and are equal to $d_{qQ}{}^{Q}(\pi/2)$,

$$c_q{}^2=\frac{1}{2^{2Q}}\frac{(2Q)!}{(Q-q)!(Q+q)!}\ . \tag{4.6}$$

This distribution peaks at $q=0$ and decreases rapidly as function of q. An example for $Q=32$ is displayed in Fig. 1. At low values of β (high temperatures) this is the important factor in the partition function, hence the low values of q will be populated. For large β the situation will change, since the lowest energy states of our Hamiltonian correspond to maximal (either positive or negative) q values. Amit *et al.*[1] have shown that in the thermodynamic limit there exists a phase transition between these two distinct regions of temperature.

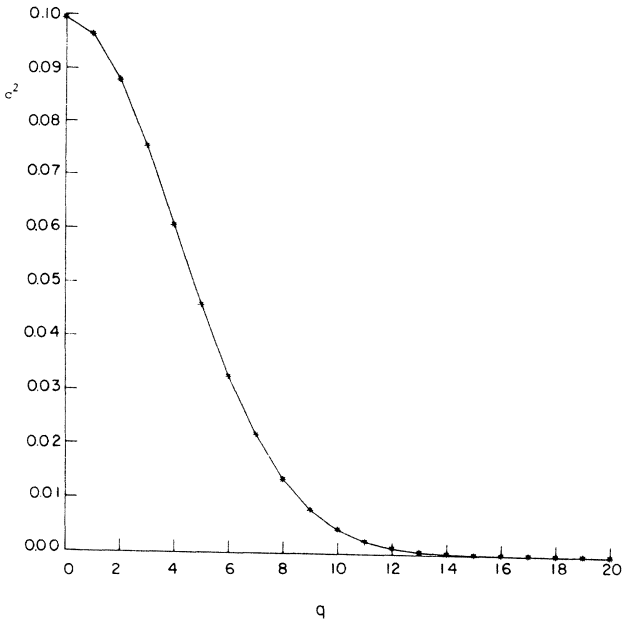

FIG. 1. The distribution of the coefficients c_q^2 as a function of $q\geq 0$ for the case $Q=32$.

As an example of a numerical evaluation of our representation of the partition function let us investigate the simplest case $p=1$. The Hamiltonian can be written as

$$H=-\frac{2}{N}S_z{}^aS_z{}^a=-\frac{2}{N}(S_z{}^+-S_z{}^-)^2\ , \tag{4.7}$$

where S^+ and S^- are collective spin operators with a range up to $Q=N/4$. We have introduced an explicit factor of 2 in the coupling in order to obtain the same normalization as Amit *et al.*,[1] taking into account the fact that the elementary spins of our problem have value $\frac{1}{2}$. The partition function can then be written in the following explicit form:

$$\bar{Z}_1=\sum_{q^+,q^-}c_{q^+}^2c_{q^-}^2\exp\left[\frac{\beta}{2Q}(q^+-q^-)^2\right]\ . \tag{4.8}$$

We have evaluated this expression for different choices of Q. The values of the free energy and specific heat (defined through the first and second derivatives of $\ln Z$ with respect to β) for $Q=8$, 16, and 32 are displayed in Fig. 2. The behavior of the specific heat shows the buildup of a phase transition. The peak moves slowly with Q to the correct location[1] $\beta_c=1$. At large β we observe that the free energy tends to the expected value of $-N/2=-2Q$, which is obtained when all spins are aligned in the direction of the binary vector ξ^a.

Since the coefficients c_q^2 are invariant under the change $q\leftrightarrow -q$ we find that the partition function $\bar{Z}_1$ of Eq. (4.8)

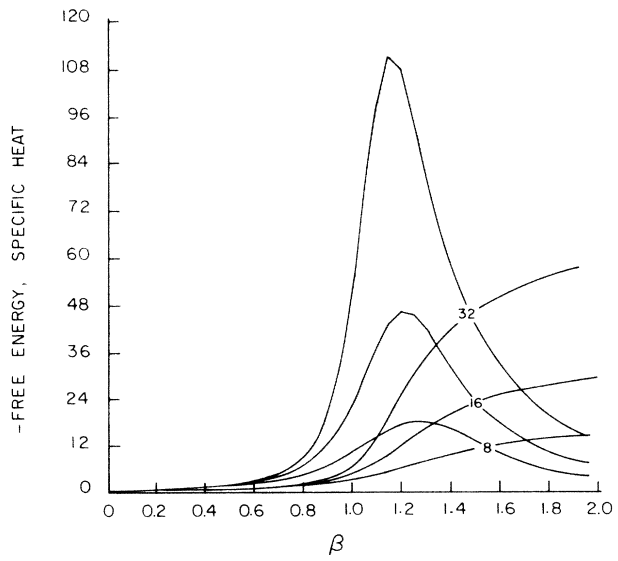

FIG. 2. Results derived from Z_1 for the cases $Q=8,16,32$. The curves describing the specific heat display a peak above $\beta=1$ which increases in magnitude and shifts towards $\beta_c=1$ as Q increases. The free-energy curves display two distinct regions. At low β the free energies of the different Q values are equal to one another, while for $\beta>1$ they are proportional to Q. At $\beta=2$ they are very close to their asymptotic values. In this figure we plot the negative of the free energy.

