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Dispersion Interactions in Confined Geometry

by

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Contents

Abstract

The nature of pair interactions between particles is a fundamental issue impinging on broad areas of physics and chemistry. The dispersion interaction is arguably one of the most universal interaction in nature, acting between any two polarizable objects. The dispersion interaction of a pair of isotropic particles positioned on the mid-plane between two dielectric plates is studied by using a semi-classical approach. Assuming an inter-particle distance R much larger than the average distance between the molecules forming the surrounding media, it is possible to recast the effect of the medium and the plates on the particles using dielectric susceptibilities corresponding to the continuum Lifshitz theory. This study reveals that confinement leads to the appearance of an intermediate region in which qualitatively different behavior is obtained, not following the classical London ($1/R^6$) and Casimir-Polder ($1/R^7$) results. In the non-retarded limit (short distances), when the dielectric ratio between the outer and inner media is smaller (larger) than one, an enhancement (suppression) in the form of a $1/R^4$ (exponential decay) law is obtained in the intermediate region. In the retarded limit (long distances), when the dielectric mismatch is larger than one, an enhancement in the form of a $1/R^5$ law is obtained in the intermediate region. The asymptotic behavior for R much larger than the distance between the plates contains in correction factors to the classical results depending on the dielectric mismatch.

1 Introduction

The dispersion interaction acts between any two polarizable objects, in particular, any two atoms or molecules, which makes it one of the most universal and important interactions in nature [1]. In many naturally occurring scenarios, particles are confined by nearby boundaries, as in, for example, porous media or biological constrictions. It is therefore of fundamental importance to investigate the effect of confinement on the dispersion interaction. This is the aim of the current work.

1.1 Dispersion interaction between oscillators: a semi-classical treatment

The theory of dispersion forces between atoms and molecules has been investigated in great detail since London [2] gave his treatment of van-der Waals forces. The dispersion interaction between two particles is linked to a process which can be described as the induction of polarization on one particle due to the instantaneous polarization field of the other. The value of the dispersion interaction energy is the expectation value of the corresponding interaction term in the Hamiltonian. An analysis of this problem was done by London, who used perturbation theory to solve the Schrödinger equation for two hydrogen atoms at large separation R (compared with the Bohr radius) including the interactions between the electrons and protons of the two atoms, and found that the interaction energy is given by

$$U(R) = -\frac{3\hbar}{\pi R^6} \int_0^\infty d\xi \alpha_1(i\xi)\alpha_2(i\xi) \tag{1}$$

where $\alpha_1(w)$ and $\alpha_2(w)$ are the polarizabilities of the two particles when w is replaced by $i\xi$. Since the interaction occurs through the electromagnetic field, an alternative viewpoint could be developed, according to which the dispersion interaction of a pair of particles could be considered to be due to the effect of the pair on the energy of the electromagnetic field. Historically, this approach was developed in a series of papers by Casimir and by Casimir and Polder [3]. The important result which was obtained by Casimir and Polder using quantum electrodynamics is the dispersion interaction energy between a pair of particles at a distance R larger than the wavelength of the radiation (the retarded limit). They showed that the interaction falls off as $1/R^7$, according to

