Two Overlapping Resonances: Production and Decay in S-Matrix Theory*

Y. DOTHAN† AND D. HORN‡
Tel-Aviv University, Tel-Aviv, Israel
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We discuss the system of two overlapping resonances along two parallel lines: (A) the general restrictions imposed on the S matrix by unitarity, and (B) the results of the Weisskopf-Wigner approximation. The equivalence between the two is used to get a physical understanding of the formalism applicable to both scattering and decay processes. We show that the connection between the overlap in channel space, χ_C, and the overlap of the two resonances, χ, is the unitarity sum rule.

IN a recent Letter[1] McGlinn and Polis raised the question of constructing an S matrix for overlapping resonances and its relation to the unitarity sum rule[2] for the $K_L K_S$ system. We disagree with the discussion and results of Ref. 1 and present here a different treatment of the subject. Our discussion proceeds along two parallel lines: (A) the general restrictions imposed on the S matrix by unitarity, and (B) the results of the Weisskopf-Wigner[3] approximation.

Some of the important implications of the S matrix unitarity for the K_S-K_L system have in the meantime been discussed by McVoy.[4] He treated the problem in an approximate way suitable for the $\Gamma_S \gg \Gamma_L$ situations. We present the general solution which is in principle applicable to other interesting cases as well, e.g., the A_2 doublet. Upon completion of our work we were informed of another discussion of the same subject by Stodolsky,[5] who follows a dynamical approach. Our discussion necessarily overlaps with parts of Refs. 4 and 5; however, we try to follow a unified approach along the two mentioned lines.

In the Secs. 1 and 2 we treat the well-known case of the single resonance to establish notation and gain insight into the problems involved. Sections 3 and 4 are devoted to the analysis of the two-resonance problem along the lines (A) and (B), respectively. In Sec. 5 we discuss the connection between the overlap in channel space and the overlap of the two resonances. This is, in fact, the unitarity sum rule. The further discussion of the discrete symmetries in Sec. 6 points out the equivalence between our scattering formalism and the usual treatment of the decay system. Section 7 summarizes the results and conclusions.

1. SINGLE-RESONANCE CASE

Let us treat first the well-known case of a single resonance. In the neighborhood of this resonance, the S matrix can be written as an $N \times N$ matrix (for N open channels with definite total angular momentum and conserved internal quantum numbers) in the form

$$S = B - i\Gamma Q/(E - \epsilon_R), \tag{1}$$

where both B and Q are energy-independent matrices and $\epsilon_R = M_R - \frac{1}{2}i\Gamma_R$ is the position of the resonance pole. One requires that Q be of rank 1 (which corresponds to a single resonance at ϵ_R). Unitarity implies that

$$S^\dagger(E^*)S(E) = 1 \tag{2}$$

and this imposes the following conditions on (1):

$$B^\dagger B = 1, \quad Q = BP, \tag{3}$$

where P is a Hermitian projection operator

$$P = P^\dagger = P^2. \tag{4}$$

A similar result follows from the Weisskopf-Wigner (WW) approximation.[3] Let us sketch it briefly. In the interaction representation, one considers the effect of an interaction Hamiltonian H' on the time development of a wave function described by the eigenfunctions of some H_0 that consist of a discrete level with probability amplitude $a_R(t)$ and a continuum with probability amplitudes $a_C(t)$. The energy eigenvalue E_R is supposed to be within the range of variation of E_C. One assumes that H' has nonvanishing matrix elements only between the resonance and the continuum. This is the WW approximation. In other words, one looks at the pure effects of the production and decay of the resonance. This corresponds to setting $B = 1$ in our S-matrix discussion.

The Schrödinger equation turns into the following set of equations:

$$\begin{aligned} i\dot{a}_C &= H_{CR}' a_R e^{i\omega_{CR}t}, \\ i\dot{a}_R &= \int \rho(E_C) dE_C H_{RC}' a_C e^{i\omega_{RC}t}, \quad H_{CR}' = H_{RC}'^*. \end{aligned} \tag{5}$$

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† Summer Visitor at Research Institute for Theoretical Physics, University of Helsinki, Finland.

‡ Summer Visitor at Center for Particle Theory, Physics Department, University of Texas, Austin, Texas 78712.

[1] W. D. McGlinn and D. Polis, Phys. Rev. Letters **22**, 908 (1969).

[2] J. S. Bell and J. Steinberger, in *Proceedings of the Oxford International Conference on Elementary Particles, 1965* (Rutherford High Energy Laboratory, Chilton, Berkshire, England, 1966), p. 195.

[3] V. Weisskopf and E. Wigner, Z. Physik **63**, 54 (1930); V. Weisskopf, Ann. Physik **9**, 23 (1931). The first paper considers the decay of atomic resonances and the second deals with the scattering processes.

[4] K. W. McVoy, Phys. Rev. Letters **23**, 56 (1969); L. Durand III and K. W. McVoy, *ibid.* **23**, 59 (1969).

[5] L. Stodolsky (unpublished report). We are thankful to Dr. Stodolsky for informing us about his work.

If we choose the boundary conditions at $t=-\infty$ as $a_R\to 0$, $a_C\to[1/\rho(E_C)]\delta(E_C-E)$, then the structure of these coefficients at $t=+\infty$ will tell us what the S matrix is. The calculation proceeds with the usual assumption about the slow variation of $\rho(E_C)|H_{RC}'|^2$ as a function of E_C or C, and leads us to the solutions

$$a_R(t)=H_{RC}'\frac{e^{i(M-E)t}}{E-M+\frac{1}{2}i\Gamma},$$
$$a_C(t)=\frac{1}{\rho}\delta(E_C-E)-\frac{i|H'|^2}{E-M+\frac{1}{2}i\Gamma}\frac{e^{i(E_C-E)t}}{i(E_C-E)}, \tag{6}$$

where

$$\Gamma=2\pi\rho|H_{RC}'|^2. \tag{7}$$

The various exponentials in (6) have to be considered as distributions obeying the well-known relation

$$\begin{aligned} e^{ixt}/(ix+\epsilon)&\to 0 \quad &&\text{as } t\to-\infty\\ &\to 2\pi\delta(x) &&\text{as } t\to\infty, \end{aligned} \tag{8}$$

which leads to the result

$$a_R\to 0,$$
$$a_C\to\left(1-\frac{i\Gamma}{E-M+\frac{1}{2}i\Gamma}\right)\frac{1}{\rho}\delta(E_C-E) \quad \text{as } t\to\infty. \tag{9}$$

Comparing (9) with (1), using the relation

$$a_C(t=+\infty)=Sa_C(t=-\infty), \tag{10}$$

we find the expected form of the S matrix of one resonance in one channel. The generalization to the N-channel problem is straightforward. One introduces a_{iC} with $i=1,\ldots,N$, as well as $\rho_i(E_C)$ and $H_{RC}'^i$. Here i is a channel index. If we assume once again that all $\rho_i(E_C)|H_{RC}'^i|^2$ are essentially constants over the relevant energy region $M\pm\Gamma$, then we can define

$$\Gamma_i=2\pi\rho_i|H_{RC}'^i|^2, \quad \Gamma=\sum_i\Gamma_i \tag{11}$$

leading to

$$a_{iC}(t=+\infty)=\left(\delta_{ij}-\frac{i2\pi\rho_jH_{CR}'^iH_{RC}'^j}{E-M+\frac{1}{2}i\Gamma}\right)a_{jC}(t=-\infty). \tag{12}$$

To compare (12) with (1) we have first to define the S matrix in a form that ensures its unitarity. This is readily achieved by the definition

$$(\sqrt{\rho_i})a_{iC}(t=+\infty)=S_{ij}(\sqrt{\rho_j})a_{jC}(t=-\infty). \tag{13}$$

The unitarity of S is implied by the conservation of the norm of the wave function

$$\sum_i\int\rho_i(E_C)|a_{iC}(E_C,t=+\infty)|^2dE_C$$
$$=\sum_i\int\rho_i(E_C)|a_{iC}(E_C,t=-\infty)|^2dE_C. \tag{14}$$

With this definition we find that (12) leads to

$$S=1-i\Gamma P/(E-\epsilon_R), \quad \epsilon_R=M-\tfrac{1}{2}i\Gamma \tag{15}$$

with P being the Hermitian projection operator

$$P_{ij}=(2\pi/\Gamma)(\rho_i\rho_j)^{1/2}H_{CR}'^iH_{RC}'^j. \tag{16}$$

2. PRODUCTION AND DECAY OF RESONANCE

The S matrix is an $N\times N$ matrix in the channel space. We will describe vectors in this space by $|i)$ and their Hermitian adjoints by $(i|$. We do not use the usual Dirac symbol $|i\rangle$ in order not to confuse the vector $|i)$ that just designates the ith channel with the wave function a_i that corresponds to the physical state.