stays unchanged under the reversal of the sign of q^- in the exponent of this expression. This means that the $p=1$ statistical problem is the same if we use the unit vector as input instead of a binary vector. This is so because any $p=1$ problem can be recast into this form by the Mattis[3] transformation $S^i \rightarrow \xi_i S^i$ (no summation implied). Note that this does not hold for the quantum-mechanical problems discussed in Sec. III.

The analysis can be easily generalized to higher p values although its evaluation can become exceedingly difficult. The case $p=2$ leads to a summation over components of four collective spins:

$$\bar{Z}=\sum_{\text{all } q} c^2_{q^{++}} c^2_{q^{+-}} c^2_{q^{-+}} c^2_{q^{--}} \times \exp\left\{\frac{2\beta}{N}[(q^{++}+q^{+-}-q^{-+}-q^{--})^2 + (q^{++}-q^{+-}+q^{-+}-q^{--})^2]\right\} . \tag{4.9}$$

This problem is decomposable into a product of two partition functions of the type (4.8). This is the same phenomenon that we discussed at the end of Sec. III: Whenever the p input vectors form a half-basis of negative parity the partition function turns into a product of p partition functions $\bar{Z}_1$. By using the Mattis transformation we can effectively change the parity of any half-basis. Hence we may conclude that the decomposition

$$\bar{Z}_p=\bar{Z}_1^p \tag{4.10}$$

will occur for all the cases in which the set of p vectors forms a half-basis. This includes, of course, the cases in which the p vectors form a full basis, because every basis can be viewed as a half-basis of positive parity in a structure which is twice the size.

The Mattis transformation can be used to reduce the size of the relevant spin algebra by a factor of 2. Starting with any set of p input binary vectors (e.g., the prime set *abd* of Table I, which defines a $p=m=3$ or $n=8$ system) one can use one of them to redefine the classical spins via a Mattis transformation ending up with a set which includes the unit element and $p-1$ binary vectors (e.g., $e=a*a$, $c=a*b$, $f=a*d$ in this example, which fits into an $n=4$ system). Hence a problem defined by p binary vectors can be described in a basis of order $n=2^m$ where $m \leqq p-1$.

Our representation of the partition function as a quantum-mechanical matrix element allows for a very suggestive decomposition into sectors of configuration space, which are quantum-mechanical states. Defining a positive and negative component of the collective spin I in the following (somewhat arbitrary) fashion:

$$|I+\rangle=\sum_{q\geqq 0} c_q \,|S_z{}^I=q\rangle, \quad |I-\rangle=\sum_{q<0} c_q \,|S_z{}^I=q\rangle , \tag{4.11}$$

we can decompose the state $|x\rangle$ of Eq. (4.1),

$$|x\rangle=\prod_{I=1}^{n}(|I+\rangle+|I-\rangle) , \tag{4.12}$$

into a coherent sum of states which have one component for each I. Correspondingly, the partition function can be written as the sum of 2^n terms, coming from orthogonal states defined by specific general directions of each collective spin. In a problem defined by p input vectors which belong to a prime set there will be $2p$ "most-favored" states which have the lowest free energy in the low-temperature region. These are the ones in which the Ith component is chosen according to the sign implied by one of the p input vectors. The factor of 2 is present since H is bilinear in the S^μ, leading to a twofold degeneracy: a most-favored state is transformed into another most-favored state by reversing the sign of all components. Using the $n-p$ binary vectors which belong to the basis but are not included in the input to H, we can construct $2(n-p)$ unfavored states which will have the highest free energy in the low-temperature region. In between we find the rest of the 2^n states with various degrees of favoritism. We encounter states which differ from the most-favored p vectors by one sign, then by two signs, and so on. In a large-p system their free energy will be a large fraction of the lowest one, decreasing in magnitude the farther they depart form any one of the most-favored p binary vectors. These different states are semistable in the low-temperature regime. In order to decay into one of the minima one has to rotate a collective spin, which may require a long series of Monte-Carlo runs in a numerical simulation.

A simple example of the different states can be given in the $p=2$ problem described by the partition function of

Eq. (4.9). The relevant basis is $n=4$. The collective spins I are designated here by $++,+-,-+,--$. The most-favored states have the I components chosen as $+,+,-,-$ or $+,-,+,-$ or their reversals. The unfavored states are defined by the choices $+,+,+,+$ and $+,-,-,+$ or their reversals. In this problem we are then left with states which have I components with three equal signs. They will all correspond to an intermediary situation. Since this $\overline{Z}_2$ problem factors into $\overline{Z}_1{}^2$ we can identify each such intermediate state as having a favored component in one $\overline{Z}_1$ and an unfavored component in the other $\overline{Z}_1$.