$$U(R) = -\frac{23}{4\pi}\hbar c\frac{\alpha_1(0)\alpha_2(0)}{R^7} \tag{2}$$

The different power law can be understood as arising from a loss of inter-correlation due to the finite velocity of light. Note that the interaction in this retarded limit depends on the low-frequency (static) values of the polarizabilities. The quantum electrodynamic approach developed by Casimir and Polder [3] was also formulated by Casimir in semi-classical terms, in which the interaction energy can be defined as the change in the zero-point energy of the electromagnetic field modes (obtained by solving Maxwell's equations). The electromagnetic field modes are perturbed by the particles through coupling of the field to the polarization currents induced on the particles. Lifshitz recast these problems in terms of interactions between continuous media of well-defined (or at least independently measurable) dielectric susceptibilities, mediated by the quantum electromagnetic field. The London and Casimir-Polder results are then recovered as limiting cases of the more general theory [4]. In a condensed medium [5] the van der Waals forces do not reduce to an interaction between separate pairs of atoms. However, since their range of action is large compared with inter-atomic distances, one can use a macroscopic approach to the problem. In this work, we are interested in the effective interaction between two neutral, non-polar particles embedded in a medium with appropriate boundary conditions. For example, two particles suspended in a slab of liquid. Two particles in empty space "see" each other and have a van der Waals attraction only because radiation emitted by one gets reflected back to the other. In a condensed medium, an embedded particle with exactly the same polarizability as the medium would be "invisible" because it would not scatter radiation differently from the medium, and it would have no tendency to move toward another embedded particle. Hence, the effective attraction energy must depend on the difference between the polarizabilities of the particle and the medium. Another factor is that the interaction between the medium and the foreign particle may locally change ϵ, the dielectric susceptibility, and thus alter α from its free-space value. However, the embedded particle can be considered as a small region in the medium where the local value of the dielectric constant has changed, say from $\epsilon(i\xi)$ to $\epsilon + \delta\epsilon$. Therefore, we define the "excess polarizability" as

$$\alpha^*(i\xi) = \frac{1}{4\pi} \int_V \delta\epsilon(\vec{r}, i\xi) d\vec{r} \tag{3}$$

integrated over the region of the particle. Now London's formula from Eq. 1 becomes modified as obtained by Pitaevskii [6] and by McLachlan [7] to

$$U(R) = -\frac{3\hbar}{\pi R^6} \int_0^\infty d\xi \frac{\alpha_1^*(i\xi)\alpha_2^*(i\xi)}{\epsilon^2(i\xi)} \tag{4}$$

and the Casimir-Polder interaction energy from Eq. 2 becomes

$$U(R) = -\frac{23}{4\pi} \hbar c \frac{\alpha_1^*(0)\alpha_2^*(0)}{\epsilon(0)^{5/2} R^7} \tag{5}$$

From now on the "excess polarizability" will be referred to as the polarizability α (omitting the star).

Mahanty and Ninham [1, 8] showed that there were important boundary effects on the dispersion interaction between a pair of oscillators in vacuum confined by perfectly conducting plates. There have also been calculations between a pair of molecules in vacuum confined by two parallel dielectric surfaces in the non-retarded limit [9]. In this work we wish to explore the effect of confinement on the dispersion interaction between two isotropic, neutral and non-polar particles, embedded in a slab of one medium, where the slab is bounded by a another medium. The calculation will consider both the retarded and the non-retarded limit. An elaboration of the semi-classical theme in zero temperature shall be given, taking the particles as oscillators interacting with the electromagnetic field with the appropriate boundary conditions for dielectric plates. Seeking a simplest derivation of the confinement effect, we neglect non-linear and non local response of the media.

The ground state energy of an assembly of N oscillators at zero temperature (for the extension to non-zero temperature see Appendix A) is given by

$$\begin{aligned} U_0 &= \frac{\hbar}{2}\sum_j w_j = \frac{\hbar}{2}\sum \text{zeros of } D_0(w) = \frac{\hbar}{2}\frac{1}{2\pi i}\oint w\frac{d}{dw}\ln D_0(w)\,dw \\ D_0(w) &= |(w_j^2 - w^2)\mathbf{I}| \end{aligned} \tag{6}$$

where $D_0(w)$ is the oscillators' secular determinant such that $D_0(w) = 0$ provides a dispersion relation defining the allowed frequencies w_j and $\mathbf{I}$ is a $3N \times 3N$ unit matrix. The integration contour goes along the imaginary axis and is closed by a semi circle in the right hand complex w plane (positive real axis). The latter contour integral is valid because $\frac{d}{dw}\ln D_0(w)$ has simple poles at the zeros of $D_0(w)$. When we look at one oscillating dipole coupled to a field there will be a change in its secular determinant, and the difference is