Let us introduce now the concept of the eigenchannel of the resonance. We describe the physical channels by orthonormal vectors $|i)$,

$$(i|j)=\delta_{ij}, \tag{17}$$

in terms of which the resonance eigenchannel will be defined as

$$|R)=\left(\frac{2\pi}{\Gamma}\right)^{1/2}\sum_i(\sqrt{\rho_i})H_{CR}'^i|i). \tag{18}$$

We note that $(R|R)=1$ and the projection operator P defined by (16) can be rewritten as

$$P=|R)(R|. \tag{19}$$

The physical significance of $|R)$ is the following: Suppose that at $t=0$, $a_R=1$ and $a_C=0$; then one can solve Eqs. (5), and one finds

$$a_R(t)=e^{-\Gamma t/2},$$
$$(\sqrt{\rho_i})a_{iC}(t)=(\sqrt{\rho_i})H_{CR}'^i\frac{1-e^{i(E_C-M+\frac{1}{2}i\Gamma)t}}{E_C-M+\frac{1}{2}i\Gamma}. \tag{20}$$

We see that as $t\to\infty$, the distribution in the channel space is the one specified by $|R)$. Thus we might say that the resonance decays into the combination of channels $|R)$. Similarly, one might speak about the production of the resonance. Using Eq. (6), one may ask which distribution of a_{iC} at $t=-\infty$ will lead to $a_R(0)=1$ and $a_C(0)=0$. If

$$(\sqrt{\rho_i})a_{iC}(-\infty)=(\sqrt{\rho_i})H_{CR}'^i/(E_C-M-\tfrac{1}{2}i\Gamma), \tag{21}$$

this goal is achieved.

In fact, starting with the wave function (21), we find

$$a_R(t)=\frac{2\pi}{\Gamma}\int\frac{dE\,e^{i(M-E)t}}{(E-M-\frac{1}{2}i\Gamma)(E-M+\frac{1}{2}i\Gamma)}$$
$$=e^{+\Gamma t/2} \quad (t<0)$$
$$=e^{-\Gamma t/2} \quad (t>0), \tag{22}$$

which is the ideal description of the production and decay of the resonance. It is also easy to see that the

right distribution of a_{iC} is achieved at $t=+\infty$, Eq. (20), since the S matrix (15) can be rewritten as

$$S=1-|R)(R|+|R)(R|\frac{E-M-\frac{1}{2}i\Gamma}{E-M+\frac{1}{2}i\Gamma}. \quad (23)$$

Thus, starting with a state proportional to $|R)/(E-M-\frac{1}{2}i\Gamma)$ at $t=-\infty$, one has to end up with a state proportional to $|R)/(E-M+\frac{1}{2}i\Gamma)$ at $t=+\infty$.

The definition of the eigenchannel of the resonance was given in the absence of any background scattering. The introduction of such a background spoils the simplicity of the interpretation. Equation (1) can be rewritten in the form

$$S=B\left(1-\frac{i\Gamma P}{E-M+\frac{1}{2}i\Gamma}\right)$$

$$=B^{1-\alpha}\left(1-\frac{i\Gamma P_\alpha}{E-M+\frac{1}{2}i\Gamma}\right)B^\alpha. \quad (24)$$

Since $B^\dagger=B^{-1}$, any $P_\alpha=B^\alpha P B^{-\alpha}$ is also a Hermitian projection operator. Writing Eq. (24) with any particular choice of α corresponds to a certain choice of splitting the scattering amplitude into an initial-state interaction, a resonance scattering, and a final-state interaction. Each such resonance will have some specific eigenchannel $B^\alpha|R)$. This artificial breaking of S into three processes is devoid of a physical meaning since all the scattering occurs simultaneously.

We see that in the presence of a background scattering one cannot, and therefore one should not, attach any particular physical meaning to the eigenchannel. Nevertheless, we would still like to have a clear definition of what is meant by a partial width since it is measured in experiments. Time-reversal invariance simplifies matters a little bit. If we work in a basis satisfying $T|i\rangle=|i\rangle$, T invariance requires a symmetric S matrix:

$$B=\tilde{B}=(B^{-1})^*, \quad Q=BP=\tilde{P}\tilde{B}=|x)(x|^*. \quad (25)$$

In the case $B=1$, we find $|x)=|R)$. We can now define the quantities γ_i:

$$\gamma_i=\Gamma(i|x)(x|i)^*=\Gamma(i|x)^2, \quad \sum_i|\gamma_i|=\Gamma. \quad (26)$$

γ_i can be interpreted as the partial-width amplitude. Note, however, that γ_i, in contrast to Γ_i, is in general a complex quantity. Its phase reflects the existence of background scattering. We can now rewrite

$$Q=\sum_i\frac{\gamma_i}{\Gamma}|i)(i|, \quad (27)$$

and we notice that in the particular case of diagonal background scattering, $B=\sum_i e^{2i\delta_i}|i)(i|$, Eq. (3) leads to Watson's final-state theorem

$$\gamma_i=|\gamma_i|e^{2i\delta_i}. \quad (28)$$

We see, therefore, that a resonance can be identified by its location ϵ_R as well as the partial widths γ_i; however, the latter are intimately connected with the background scattering. The arbitrariness in the definition of the eigenchannel reflects the fact that the true physics is given by the background plus the resonance and that they have to be considered together. In this connection let us mention the paper of Weidenmüller,[6] who pointed out that in the presence of a background term none of the eigenphases of the S matrix increases by π over the resonance. Only the sum of the eigenphases increases by π. This is another way of seeing that the analysis of an S matrix in terms of its eigenchannels is unfruitful in this case.

One has to emphasize that all that was said here applies to an S matrix of definite total angular momentum (usually denoted by S^J). When an experimentalist measures total cross sections, he encounters the incoherent sum of all these S^J amplitudes. Therefore, it is hard to tell whether a resonance seen in some total cross section, say, is or is not accompanied by a background in its own S^J. This can be resolved by phase-shift analysis. The recent CERN π-N phase shifts[7] show that in many cases the background is small and the WW conditions are met.

3. S MATRIX FOR TWO RESONANCES

In this section we construct an S matrix in the region of two overlapping resonances. We consider the case of $B=1$. If a background amplitude has to be taken into account, then it should multiply the S matrix presented here. We begin by writing the S matrix in the form

$$S(E)=1-\frac{i\Gamma_1 Q_1}{E-\epsilon_1}-\frac{i\Gamma_2 Q_2}{E-\epsilon_2}, \quad (29)$$

$$\epsilon_1=M_1-\tfrac{1}{2}i\Gamma_1, \quad \epsilon_2=M_2-\tfrac{1}{2}i\Gamma_2, \quad \epsilon_1\neq\epsilon_2.$$

Q_1 and Q_2 are two matrices of rank 1:

$$(\sqrt{\nu_1})Q_1=|a_1)(b_1|, \quad (\sqrt{\nu_2})Q_2=|a_2)(b_2|. \quad (30)$$

We choose all four vectors in the channel space to be normalized to 1. $\sqrt{\nu_i}$ are normalization factors of Q_i. We assume here that $\epsilon_1\neq\epsilon_2$. The case of degenerate poles is treated in the Appendix.