V. DISCUSSION

Simple spin systems are favorite models in statistical mechanics, simulating the structures of many physical systems. The classical Ising model and the quantum-mechanical Heisenberg model are well-known textbook examples. Being based on nearest-neighbor interactions they represent very different structures from the ones that we have studied here. In a one-dimensional system, where

$$J_{ij}=\delta_{i,j+1}+\delta_{i,j-1}\ , \tag{5.1}$$

with $i=1,2,\ldots,N$ ordered cyclicly, it is advantageous to construct the spin-wave operators

$$S(k)=\sum_j e^{ikj}S^j\ , \tag{5.2}$$

which obey the algebra

$$[S_\alpha(k),S_\beta(p)]=i\epsilon_{\alpha\beta\gamma}S_\gamma(k+p)\ . \tag{5.3}$$

The momenta k take on discrete values appropriate to the lattice. As N increases they become dense and the algebra (5.3) turns into a Kac-Moody algebra. The Hamiltonian can be rewritten in a bilinear form in terms of the spin waves, hence they determine the structure of the spectrum. The excitations are labeled by momenta, reflecting the translation invariance of these models.

The models that we discussed here belong to another extreme. They represent interactions of all spins with one another. Therefore there is no apparent space structure involved. Moreover, the interaction is parametrized in a binary basis. This led us to consider the generalized spin operators

$$S^\mu=\sum_i \xi_i{}^\mu S^i \tag{5.4}$$

instead of the Fourier transforms of the nearest-neighbor interaction. The use of binary bases as a complete set of functions is well known in some computer-science and electrical-engineering areas and goes under the name of Walsh functions.[4] This is the right tool for handling information bits and it seems therefore appropriate to replace in this problem the Fourier transform with Walsh transforms.

The algebra of the generalized spin operators

$$[S_\alpha{}^a,S_\beta{}^b]=i\epsilon_{\alpha\beta\gamma}S_\gamma{}^{a*b} \tag{5.5}$$

is quite different from that of the spin waves, Eq. (5.3). The most important difference is its decomposition into many subalgebras. Every three operators that participate in the relation (5.5) constitute, together with S^e, an algebra of $[SU(2)]^4$. This simplifies considerably the analysis of the models that we discussed.

A general symmetric coupling matrix J_{ij} can be expressed as such in the configuration space basis, labeled by the site index i, or in the binary basis

$$J_{ij}=\sum_{\mu,\nu}\omega_{\mu\nu}\xi_i{}^\mu\xi_j{}^\nu\ , \tag{5.6}$$

where ω is a real symmetric matrix. The generalized spin operators come into play when ω is a diagonal matrix. This is the case discussed in Secs. III and IV. We have mostly looked into the even more restricted situation in which the diagonal matrix ω had two degenerate eigenvalues: most diagonal elements vanished, and the few nonzero ones were equal to one another. In Sec. III we have discussed the resulting Heisenberg models and demonstrated the usefulness of the new generalized spin algebra. By rewriting the generalized spin operators in terms of collective spin variables we have, in fact, diagonalized these quantum-mechanical models. The same approach was applied in Sec. IV to models in statistical mechanics by employing a representation of the classical partition function as a quantum-mechanical expectation value.

The $p=1$ model was proposed and analyzed by Mattis.[3] Using our techniques we were able to give an explicit representation of the $p=1$ problem in terms of a series of two-integer variables. Moreover, we have seen that there are many other models, namely those where the p binary vectors are Mattis transformations of a basis, which reduce into products of the $p=1$ problem.

Van Hemmen[5] discussed a particular $p=3$ problem whose analog, in our language, involves the vectors e, a, and b. Using a different coupling for S^eS^e than for the other generalized spin operators, he obtains a rich phase structure for this system. This result relies on the average over the random variables in the thermodynamic limit, but we may expect a finite system to display analogous behavior. Provost and Vallee[6] have generalized Van Hemmen's model into a general p structure, analyzing it with both mean-field and replica methods. Our new technique may allow numerical and algebraic investigations of such models using the explicit forms for the partition function.

The general finite-p problem was discussed by Amit *et al.*[1] in the $N\to\infty$ limit. Their input vectors were generated randomly. This allowed them to derive physical conclusions in the thermodynamic limit. Our input is more rigid since we employ only vectors which belong to a binary basis. When the randomly generated input vectors are expressed in an appropriate binary basis, ω should turn out to have large diagonal and small nondiagonal matrix elements. In such a situation our formalism should still be applicable, but perturbative corrections have to be invoked.

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