$$\Delta U = \frac{\hbar}{2}\frac{1}{2\pi i}\oint w\frac{d}{dw}\ln\frac{D_1(w)}{D_0(w)}\,dw \tag{7}$$

where $D_1(w)$ is the secular determinant in the coupled situation obtained from the field equations. Thus, ΔU is the self energy of the oscillator in the field. When two such oscillators are coupled to the field, the interaction energy is the difference between the energy of the pair and the self energies of the two oscillators

$$U(1,2) = \frac{\hbar}{4\pi i} \oint w \frac{d}{dw} \ln \frac{D_{12}(w)/D_0(w)}{[D_1(w)/D_0(w)][D_2(w)/D_0(w)]} \, dw \tag{8}$$

Here $D_2(w)$ and $D_{12}(w)$ are the secular determinants when the second oscillator is coupled to the field, and when both are coupled to the field, respectively. An integration by parts and a suitable choice of the contour including the imaginary axis makes it possible to write Eq. 8 in the form

$$\begin{aligned} U(1,2) &= -\frac{\hbar}{4\pi i} \oint \ln \frac{D_{12}(w)/D_0(w)}{[D_1(w)/D_0(w)][D_2(w)/D_0(w)]} \, dw \\ &= \frac{\hbar}{4\pi i} \int_{-i\infty}^{i\infty} \ln \frac{D_{12}(w)/D_0(w)}{[D_1(w)/D_0(w)][D_2(w)/D_0(w)]} \, dw \end{aligned} \tag{9}$$

where we use the result $D_j(w) \to 1$ on the semicircular path [1].

Using the fact that the secular determinant is even in w (a consequence of the time reversal symmetry of Maxwell's equations) and defining $w = i\xi$ gives

$$U(1,2) = \frac{\hbar}{2\pi} \int_0^{\infty} \ln \frac{D_{12}(i\xi)/D_0(i\xi)}{[D_1(i\xi)/D_0(i\xi)][D_2(i\xi)/D_0(i\xi)]} \, d\xi \tag{10}$$

Eq. 10 is the basis of the following treatment.

1.2 Interaction between point-particles

The effect of the electromagnetic field on an embedded particle is mainly in the form of induction of polarization on it. We shall also assume that the particles are essentially point-like, so that in the presence of an electric field $\vec{E}(\vec{r}, w)$ the induced polarization density can

[1] This result is used in other branches of physics such as lattice dynamics [10].

be written as

$$\vec{p}(\vec{r}, w) = [\boldsymbol{\alpha}(w) \cdot \vec{E}(\vec{r}, w)]\delta(\vec{r} - \vec{R}) \tag{11}$$

where $\vec{R}$ is the position coordinate of the particle, and the tensor $\boldsymbol{\alpha}(w)$ is the polarizability derived, for example, in standard texts on quantum mechanics. To consider the effect of the coupling between a pair of particles at $\vec{R_1}$ and $\vec{R_2}$ with polarizabilities $\boldsymbol{\alpha}_1(w)$ and $\boldsymbol{\alpha}_2(w)$, and the electromagnetic field, the polarization current induced on the particles is taken as

$$\begin{aligned}\vec{J}(\vec{r}, w) &= iw\vec{p}(\vec{r}, w) \\ &= iw[\boldsymbol{\alpha}_1(w)\vec{E}(\vec{R_1}, w)\delta(\vec{r} - \vec{R_1}) + \boldsymbol{\alpha}_2(w)\vec{E}(\vec{R_2}, w)\delta(\vec{r} - \vec{R_2})]\end{aligned} \tag{12}$$