[6] H. A. Weidenmüller, Phys. Letters **24B**, 441 (1967).
[7] A. Donnachie, R. G. Kirsopp, and C. Lovelace, Phys. Letters **26B**, 161 (1968).

Let us now impose unitarity, Eq. (2), on this S matrix. It results in the set of two equations

$$\begin{aligned}(\sqrt{\nu_1})|a_1)(b_1|-(a_1|a_1)|b_1)(b_1|\\ +i\frac{\Gamma_2}{\epsilon_1-\epsilon_2^*}(a_2|a_1)|b_2)(b_1|=0,\\ (\sqrt{\nu_2})|a_2)(b_2|-(a_2|a_2)|b_2)(b_2|\\ +i\frac{\Gamma_1}{\epsilon_2-\epsilon_1^*}(a_1|a_2)|b_1)(b_2|=0.\end{aligned} \tag{31}$$

Since we chose all four vectors to be normalized to 1, i.e.,

$$(a_1|a_1)=(a_2|a_2)=(b_1|b_1)=(b_2|b_2)=1, \tag{32}$$

we can easily solve Eq. (31) for $|b_1)$ and $|b_2)$ in terms of $|a_1)$ and $|a_2)$, and find

$$\begin{aligned}|b_1)&=\frac{1}{\sqrt{\nu}}\left(|a_1)+\frac{i\Gamma_2}{\epsilon_1-\epsilon_2^*}\chi_C|a_2)\right),\\ |b_2)&=\frac{1}{\sqrt{\nu}}\left(|a_2)+\frac{i\Gamma_1}{\epsilon_2-\epsilon_1^*}\chi_C^*|a_1)\right),\end{aligned} \tag{33}$$

where

$$\nu_1=\nu_2=\nu=1-\frac{\Gamma_1\Gamma_2}{|\epsilon_2-\epsilon_1^*|^2}|\chi_C|^2, \tag{34}$$

$$\chi_C=(a_2|a_1). \tag{35}$$

χ_C is the overlap between the two channel vectors $|a_2)$ and $|a_1)$. Its relation to what is usually called the overlap between the two resonances will be discussed later. The two quantities χ_C and ν obey the inequalities

$$|\chi_C|\leq 1, \quad \nu\leq 1. \tag{36}$$

The substitution of (33) back into (29) gives us the desired solution for the S matrix. We conclude, therefore, that once the location of the two resonances in the E plane is given, then a unitary $S(E)$ is determined up to two vectors $|a_1)$ and $|a_2)$ in the channel space.

Before going on to the physical interpretation of the various quantities, which will be achieved by solving the S matrix in the WW approximation, let us establish some properties of the two-dimensional space encountered here. The vectors $|a_1)$ and $|a_2)$ span a two-dimensional space, but they are not orthogonal. Let us define their reciprocal vectors[8] obeying

$$\begin{aligned}(a_2'|a_1)=0, \quad (a_1'|a_1')=1,\\ (a_1'|a_2)=0, \quad (a_2'|a_2')=1,\end{aligned} \tag{37}$$

where both $|a_1')$ and $|a_2')$ are supposed to belong to the two-dimensional space spanned by $|a_1)$ and $|a_2)$, i.e.,

$$\mathbf{I}=|a_1)(a_1|+|a_2')(a_2'|=|a_2)(a_2|+|a_1')(a_1'|, \tag{38}$$

where $\mathbf{I}$ is the unit matrix in this two-dimensional space. Since $\chi_C=(a_2|a_1)$, we find the following solution:

$$|a_1')=\frac{|a_1)-\chi_C|a_2)}{(1-|\chi_C|^2)^{1/2}}, \quad |a_2')=\frac{|a_2)-\chi_C^*|a_1)}{(1-|\chi_C|^2)^{1/2}}. \tag{39}$$

Using these properties, we find

$$\begin{aligned}(\sqrt{\nu})|b_1)&=\left(1+\frac{i\Gamma_2|\chi_C|^2}{\epsilon_1-\epsilon_2^*}\right)|a_1)\\ &\quad+\frac{i\Gamma_2\chi_C(1-|\chi_C|^2)^{1/2}}{\epsilon_1-\epsilon_2^*}|a_2'),\\ (\sqrt{\nu})|b_1)&=\left(1+\frac{i\Gamma_1|\chi_C|^2}{\epsilon_2-\epsilon_1^*}\right)|a_2)\\ &\quad+\frac{i\Gamma_1\chi_C^*(1-|\chi_C|^2)^{1/2}}{\epsilon_2-\epsilon_1^*}|a_1').\end{aligned} \tag{40}$$

Note that for fixed χ_C each resonance becomes purer as we let the distance $\epsilon_2^*-\epsilon_1$ grow. From Eq. (40), one can easily check the norm of the vectors $|b_1)$ and $|b_2)$. This leads again to

$$\nu=\left(1-\frac{\Gamma_1\Gamma_2|\chi_C|^2}{|\epsilon_1-\epsilon_2^*|^2}\right), \tag{41}$$

as expected.

Still another way to represent the S matrix of the two resonances would be in the form

$$S=\left(1-\frac{i\Gamma_1P_1}{E-\epsilon_1}\right)\left(1-\frac{i\Gamma_2P_2}{E-\epsilon_2}\right), \tag{42}$$

since each bracket is unitary and, therefore, the whole S matrix is unitary. P_1 and P_2 are Hermitian projection operators and do not commute in general. We could, of course, just as well reverse the order of the two brackets, which would call for a redefinition of P_1 and P_2. This points out the fact that one cannot attach a clear physical significance to P_1 and P_2 for the S matrix of (42) unless $P_1P_2=P_2P_1$. The form (42) in the single-channel case was discussed by Coleman.[9]

4. WEISSKOPF-WIGNER APPROXIMATION FOR TWO OVERLAPPING RESONANCES

We generalize Eq. (5) to the case of two resonance levels. We replace a_R by a_1 and a_2 designating the probability amplitudes of these two levels. We allow H' to have matrix elements between 1 and 2 as well as between them and the continuum channels. Let us assume that the original unperturbed masses of 1 and 2 were equal; thus $\omega_{1C}=\omega_{2C}=\omega_{RC}$. There is no loss of generality here since we can always put the mass

[8] The concept of reciprocal vectors is widely used in solid state physics. See, e.g., C. Kittel, *Introduction to Solid State Physics* (Wiley, New York, 1968), p. 49.