When the continuum assumption can be employed, $\vec{D}(\vec{r}, w) = \epsilon(w)\vec{E}(\vec{r}, w)$, the frequency-dependent wave equation for the electric field given by Maxwell's equations is

$$\begin{aligned}\nabla \times \nabla \times \vec{E}(\vec{r}, w) - \epsilon(w)\frac{w^2}{c^2}\vec{E}(\vec{r}, w) &= -\frac{4\pi i w}{c^2}\vec{J}(\vec{r}, w) \\ &= \frac{4\pi w^2}{c^2}[\boldsymbol{\alpha}_1(w)\vec{E}(\vec{R_1}, w)\delta(\vec{r} - \vec{R_1}) + \boldsymbol{\alpha}_2(w)\vec{E}(\vec{R_2}, w)\delta(\vec{r} - \vec{R_2})]\end{aligned} \tag{13}$$

where we have substituted Eq. 12 in Eq. 13. Hence, the solution for $\vec{E}(\vec{r}, w)$ is given by

$$\vec{E}(\vec{r}, w) = \frac{4\pi w^2}{c^2}[\boldsymbol{\alpha}_1(w)\vec{E}(\vec{R_1}, w)\mathbf{G}_e(\vec{r} - \vec{R_1}, w) + \boldsymbol{\alpha}_2(w)\vec{E}(\vec{R_2}, w)\mathbf{G}_e(\vec{r} - \vec{R_2}, w)] \tag{14}$$

where $\mathbf{G}_e(\vec{r}, \vec{r'}, w)$ is the dyadic Green function of the equation

$$\nabla \times \nabla \times \mathbf{G}_e(\vec{r}, \vec{r'}, w) - \epsilon(w)\frac{w^2}{c^2}\mathbf{G}_e(\vec{r}, \vec{r'}, w) = \mathbf{I}_{3\times 3}\delta(\vec{r} - \vec{r'}) \tag{15}$$

with the appropriate boundary conditions. Dyadic functions (tensors) are marked here as bold symbols. From Eq. 14 the equations for $\vec{E}(\vec{R_1}, w)$ and $\vec{E}(\vec{R_2}, w)$ are

$$\begin{aligned}\vec{E}(\vec{R_1}, w) &= \frac{4\pi w^2}{c^2}[\boldsymbol{\alpha}_1(w)\vec{E}(\vec{R_1}, w)\mathbf{G}_e(0, w) + \boldsymbol{\alpha}_2(w)\vec{E}(\vec{R_2}, w)\mathbf{G}_e(\vec{R_1} - \vec{R_2}, w)] \\ \vec{E}(\vec{R_2}, w) &= \frac{4\pi w^2}{c^2}[\boldsymbol{\alpha}_1(w)\vec{E}(\vec{R_1}, w)\mathbf{G}_e(\vec{R_2} - \vec{R_1}, w) + \boldsymbol{\alpha}_2(w)\vec{E}(\vec{R_2}, w)\mathbf{G}_e(0, w)]\end{aligned} \tag{16}$$

and in order for non trivial solutions to exist, the secular determinant formed out of the coefficients must vanish. Thus, the secular 6×6 determinant is

$$\frac{D_{12}(w)}{D_0(w)} = \begin{vmatrix} \mathbf{I} - \frac{4\pi w^2}{c^2}\boldsymbol{\alpha}_1(w)\mathbf{G}_e(0,w) & -\frac{4\pi w^2}{c^2}\boldsymbol{\alpha}_2(w)\mathbf{G}_e(\vec{R},w) \\ -\frac{4\pi w^2}{c^2}\boldsymbol{\alpha}_1(w)\mathbf{G}_e(-\vec{R},w) & \mathbf{I} - \frac{4\pi w^2}{c^2}\boldsymbol{\alpha}_2(w)\mathbf{G}_e(0,w) \end{vmatrix} \tag{17}$$

where $\vec{R} = \vec{R}_1 - \vec{R}_2$.