[9] S. Coleman, in 1968 International School of Physics "Ettore Majorana" (unpublished).

differences into H'. We assume, as before, that E_R lies within the range of variation of E_C. The Schrödinger equation now gives

$$i\dot{a}_{iC}=(H_{C1}{}'{}^{i}a_1+H_{C2}{}'{}^{i}a_2)e^{i\omega_{CR}t},$$

$$i\dot{a}_1=\sum_i\int\rho_i(E_C)dE_C H_{1C}{}'{}^{i}a_{iC}e^{i\omega_{RC}t}+H_{11}{}'a_1+H_{12}{}'a_2, \quad (43)$$

$$i\dot{a}_2=\sum_i\int\rho_i(E_C)dE_C H_{2C}{}'{}^{i}a_{iC}e^{i\omega_{RC}t}+H_{21}{}'a_1+H_{22}{}'a_2.$$

It seems advantageous to group a_1 and a_2 into a two-dimensional column vector designated by a_R:

$$a_R=\begin{pmatrix}a_1\\a_2\end{pmatrix}.$$

Then Eqs. (43) reduce to

$$i\dot{a}_{iC}=(H_{C1}{}'{}^{i}H_{C2}{}'{}^{i})a_R e^{i\omega_{CR}t},$$

$$i\dot{a}_R=\begin{pmatrix}H_{11}{}' & H_{12}{}'\\ H_{21}{}' & H_{22}{}'\end{pmatrix}a_R+\sum_i\int\rho_i(E_C)dE_C e^{i\omega_{RC}t}\begin{pmatrix}H_{1C}{}'{}^{i}\\ H_{2C}{}'{}^{i}\end{pmatrix}a_{iC}. \quad (44)$$

The solution to Eq. (44) is achieved in a similar way to that of Eq. (5). a_{iC} is obtained by formal integration of the first equation and then plugged into the second equation. If one uses the boundary conditions $a_{iC}=0$ at $t=0$, then this substitution leads to the following equation[10] for a_R at positive t:

$$i\dot{a}_R=\begin{pmatrix}H_{11}{}' & H_{12}{}'\\ H_{21}{}' & H_{22}{}'\end{pmatrix}a_R-\tfrac{1}{2}i\sum_i\rho_i 2\pi\begin{pmatrix}H_{1C}{}'{}^{i}H_{C1}{}'{}^{i} & H_{1C}{}'{}^{i}H_{C2}{}'{}^{i}\\ H_{2C}{}'{}^{i}H_{C1}{}'{}^{i} & H_{2C}{}'{}^{i}H_{C2}{}'{}^{i}\end{pmatrix}a_R. \quad (45)$$

In order to write the solution in a compact form, one defines the two Hermitian matrices

$$\mathbf{M}=\begin{pmatrix}M_0+H_{11}{}' & H_{12}{}'\\ H_{21}{}' & M_0+H_{22}{}'\end{pmatrix},$$
$$\mathbf{\Gamma}=\sum_i 2\pi\rho_i\begin{pmatrix}H_{1C}{}'{}^{i}H_{C1}{}'{}^{i} & H_{1C}{}'{}^{i}H_{C2}{}'{}^{i}\\ H_{2C}{}'{}^{i}H_{C1}{}'{}^{i} & H_{2C}{}'{}^{i}H_{C2}{}'{}^{i}\end{pmatrix}, \quad (46)$$

where M_0 is the degenerate eigenvalue of H_0 in the resonance subspace. Then it follows that

$$i\dot{a}_R=-M_0 a_R+(\mathbf{M}-\tfrac{1}{2}i\mathbf{\Gamma})a_R, \quad t\geq 0. \quad (47)$$

[10] T. D. Lee, R. Oehme, and C. N. Yang, Phys. Rev. 106, 340 (1957).

The matrix $\mathbf{M}-\frac{1}{2}i\mathbf{\Gamma}$ is usually referred to as the effective Hamiltonian. Its eigenvalues are $\epsilon_1=M_1-\frac{1}{2}i\Gamma_1$ and $\epsilon_2=M_2-\frac{1}{2}i\Gamma_2$, the actual locations of the two resonances. That this is the case can be seen by solving Eq. (44) with the boundary conditions

$$a_R\to 0,\qquad a_{iC}\to(1/\rho_i)\delta(E_C-E)\delta_{ik}, \quad t\to-\infty \quad (48)$$

appropriate for the calculation of the S matrix. The resultant equation for a_R is now an inhomogeneous one and can be solved by standard methods to give

$$a_R(t)=e^{i(M_0-E)t}(E-\mathbf{M}+\tfrac{1}{2}i\mathbf{\Gamma})^{-1}\begin{pmatrix}H_{1C}{}'{}^{k}\\ H_{2C}{}'{}^{k}\end{pmatrix}. \quad (49)$$

The distribution at $t=+\infty$ is then

$$a_{ic}\to\left[\delta_{ij}-i2\pi\rho_j(H_{C1}{}'{}^{i}H_{C2}{}'{}^{i})(E-\mathbf{M}+\tfrac{1}{2}i\mathbf{\Gamma})^{-1}\begin{pmatrix}H_{1C}{}'{}^{j}\\ H_{2C}{}'{}^{j}\end{pmatrix}\right]\times\frac{1}{\rho_k}\delta_{jk}\delta(E_C-E), \quad t\to+\infty. \quad (50)$$

From this we learn that ϵ_1 and ϵ_2 are indeed the two desired poles.

In order to bring the S matrix into a clear mathematical form, we have to study the properties of the effective Hamiltonian $\mathbf{M}-\frac{1}{2}i\mathbf{\Gamma}$. As already mentioned, we designate its eigenvalues by ϵ_1 and ϵ_2 (assuming $\epsilon_1\neq\epsilon_2$), and thus

$$(\mathbf{M}-\tfrac{1}{2}i\mathbf{\Gamma})|\mathrm{I}\rangle=\epsilon_1|\mathrm{I}\rangle,\qquad(\mathbf{M}-\tfrac{1}{2}i\mathbf{\Gamma})|\mathrm{II}\rangle=\epsilon_2|\mathrm{II}\rangle, \quad \epsilon_1\neq\epsilon_2 \quad (51)$$

where $|\mathrm{I}\rangle$ and $|\mathrm{II}\rangle$ are the two right eigenvectors. The same matrix also has two left eigenvectors with the same eigenvalues:

$$\langle\mathrm{I}'|(\mathbf{M}-\tfrac{1}{2}i\mathbf{\Gamma})=\langle\mathrm{I}'|\epsilon_1,\quad\langle\mathrm{II}'|(\mathbf{M}-\tfrac{1}{2}i\mathbf{\Gamma})=\langle\mathrm{II}'|\epsilon_2. \quad (52)$$

Only when the two matrices $\mathbf{M}$ and $\mathbf{\Gamma}$ commute can they be diagonalized simultaneously by a unitary matrix. Since, in general, that is not the case, the diagonalization cannot be achieved by a unitary transformation, and the right eigenvectors are different from the left eigenvectors. The set $|\mathrm{I}'\rangle|\mathrm{II}'\rangle$ is the reciprocal set to $|\mathrm{I}\rangle|\mathrm{II}\rangle$ in the same sense as in Sec. 3, i.e.,

$$\langle\mathrm{I}'|\mathrm{II}\rangle=0,\quad\langle\mathrm{II}'|\mathrm{I}\rangle=0. \quad (53)$$

If we normalize all the eigenvectors to 1, and denote

$$\chi=\langle\mathrm{II}|\mathrm{I}\rangle, \quad (54)$$

then

$$|\mathrm{I}'\rangle=\frac{|\mathrm{I}\rangle-\chi|\mathrm{II}\rangle}{(1-|\chi|^2)^{1/2}},\quad|\mathrm{II}'\rangle=\frac{|\mathrm{II}\rangle-\chi^*|\mathrm{I}\rangle}{(1-|\chi|^2)^{1/2}}, \quad (55)$$
$$\langle\mathrm{I}'|\mathrm{I}\rangle=\langle\mathrm{II}'|\mathrm{II}\rangle=(1-|\chi|^2)^{1/2},\quad\langle\mathrm{II}'|\mathrm{I}'\rangle=-\chi,$$

and

$$\mathbf{I}=|\mathrm{I}\rangle\langle\mathrm{I}|+|\mathrm{II}'\rangle\langle\mathrm{II}'|=|\mathrm{II}\rangle\langle\mathrm{II}|+|\mathrm{I}'\rangle\langle\mathrm{I}'|$$
$$=\frac{1}{(1-|\chi|^2)^{1/2}}(|\mathrm{I}\rangle\langle\mathrm{I}'|+|\mathrm{II}\rangle\langle\mathrm{II}'|)$$
$$=\frac{1}{(1-|\chi|^2)^{1/2}}(|\mathrm{I}'\rangle\langle\mathrm{I}|+|\mathrm{II}'\rangle\langle\mathrm{II}|). \tag{56}$$