The zeros of Eq. 17 produce the perturbed frequencies of the modes of the electromagnetic field, where the perturbation is due to the two particles. In this case $D_1(w)$ and $D_2(w)$ have the form

$$\begin{aligned} \frac{D_1(w)}{D_0(w)} &= |\mathbf{I} - \frac{4\pi w^2}{c^2}\boldsymbol{\alpha}_1(w)\mathbf{G}_e(0,w)| \\ \frac{D_2(w)}{D_0(w)} &= |\mathbf{I} - \frac{4\pi w^2}{c^2}\boldsymbol{\alpha}_2(w)\mathbf{G}_e(0,w)| \end{aligned} \tag{18}$$

Using the contour integral representation, Eq. 9, the change in the zero-point energy of the field can be written as

$$U(1,2) = -\frac{\hbar}{4\pi i}\oint \ln \frac{\left| \mathbf{I} - \begin{pmatrix} \frac{4\pi w^2}{c^2}\boldsymbol{\alpha}_1(w)\mathbf{G}_e(0,w) & \frac{4\pi w^2}{c^2}\boldsymbol{\alpha}_2(w)\mathbf{G}_e(\vec{R},w) \\ \frac{4\pi w^2}{c^2}\boldsymbol{\alpha}_1(w)\mathbf{G}_e(-\vec{R},w) & \frac{4\pi w^2}{c^2}\boldsymbol{\alpha}_2(w)\mathbf{G}_e(0,w) \end{pmatrix} \right|}{|\mathbf{I} - \frac{4\pi w^2}{c^2}\boldsymbol{\alpha}_1(w)\mathbf{G}_e(0,w)||\mathbf{I} - \frac{4\pi w^2}{c^2}\boldsymbol{\alpha}_2(w)\mathbf{G}_e(0,w)|} dw \tag{19}$$

In principal, the full expression for the dispersion energy between two neutral particles should also include the contribution from the magnetic dipole fluctuations. We employ a common assumption that this contribution is negligible. At this point, it should be stated that $\mathbf{G}_e(0,w)$ is an ill-defined quantity for a point particle, because the components of the tensor $\mathbf{G}_e(0,w)$ diverge for $\vec{R} \to 0$. However actual particles are not geometrical points, and it can be shown that the finite size keeps $\mathbf{G}_e(0,w)$ finite. Since we are interested only in the interaction energy for inter-particle distances much larger than the size of the particle, we can use the formula: $\ln|\mathbf{I}+\mathbf{A}| = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \mathrm{Tr}[\mathbf{A}^n]$ to the order of $\boldsymbol{\alpha}^2$, and by assuming that the oscillators are isotropic so that $\boldsymbol{\alpha}_i(w) = \alpha_i(w)\mathbf{I}, i = 1, 2$, the formula in Eq. 10 for $U(1,2)$ becomes

$$U(\vec{R}) = -8\pi\hbar \int_o^\infty \alpha_1(i\xi)\alpha_2(i\xi)\mathrm{Tr}\left[\frac{\xi^2}{c^2}\mathbf{G}_e(\vec{R}, i\xi) \cdot \frac{\xi^2}{c^2}\mathbf{G}_e(-\vec{R}, i\xi)\right] d\xi \tag{20}$$

The non-retarded limit corresponds to the situation where the distance of separation R is less than the wavelengths of the radiation associated with the transitions of the particles λ_0. This limit is also obtained for $c \to \infty$, where by Eq. 20 becomes

$$U(\vec{R}) = -8\pi\hbar \int_0^\infty \alpha_1(i\xi)\alpha_2(i\xi)\mathrm{Tr}\left[\lim_{c\to\infty}\frac{\xi^2}{c^2}\mathbf{G}_e(\vec{R}, i\xi) \cdot \lim_{c\to\infty}\frac{\xi^2}{c^2}\mathbf{G}_e(-\vec{R}, i\xi)\right] d\xi \tag{21}$$

The following analysis will be based on Eqs. 20 and 21.

The above derivation is based on Refs. [1, 11].

2 The structure of the Green functions

2.1 Dyadic Green functions

This section is based on the derivation presented in Refs. [12, 13]. In order to introduce the dyadic Green functions in electromagnetic theory, Maxwell's equations need to be elevated to a dyadic form first. Let us consider three sets of harmonically oscillating fields with the same frequency and in the same environment which are produced by three distinct current distributions $\vec{J}_j$ with $j = (1, 2, 3)$. Maxwell's equations for these fields can then be written in the dyadic form

$$\begin{aligned} &\nabla \times \mathbf{E} = -\frac{iw}{c}\mathbf{B} && \nabla \cdot \mathbf{D} = 4\pi\vec{\rho} \\ &\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{iw}{c}\mathbf{D} && \nabla \cdot \mathbf{B} = 0 \\ &\nabla \cdot \mathbf{J} = -iw\vec{\rho} \end{aligned} \tag{22}$$

Assuming $\mathbf{D} = \epsilon\mathbf{E}$ where $\epsilon = \epsilon(w)$ is the dielectric constant and $\mathbf{B} = \mathbf{H}$ (where the magnetic permeability is taken to be one) the rotor equations become

$$\begin{aligned} \nabla \times \nabla \times \mathbf{E}(\vec{r}, w) - \frac{\epsilon w^2}{c^2}\mathbf{E}(\vec{r}, w) &= -\frac{4\pi i w}{c^2}\mathbf{J}(\vec{r}, w) \\ \nabla \times \nabla \times \mathbf{H}(\vec{r}, w) - \frac{\epsilon w^2}{c^2}\mathbf{H}(\vec{r}, w) &= \frac{4\pi}{c}\nabla \times \mathbf{J}(\vec{r}, w) \end{aligned} \tag{23}$$

The next step is to normalize the current such that $-\frac{4\pi i w}{c^2}\mathbf{J} = \mathbf{I}\delta(\vec{r} - \vec{r'})$, where $\mathbf{I}$ is the idem factor (the dyadic analogue of unity). Under this condition, new notations for the dyadic functions can be used

$$\begin{aligned} \mathbf{G}_e &= \mathbf{E} \\ \mathbf{G}_m &= -\frac{iw}{c}\mathbf{H} \end{aligned} \tag{24}$$

With this change of notation Eq. 22 can be written in the form

$$\begin{aligned}
&\nabla \times \mathbf{G}_e = \mathbf{G}_m \qquad\qquad \nabla \cdot \mathbf{G}_e = -\frac{1}{k^2}\nabla\delta(\vec{r}-\vec{r'}) \\
&\nabla \times \mathbf{G}_m = \mathbf{I}\delta(\vec{r}-\vec{r'}) + k^2\mathbf{G}_e \qquad \nabla \cdot \mathbf{G}_m = 0 \\
&\text{where} \quad k^2 = \epsilon(w)\frac{w^2}{c^2}
\end{aligned} \tag{25}$$

The rotor equations can now be written in the form

$$\begin{aligned}
&\nabla \times \nabla \times \mathbf{G}_e(\vec{r},\vec{r'}) - k^2\mathbf{G}_e(\vec{r},\vec{r'}) = \mathbf{I}\delta(\vec{r}-\vec{r'}) \\
&\nabla \times \nabla \times \mathbf{G}_m(\vec{r},\vec{r'}) - k^2\mathbf{G}_m(\vec{r},\vec{r'}) = \nabla \times [\mathbf{I}\delta(\vec{r}-\vec{r'})]
\end{aligned} \tag{26}$$

In addition to Maxwell's equations, the boundary conditions need also be cast into dyadic form. The boundary condition for the tangential electric and magnetic fields are

$$\begin{aligned}
&\hat{n} \times (\vec{E}^+ - \vec{E}^-) = 0 \\
&\hat{n} \times (\vec{H}^+ - \vec{H}^-) = \vec{J}_s
\end{aligned} \tag{27}$$

where $\hat{n}$ denotes the unit normal vector pointing from an interface to the positive side of that surface and $\vec{J}_s$ denotes the surface current density. By considering three sets of electric fields due to three orthogonal infinitesimal electric dipoles, the dyadic form of the tangential electric boundary condition becomes

$$\hat{n} \times (\mathbf{G}_e^+ - \mathbf{G}_e^-) = 0 \tag{28}$$

When the surface current density function $\mathbf{J}_s$ corresponds to two tangential infinitesimal electric dipoles, the dyadic surface current density has the form