We can now represent the effective Hamiltonian in the form[11]

$$\mathbf{M}-\tfrac{1}{2}i\mathbf{\Gamma}=\epsilon_1\frac{|\mathrm{I}\rangle\langle\mathrm{I}'|}{(1-|\chi|^2)^{1/2}}+\epsilon_2\frac{|\mathrm{II}\rangle\langle\mathrm{II}'|}{(1-|\chi|^2)^{1/2}}. \tag{57}$$

With these definitions at hand, it can be readily established that

$$S_{ij}=\delta_{ij}-\frac{i2\pi(\rho_i\rho_j)^{1/2}}{(1-|\chi|^2)^{1/2}}\left(\frac{\langle i|H'|\mathrm{I}\rangle\langle\mathrm{I}'|H'|j\rangle}{E-\epsilon_1}+\frac{\langle i|H'|\mathrm{II}\rangle\langle\mathrm{II}'|H'|j\rangle}{E-\epsilon_2}\right). \tag{58}$$

In order to establish the connection between Eqs. (58) and (29), we have first to determine Γ_1 and Γ_2 in terms of the various matrices defined above. This is simply done by looking at the Hermitian conjugate of Eq. (57):

$$\mathbf{M}+\tfrac{1}{2}i\mathbf{\Gamma}=\epsilon_1^*\frac{|\mathrm{I}'\rangle\langle\mathrm{I}|}{(1-|\chi|^2)^{1/2}}+\epsilon_2^*\frac{|\mathrm{II}'\rangle\langle\mathrm{II}|}{(1-|\chi|^2)^{1/2}}. \tag{59}$$

Linear combinations of (57) and (59) lead us to

$$\begin{aligned}\langle\mathrm{I}|\mathbf{\Gamma}|\mathrm{I}\rangle&=\langle\mathrm{I}'|\mathbf{\Gamma}|\mathrm{I}'\rangle=\Gamma_1,\\ \langle\mathrm{II}|\mathbf{\Gamma}|\mathrm{II}\rangle&=\langle\mathrm{II}'|\mathbf{\Gamma}|\mathrm{II}'\rangle=\Gamma_2,\\ \langle\mathrm{I}|\mathbf{M}|\mathrm{I}\rangle&=\langle\mathrm{I}'|\mathbf{M}|\mathrm{I}'\rangle=M_1,\\ \langle\mathrm{II}|\mathbf{M}|\mathrm{II}\rangle&=\langle\mathrm{II}'|\mathbf{M}|\mathrm{II}'\rangle=M_2.\end{aligned} \tag{60}$$

If we now define

$$\begin{aligned}(\gamma_{\mathrm{I}}{}^i)^{1/2}&=(2\pi\rho_i)^{1/2}\langle i|H'|\mathrm{I}\rangle,\\ (\gamma_{\mathrm{II}}{}^i)^{1/2}&=(2\pi\rho_i)^{1/2}\langle i|H'|\mathrm{II}\rangle,\\ (\gamma_{\mathrm{I}'}{}^i)^{1/2}&=(2\pi\rho_i)^{1/2}\langle i|H'|\mathrm{I}'\rangle,\\ (\gamma_{\mathrm{II}'}{}^i)^{1/2}&=(2\pi\rho_i)^{1/2}\langle i|H'|\mathrm{II}'\rangle,\end{aligned} \tag{61}$$

then, as a consequence of (60), the following identities hold:

$$\begin{aligned}\sum_i|\gamma_{\mathrm{I}}{}^i|&=\sum_i|\gamma_{\mathrm{I}'}{}^i|=\Gamma_1,\\ \sum_i|\gamma_{\mathrm{II}}{}^i|&=\sum_i|\gamma_{\mathrm{II}'}{}^i|=\Gamma_2.\end{aligned} \tag{62}$$

[11] A somewhat similar analysis from a different approach can be found in the paper of R. Jacob and R. G. Sachs, Phys. Rev. **121**, 350 (1961).

Comparing now the S matrices derived in the two fashions, we conclude that

$$\begin{aligned}\frac{1}{\sqrt{\nu}}(i|a_1)(b_1|j)&=\frac{1}{(1-|\chi|^2)^{1/2}}\frac{1}{\Gamma_1}(\gamma_{\mathrm{I}}{}^i\gamma_{\mathrm{I}'}{}^{j*})^{1/2},\\ \frac{1}{\sqrt{\nu}}(i|a_2)(b_2|j)&=\frac{1}{(1-|\chi|^2)^{1/2}}\frac{1}{\Gamma_2}(\gamma_{\mathrm{II}}{}^i\gamma_{\mathrm{II}'}{}^{j*})^{1/2}.\end{aligned} \tag{63}$$

Since $|a_1)$, $|a_2)$, $|b_1)$, and $|b_2)$ were normalized to 1, we can set

$$\begin{aligned}(i|a_1)&=(\gamma_{\mathrm{I}}{}^i/\Gamma_1)^{1/2},\quad (i|a_2)=(\gamma_{\mathrm{II}}{}^i/\Gamma_2)^{1/2},\\ (i|b_1)&=(\gamma_{\mathrm{I}'}{}^i/\Gamma_1)^{1/2},\quad (i|b_2)=(\gamma_{\mathrm{II}'}{}^i/\Gamma_2)^{1/2}.\end{aligned} \tag{64}$$

Moreover, comparing this with Eq. (41), we are led to

$$|\chi|^2=\frac{\Gamma_1\Gamma_2}{|\epsilon_1-\epsilon_2^*|^2}|\chi_C|^2. \tag{65}$$

Note that in Eq. (64) we made a certain choice of the phases of the various vectors that is not a necessary outcome of Eq. (63). However, the phases of the γ^i are still arbitrary. We will come back to this question in Sec. 6.

5. THE MEANING OF IT ALL

After working out the elaborate algebra, we are now at a stage where we can relax and enjoy the beauty of the formalism. Let us start with Eq. (65). It has a very simple and intuitive meaning. There are two different overlaps that one encounters in this game. One is the overlap in the channel space $\chi_C=(a_2|a_2)$; the other is the overlap between the two resonance states $\chi=\langle 2|1\rangle$. Unitarity assures us that the overlap χ can be determined by the distribution of the decay products. There it can be easily calculated by summing the overlaps of the wave functions in each channel. In the WW approximation the wave function is a simple Breit-Wigner (BW) distribution. Hence, χ should be given by the overlap of two normalized BW distributions multiplied by χ_C:

$$\chi=\chi_C\frac{(\Gamma_1\Gamma_2)^{1/2}}{2\pi}\int\frac{dE}{(E-\epsilon_2^*)(E-\epsilon_1)}=\frac{i(\Gamma_1\Gamma_2)^{1/2}}{\epsilon_2^*-\epsilon_1}\chi_C. \tag{66}$$

This leads then automatically to Eq. (65). This distinction between the two overlap functions is missing in Ref. 1 and answers essentially the question that is raised at the beginning of this paper: χ_C is indeed independent of the distance between the resonances; however, χ does depend on it. In fact,

$$|\chi|\leq\frac{(\Gamma_1\Gamma_2)^{1/2}}{|\epsilon_2^*-\epsilon_1|}, \tag{67}$$

the limit corresponding to that of identical channels, i.e., $P_1=P_2$ in Eq. (42).

Equation (66) can be regarded as a concise form of what is usually called the unitarity sum rule (USR).[2] The reason for this is that the overlap $\chi_C=(a_2|a_1)$ can be simply calculated in terms of the quantities γ^i defined by (61), namely,

$$\chi_C=\sum_i (a_2|i)(i|a_1)=\sum_i \frac{(\gamma_{\rm II}{}^{i*}\gamma_{\rm I}{}^i)^{1/2}}{(\Gamma_1\Gamma_2)^{1/2}}$$

$$=\sum_i \frac{2\pi\rho_i}{(\Gamma_1\Gamma_2)^{1/2}}\langle i|H'|{\rm I}\rangle\langle i|H'|{\rm II}\rangle^*. \tag{68}$$

Hence Eq. (66) can be rewritten in the form

$$\chi=\frac{2\pi i}{\epsilon_2{}^*-\epsilon_1}\sum_i \rho_i\langle i|H'|{\rm I}\rangle\langle i|H'|{\rm II}\rangle^*, \tag{69}$$

which is a more familiar form of the USR.

An alternative way of deriving the USR from the formalism developed[5] in Sec. 4 is by noting that, in addition to Eq. (60), one can also write

$$-i\langle{\rm II}|\boldsymbol{\Gamma}|{\rm I}\rangle=\chi(\epsilon_1-\epsilon_2{}^*). \tag{70}$$

Now, by definition,

$$\langle{\rm II}|\boldsymbol{\Gamma}|{\rm I}\rangle=\sum_i 2\pi\rho_i\langle{\rm II}|H'|i\rangle\langle i|H'|{\rm I}\rangle, \tag{71}$$

which leads one back to the USR (69).

Let us now return to the question of what all the various states are. If we start from some initial channel at $t=-\infty$, say, $|k)$, then we find the distribution in channel space at $t=+\infty$ from Eq. (58) to be determined by the states $|{\rm I}\rangle$ and $|{\rm II}\rangle$. Moreover, if we continue analytically to the poles ϵ_1 and ϵ_2, we find it to be $|a_1)$ and $|a_2)$, respectively. On the other hand, we might view this distribution as resulting from the decay of the physical states $|{\rm I}\rangle$ and $|{\rm II}\rangle$. Thus we may use $|{\rm I}\rangle$ and $|{\rm II}\rangle$ to designate the physical decaying states. In the same spirit, one might think about $|{\rm I}'\rangle$ and $|{\rm II}'\rangle$ as the physical growing states since, if we reverse the time direction, they exchange their roles with $|{\rm I}\rangle$ and $|{\rm II}\rangle$, respectively.

To understand things better, we look at the time-dependent solution of a_R, Eq. (49). It can now be rewritten as

$$a_R(t)=\frac{e^{i(M-E)t}}{(1-|\chi|^2)^{1/2}}\left(\frac{|{\rm I}\rangle\langle{\rm I}'|}{E-\epsilon_1}+\frac{|{\rm II}\rangle\langle{\rm II}'|}{E-\epsilon_2}\right)H'|k\rangle. \tag{72}$$

If we start at $t=-\infty$ with a normalized distribution corresponding to $|b_1)$, namely,

$$(\sqrt{\rho_i})a_{iC}(-\infty)=\left(\frac{\gamma_{\rm I}{}^i}{2\pi}\right)^{1/2}\frac{1}{E_C-\epsilon_1{}^*}$$

$$=(i|b_1)\left(\frac{\Gamma_1}{2\pi}\right)^{1/2}\frac{1}{E_C-\epsilon_1{}^*}, \tag{73}$$

then

$$a_R(t)=\sum_i\int\frac{e^{i(M-E)t}}{2\pi(1-|\chi|^2)^{1/2}}\frac{dE}{E-\epsilon_1{}^*}$$

$$\times\left(\frac{|{\rm I}\rangle|\gamma_{\rm I}{}^i|}{E-\epsilon_1}+\frac{|{\rm II}\rangle(\gamma_{\rm I}{}^i\gamma_{\rm II}{}^{i*})^{1/2}}{E-\epsilon_2}\right). \tag{74}$$

Now one can easily check that

$$\sum_i(\gamma_{\rm I}{}^i\gamma_{\rm II}{}^{i*})^{1/2}=(b_2|b_1)(\Gamma_1\Gamma_2)^{1/2}=(\Gamma_1\Gamma_2)^{1/2}\chi_C e^{i\phi}, \tag{75}$$

where ϕ is a real phase defined by $(b_2|b_1)=e^{i\phi}(a_2|a_1)$. We find then that

$$\begin{aligned}|a_{\rm I}(t)|^2&=\frac{1}{1-|\chi|^2}e^{\Gamma_1 t}, \quad t<0\\ &=\frac{1}{1-|\chi|^2}e^{-\Gamma_1 t}, \quad t>0\\ |a_{\rm II}(t)|^2&=\frac{|\chi|^2}{1-|\chi|^2}e^{\Gamma_1 t}, \quad t<0\\ &=\frac{|\chi|^2}{1-|\chi|^2}e^{-\Gamma_2 t}, \quad t>0.\end{aligned} \tag{76}$$

Since $|{\rm I}\rangle$ and $|{\rm II}\rangle$ are not orthogonal, the sum of their norms can exceed unity. Nevertheless, $|a_R(t)|^2$ must be smaller than or equal to 1. We find from (74) that

$$a_R(0)=\frac{1}{(1-|\chi|^2)^{1/2}}\left[|{\rm I}\rangle+e^{i\phi}\chi\frac{\epsilon_2{}^*-\epsilon_1}{\epsilon_1{}^*-\epsilon_2}|{\rm II}\rangle\right],$$

$$|a_R(0)|^2=\frac{1}{1-|\chi|^2}\Bigg[1+|\chi|^2 \tag{77}$$

$$+|\chi|^2\left(e^{i\phi}\frac{\epsilon_2{}^*-\epsilon_1}{\epsilon_1{}^*-\epsilon_2}+e^{-i\phi}\frac{\epsilon_2-\epsilon_1{}^*}{\epsilon_1-\epsilon_2{}^*}\right)\Bigg].$$

Hence we conclude that $|a_R(0)|=1$, and

$$e^{i\phi}=\frac{\epsilon_2-\epsilon_1{}^*}{\epsilon_2{}^*-\epsilon_1}=\frac{(b_2|b_1)}{(a_2|a_1)}. \tag{78}$$

This is another relation that comes directly from unitarity and we will, therefore, call it the unitarity relation (UR). It can, of course, be derived directly from Eq. (33).

From the preceding discussion we learn that the wave function (73) is suitable for building up the resonance. At $t=0$, $|a_R|=1$ and $|a_{iC}|=0$; thus it was absorbed completely in the resonances. The relative sizes of the resonances are

$$\langle{\rm I}|{\rm I}\rangle:\langle{\rm II}|{\rm II}\rangle=1:|\chi|^2,$$

as expected.

Before concluding this section, let us remark that a USR relating $(b_2|b_1)$ to $\langle \mathrm{II}'|\mathrm{I}'\rangle$ can be written in an identical form to Eq. (66). The UR equation (78) tells us that the phase of $(b_2|b_1)$ is determined by the masses and widths of the resonances, once the phase of χ_C is known.

One can pass directly from the UR equation (78) to the USR equation (66). This can be achieved by using the relation (55) between the I′, II′ and the I, II states and the definitions (64) of the channel vectors. Then, starting from (64), and using the UR (78), the USR follows.

6. DISCRETE SYMMETRIES

In this section we discuss the effects of T-invariance and CPT-invariance requirements on the formalism developed in the Sec. 5.