$$-\frac{4\pi i w}{c^2}\mathbf{J}_s = \mathbf{I}_s\delta(\vec{r}_s - \vec{r'}_s) \tag{29}$$

where $\mathbf{I}_s$ denotes the two-dimensional idem factor defined by $\mathbf{I}_s = \mathbf{I} - \hat{n}\hat{n}$, and $\delta(\vec{r}_s - \vec{r'}_s)$ denotes the two-dimensional delta function such that $\iint \delta(\vec{r}_s - \vec{r'}_s)dS = 1$ where the region of integration includes the point $\vec{r'}_s$ on the surface. The magnetic boundary condition can now be elevated into a dyadic form

$$\hat{n} \times (\mathbf{G}_m^+ - \mathbf{G}_m^-) = \mathbf{I}_s \delta(\vec{r}_s - \vec{r'}_s) \tag{30}$$

These two dyadic boundary conditions will be used in the following derivation.

2.2 Introducing the problem for dielectric media

The structure of our problem is shown in figure 1 where the point source is located in the middle region (region 2).

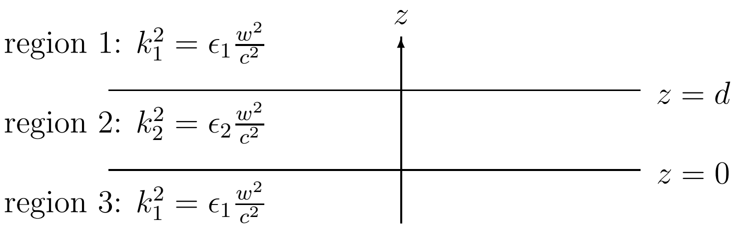

Figure 1: Dielectric medium confined by two plates with different dielectric susceptibility

There are three regions $i = 1, 2, 3$ which satisfy

$$\begin{aligned} &\nabla \times \nabla \times \mathbf{G}_e^{(1)}(\vec{r}, \vec{r'}) - k_1^2 \mathbf{G}_e^{(1)}(\vec{r}, \vec{r'}) = 0 && z \geq d \\ &\nabla \times \nabla \times \mathbf{G}_e^{(2)}(\vec{r}, \vec{r'}) - k_2^2 \mathbf{G}_e^{(2)}(\vec{r}, \vec{r'}) = \mathbf{I}\delta(\vec{r} - \vec{r'}) && 0 \leq z \leq d \\ &\nabla \times \nabla \times \mathbf{G}_e^{(3)}(\vec{r}, \vec{r'}) - k_1^2 \mathbf{G}_e^{(3)}(\vec{r}, \vec{r'}) = 0 && z \leq 0 \end{aligned} \tag{31}$$

where there are nine electric dyadic Green functions for each region. The general solutions to Eq. 31 are found using the method of scattering superposition, where $\mathbf{G}_{es}^i(\vec{r}, \vec{r'})$ is the scattered part (homogeneous solution) and $\mathbf{G}_{e0}^2(\vec{r}, \vec{r'})$ is the free dyadic Green function for a point source located in the middle region.

$$
\begin{aligned}
&\mathbf{G}_e^{(1)}(\vec{r},\vec{r'}) = \mathbf{G}_{es}^{(1)}(\vec{r},\vec{r'}) \\
&\mathbf{G}_e^{(2)}(\vec{r},\vec{r'}) = \mathbf{G}_{e0}^{(2)}(\vec{r},\vec{r'}) + \mathbf{G}_{es}^{(2)}(\vec{r},\vec{r'}) \\
&\mathbf{G}_e^{(3)}(\vec{r},\vec{r'}) = \mathbf{G}_{es}^{(3)}(\vec{r},\vec{r'})
\end{aligned}
\tag{32}
$$