If T invariance is to hold, then the S matrix has to be symmetric in the channel space. This means that

$$(i|a_{1,2})(b_{1,2}|j)=(j|a_{1,2})(b_{1,2}|i)\,, \tag{79}$$

which is equivalent to

$$\gamma_{\mathrm{I},\mathrm{II}}{}^{i}\gamma_{\mathrm{I}',\mathrm{II}'}{}^{j*}=\gamma_{\mathrm{I},\mathrm{II}}{}^{j}\gamma_{\mathrm{I}',\mathrm{II}'}{}^{i*}. \tag{80}$$

This can be rewritten as

$$\gamma_{\mathrm{I},\mathrm{II}}{}^{i}=e^{2i\delta_{1,2}}\gamma_{\mathrm{I}',\mathrm{II}'}{}^{i*}. \tag{81}$$

Note that we had also a freedom of phase in the definition of $a_{1,2}$, Eq. (64). We can specify all the phases in the following way: Consider the matrix element $\langle i|H'|\mathrm{I}\rangle$. If H' commutes with the antiunitary time-reversal operator T, then

$$\begin{aligned}\langle i|H'|\mathrm{I}\rangle&=\langle Ti|TH'T^{-1}|T\mathrm{I}\rangle^*=\langle Ti|H'|T\mathrm{I}\rangle^*\\&=\langle i|H'|T\mathrm{I}\rangle^*.\end{aligned} \tag{82}$$

In the last equality we assumed that $T|i\rangle=|i\rangle$. This is also necessary for obtaining (79) and can be achieved easily.[12] This leads us to the conclusion that

$$T|\mathrm{I}\rangle=e^{-i\delta_1}|\mathrm{I}'\rangle,\quad T|\mathrm{II}\rangle=e^{-i\delta_2}|\mathrm{II}'\rangle, \tag{83}$$

which is consistent with (82). We will now choose the phases to be

$$e^{-i\delta_1}=e^{-i\delta_2}=1. \tag{84}$$

Making this choice, we are led to

$$\sum_i(\gamma_{\mathrm{I}'}{}^{i}\gamma_{\mathrm{II}'}{}^{i*})^{1/2}=\sum_i(\gamma_{\mathrm{I}}{}^{i*}\gamma_{\mathrm{II}}{}^{i})^{1/2}. \tag{85}$$

This means that

$$(b_2|b_1)=(a_1|a_2),\quad e^{i\phi}=\chi_C{}^*/\chi_C, \tag{86}$$

which leads to

$$1=\frac{\chi_C}{\chi_C{}^*}e^{i\phi}=\frac{\chi_C}{\chi_C{}^*}\frac{\epsilon_2-\epsilon_1{}^*}{\epsilon_2{}^*-\epsilon_1}=-\frac{\chi}{\chi^*}, \tag{87}$$

$$\mathrm{Re}\chi=0. \tag{88}$$

[12] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

So, with our choice of phases, T invariance leads to a purely imaginary overlap. Note that the choice (84) fixes the phase of χ_C to be $e^{-i\phi/2}$.

The S matrix can then be written in the form

$$S=1-\frac{i\Gamma_1}{E-\epsilon_1}\frac{|a_1)(a_1|^*}{\sqrt{\nu}}-\frac{i\Gamma_2}{E-\epsilon_2}\frac{|a_2)(a_2|^*}{\sqrt{\nu}}, \tag{89}$$

where $(a_1|^*$ means that

$$[(a_1|^*]|j)=(a_1|j)^*. \tag{90}$$

An alternative way of checking the time-reversal property is by noting that the choice

$$\begin{aligned}T|\mathrm{I}\rangle&=|\mathrm{I}'\rangle,\quad T|\mathrm{I}'\rangle=|\mathrm{I}\rangle,\\T|\mathrm{II}\rangle&=|\mathrm{II}'\rangle,\quad T|\mathrm{II}'\rangle=|\mathrm{II}\rangle\end{aligned} \tag{91}$$

guarantees that if one looks at the matrices $\boldsymbol{\Gamma}$ and $\mathbf{M}$ in a T-invariant basis (i.e., $T|1\rangle=|1\rangle$, $T|2\rangle=|2\rangle$), then they are symmetric:

$$M_{12}=M_{12}{}^*=M_{21},\quad \Gamma_{12}=\Gamma_{12}{}^*=\Gamma_{21}. \tag{92}$$

This follows directly from the expansions (57) and (59), since now

$$\langle 1|\mathrm{I}\rangle\langle \mathrm{I}'|2\rangle=\langle 2|\mathrm{I}\rangle\langle \mathrm{I}'|1\rangle,\quad \text{etc.}$$

The symmetry (92) guarantees the T-invariance property of $a_R(t)$. This is the standard derivation of T invariance.[13]

From this analysis it is quite obvious how to proceed with CPT invariance. We have in mind, of course, the K_SK_L system that has mixed in it two different CP eigenvalues. In this case we can simply choose

$$\begin{aligned}CPT|\mathrm{I}\rangle&=|\mathrm{I}'\rangle,\quad CPT|\mathrm{II}\rangle=-|\mathrm{II}'\rangle,\\CPT|\mathrm{I}'\rangle&=|\mathrm{I}\rangle,\quad CPT|\mathrm{II}'\rangle=-|\mathrm{II}\rangle.\end{aligned} \tag{93}$$

This assures us that

$$\mathrm{Im}\chi=0; \tag{94}$$

therefore, if both T and CP are conserved, $\chi=0$ and the two resonances are orthogonal. Equations (93) assure us that if we expand $\mathbf{M}$ and $\boldsymbol{\Gamma}$ in a basis obeying $CPT|1\rangle=|1\rangle$ and $CPT|2\rangle=|2\rangle$, then

$$M_{11}=M_{22},\quad \Gamma_{11}=\Gamma_{22}, \tag{95}$$

which are the usual conditions for CPT invariance.[13]

The continuum states $|i\rangle$ in the K_SK_L problem can be chosen as eigenstates of CP with either $+1$ or -1 as eigenvalues. For the eigenvalue $+1$, we find

$$\sqrt{\gamma_{\mathrm{I}}{}^{i}}=\sqrt{\gamma_{\mathrm{I}'}{}^{i*}},\quad \sqrt{\gamma_{\mathrm{II}}{}^{i}}=-\sqrt{\gamma_{\mathrm{II}'}{}^{i*}},\quad CP|i\rangle=|i\rangle \tag{96}$$

and, for the eigenvalue -1,

$$\sqrt{\gamma_{\mathrm{I}}{}^{j}}=-\sqrt{\gamma_{\mathrm{I}'}{}^{j*}},\quad \sqrt{\gamma_{\mathrm{II}}{}^{j}}=\sqrt{\gamma_{\mathrm{II}'}{}^{j*}},\quad CP|j\rangle=-|j\rangle. \tag{97}$$

[13] See, e.g., T. D. Lee and C. S. Wu, Ann. Rev. Nucl. Sci. **16**, 511 (1966).

Thus we can write

$$|a_{1,2}\rangle = |a_{1,2}^{+}\rangle + |a_{1,2}^{-}\rangle, \tag{98}$$

$$|a_1^{+}\rangle = \sum_i \frac{\sqrt{\gamma_1{}^i}}{\sqrt{\Gamma_1}}|i\rangle, \quad |a_1^{-}\rangle = \sum_j \frac{\sqrt{\gamma_1{}^j}}{\sqrt{\Gamma_1}}|j\rangle, \quad \text{etc.}, \tag{99}$$

where

$$CPT|i\rangle = |i\rangle, \quad CPT|j\rangle = -|j\rangle,$$

which leads us to

$$S = 1 - \frac{i\Gamma_1}{E-\epsilon_1}\frac{|a_1^{+})(a_1^{+}|^{*} - |a_1^{-})(a_1^{-}|^{*}}{\sqrt{\nu}} - \frac{i\Gamma_2}{E-\epsilon_2}\frac{-|a_2^{+})(a_2^{+}|^{*} + |a_2^{-})(a_2^{-}|^{*}}{\sqrt{\nu}}, \tag{100}$$

exhibiting the CPT-invariance properties with the specific choice of phases (93).

The forms of the S matrix in the two cases of T and CPT invariance, Eqs. (89) and (100), are essentially included in the discussion of Ref. 4. However, the connection between them and the usual consequences of these invariance principles in the decay formalism is not worked out there. We stressed this point in the present section in order to show that the two formalisms are indeed equivalent.

7. DISCUSSION AND SUMMARY

The work carried out here is essentially an exercise in quantum mechanics. This is so because no basic new idea was introduced. We tried to exploit the basic notions of quantum mechanics in order to establish a mathematical framework that will enable us to gain physical insight into related scattering and decay problems.

As expected, the WW approximation gave us the same S-matrix structure as the general approach based on unitarity. Nevertheless, we have to rely on the WW method in order to understand how the resonance grows and decays and to relate this process to the various parameters that appear in the S-matrix formalism. This gives us the connection between the channel-space vectors that appear in the S matrix and the matrix elements of the interaction Hamiltonian H'.

The formalism developed here can be readily applied to the $K_S K_L$ system. The results of the discrete symmetries in Sec. 6 show clearly how this can be achieved. In any particular basis we can choose $|\mathrm{I}\rangle = |K_S\rangle$, $|\mathrm{II}\rangle = |K_L\rangle$, and obtain all usual results. Since we cannot observe the weak-production mechanism, we have no direct information about $|\mathrm{I}'\rangle$ and $|\mathrm{II}'\rangle$. Nevertheless, we do know all their scalar products via Eq. (55). Thus, were we able to measure this process in real scattering, we would have direct physical data and could test, say, the UR (78). Since we have to rely on Eq. (55), we gain practically nothing by dealing with the $|\mathrm{I}'\rangle$ and $|\mathrm{II}'\rangle$ states. Therefore, the usual treatment of the decay states is complete.

The same formalism can also apply to a strong production and decay process. This is so since we did not assume anything about the strength of H'. Then, in principle, the whole structure of the S matrix is observable and all four channel vectors can be constructed. This formalism may be applied to the A_2 doublet if both parts do have the same quantum numbers. In this case both H_0 and H' belong to the strong interactions.

Throughout the paper we used the assumption that the number of open channels N is constant within the relevant energy range. Only if the various $\rho_i |H_{CR'}{}^i|^2$ factors are constant does one get our solutions. Should a new channel open up and couple strongly to the resonance, it will modify our results.

We introduced the concept of χ_C that plays a key role in our formalism. Note that in the case of the $K_S K_L$ system there is a difference of an order of magnitude between χ and χ_C. Inserting the known values of the masses and widths, we find that $\chi \simeq 4(1+i)\times 10^{-2}\chi_C$. Both χ and χ_C are natural dimensionless parameters that describe the system of overlapping resonances. The distinction, as well as the relation between them, helps us to understand the connection between the scattering and the decay formalism.

Throughout our work we assumed a constant or no background, and single or double poles. We did not consider threshold effects or other possible variations with energy which may appear in different situations. In addition, one has to emphasize that the work presented here is formulated in a way suitable for applications to particle physics. Other formulations may be more suitable for applications in other fields of physics. See, e.g., the work of Mower.[14]

Note added in manuscript. After this paper was submitted for publication, a related work by G. C. Wick appeared as a CERN report. This work was published in the meantime in Phys. Letters **30B**, 126 (1969). The discussion and results of Wick's paper are in complete agreement with ours.

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[14] L. Mower, Phys. Rev. **142**, 799 (1966).

APPENDIX: DEGENERATE RESONANCES

We treat here the case that $\epsilon_1=\epsilon_2=\epsilon$. Using the formalism of Sec. 3 and going to this limit, we find two different possibilities. The first comes from the limit of Eq. (29) that leads to

$$S=1-i\Gamma Q/(E-\epsilon), \tag{A1}$$

where

$$Q=Q^\dagger=Q^2, \tag{A2}$$

as in Eq. (3). The difference between (A1) and (1) is that now Q will be of rank 2. This means that det$S$ will have a double pole. Each element of S_{ij} will have at most a simple pole at $E=\epsilon$, but the residues no longer obey factorization.

Going to the limit from Eq. (42), we find the form

$$S=1-\frac{i\Gamma(P_1+P_2)}{E-\epsilon}-\frac{\Gamma^2P_1P_2}{(E-\epsilon)^2}, \tag{A3}$$

which is obviously different from (A2). The common feature is that since P_1 and P_2 are projection operators of rank 1, it turns out that det$S$ still has only a double pole at $E=\epsilon$. However, here each individual S_{ij} may have a double pole at this energy.

An alternative way of treating these two possibilities arises from a discussion of the effective Hamiltonian $\mathbf{M}-\frac{1}{2}i\mathbf{\Gamma}$. As is well known, if this matrix has degenerate eigenvalues, then it is either proportional to the unit matrix $\epsilon\mathbf{I}$, or can be brought to the Jordan form

$$\begin{pmatrix}\epsilon & \eta\\ 0 & \epsilon\end{pmatrix} \tag{A4}$$

by a similarity transformation. The first possibility ($\eta=0$) leads, of course, directly to Eq. (A1). Let us now see how the more general case (A4) leads to a structure of the form (A3).

We may choose to describe the Jordan form in an orthonormal basis, in which case the effective Hamiltonian can be written as

$$\begin{aligned}\mathbf{M}-\tfrac{1}{2}i\mathbf{\Gamma}&=\epsilon(|1\rangle\langle 1|+|2\rangle\langle 2|)+\eta|1\rangle\langle 2|\\&=\epsilon\mathbf{I}+\eta|1\rangle\langle 2|,\end{aligned} \tag{A5}$$

which is equivalent to (A4). The S matrix will be given as before in the form

$$S_{ij}=\delta_{ij}-i2\pi(\rho_i\rho_j)^{1/2}\langle i|H'\frac{1}{E-\mathbf{M}+\frac{1}{2}i\mathbf{\Gamma}}H'|j\rangle. \tag{A6}$$

From (A5), we find

$$\begin{aligned}\frac{1}{E-\mathbf{M}+\frac{1}{2}i\mathbf{\Gamma}}&=\frac{1}{E-\epsilon}(|1\rangle\langle 1|+|2\rangle\langle 2|)\\&\quad+\frac{\eta}{(E-\epsilon)^2}|1\rangle\langle 2|,\end{aligned} \tag{A7}$$

which leads to the S matrix

$$\begin{aligned}S_{ij}=\delta_{ij}&-i2\pi(\rho_i\rho_j)^{1/2}\\&\times\left(\frac{\langle i|H'|1\rangle\langle 1|H'|j\rangle+\langle i|H'|2\rangle\langle 2|H'|j\rangle}{E-\epsilon}\right.\\&\left.+\eta\frac{\langle i|H'|1\rangle\langle 2|H'|j\rangle}{(E-\epsilon)^2}\right).\end{aligned} \tag{A8}$$

The equivalence with (A3) is obvious. If one defines

$$\begin{aligned}(i|a_1)&=\langle i|H'|1\rangle(2\pi\rho_i/\Gamma)^{1/2},\\(i|a_2)&=\langle i|H'|2\rangle(2\pi\rho_i/\Gamma)^{1/2},\end{aligned} \tag{A9}$$

then, in the notation of (A3),

$$P_1=|a_1)(a_1|, \quad P_2=|a_2)(a_2|. \tag{A10}$$

It is trivial to see how the single pole matches in the two formulas. The double pole matches too since

$$(a_1|a_2)=(1/\Gamma)\langle 1|\mathbf{\Gamma}|2\rangle=i\eta/\Gamma, \tag{A11}$$

as can be easily verified from Eq. (A5).

The formula (A3) has a particularly simple form in the case $P_1=P_2=P$, namely,

$$S=1-P2i\Gamma(E-M)/(E-M+\tfrac{1}{2}i\Gamma)^2, \tag{A12}$$

which is referred to in the literature as a dipole.[5,15] In terms of the parameters of (A4), this situation is achieved if

$$\eta=-i\Gamma, \tag{A13}$$

as can be seen from Eq. (A11).

[15] The application of this form to the A_2 doublet is discussed by G. Chikovani *et al.*, Phys. Letters **25B**, 44 (1967); K. E. Lassila and P. V. Ruuskanen, Phys. Rev. Letters **19**, 762 (1967).