By using Eqs. 25, 28 and 30, the boundary conditions at the interfaces are:

$$
\begin{aligned}
&\hat{z} \times [\mathbf{G}_e^{(1)}(\vec{r},\vec{r'}) - \mathbf{G}_e^{(2)}(\vec{r},\vec{r'})]_{z=d} = 0 \\
&\hat{z} \times [\mathbf{G}_e^{(3)}(\vec{r},\vec{r'}) - \mathbf{G}_e^{(2)}(\vec{r},\vec{r'})]_{z=0} = 0 \\
&\hat{z} \times [\nabla \times \mathbf{G}_e^{(1)}(\vec{r},\vec{r'}) - \nabla \times \mathbf{G}_e^{(2)}(\vec{r},\vec{r'})]_{z=d} = 0 \\
&\hat{z} \times [\nabla \times \mathbf{G}_e^{(3)}(\vec{r},\vec{r'}) - \nabla \times \mathbf{G}_e^{(2)}(\vec{r},\vec{r'})]_{z=0} = 0
\end{aligned}
\tag{33}
$$

The derivation of the dyadic Green functions is very technical and can be found in Appendix B.

It yields for region 2 the following Green functions :

$$\frac{\xi^2}{c^2}\mathbf{G}_e^{(2)}(R) = \frac{1}{4\pi R^3 \epsilon_2}\int_0^\infty dx \begin{pmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \end{pmatrix}$$

$$\begin{aligned}
G_1(R) =& (\sinh\left[\frac{dy_2}{R}\right] y_1 y_2(\epsilon_1+\epsilon_2)((xJ_0(x)-J_1(x))y_2^2 + \frac{R^2\xi^2\epsilon_2}{c^2}J_1(x)) \\
&+ \cosh\left[\frac{dy_2}{R}\right]((xJ_0(x)-J_1(x))y_2^2 + \frac{R^2\xi^2\epsilon_2}{c^2}J_1(x))(y_2^2\epsilon_1 + y_1^2\epsilon_2) \\
&- (y_2^2\epsilon_1 - y_1^2\epsilon_2)(xJ_0(x)y_2^2 - J_1(x)(y_2^2 + \frac{R^2\xi^2\epsilon_2}{c^2}))) \\
&/(2y_2(\cosh\left[\frac{dy_2}{2R}\right] y_1 + \sinh\left[\frac{dy_2}{2R}\right] y_2)(\cosh\left[\frac{dy_2}{2R}\right] y_2\epsilon_1 + \sinh\left[\frac{dy_2}{2R}\right] y_1\epsilon_2)) \\
G_2(R) =& (\sinh\left[\frac{dy_2}{R}\right] y_1 y_2(\epsilon_1+\epsilon_2)(J_1(x)y_2^2 + \frac{R^2\xi^2\epsilon_2}{c^2}(xJ_0(x)-J_1(x)))+ \\
&(J_1(x)y_2^2 + \frac{R^2\xi^2\epsilon_2}{c^2}(-xJ_0(x)+J_1(x)))(-y_2^2\epsilon_1 + y_1^2\epsilon_2)+ \\
&\cosh\left[\frac{dy_2}{R}\right](J_1(x)y_2^2 + \frac{R^2\xi^2\epsilon_2}{c^2}(xJ_0(x)-J_1(x)))(y_2^2\epsilon_1 + y_1^2\epsilon_2)) \\
&/(2y_2(\cosh\left[\frac{dy_2}{2R}\right] y_1 + \sinh\left[\frac{dy_2}{2R}\right] y_2)(\cosh\left[\frac{dy_2}{2R}\right] y_2\epsilon_1 + \sinh\left[\frac{dy_2}{2R}\right] y_1\epsilon_2)) \\
G_3(R) =& -\frac{x^3 J_0(x)(\cosh\left[\frac{dy_2}{2R}\right] y_2\epsilon_1 + \sinh\left[\frac{dy_2}{2R}\right] y_1\epsilon_2)}{y_2(\sinh\left[\frac{dy_2}{2R}\right] y_2\epsilon_1 + \cosh\left[\frac{dy_2}{2R}\right] y_1\epsilon_2)}
\end{aligned} \tag{34}$$

where $J_0(x)$ and $J_1(x)$ are the zero and first order Bessel functions and $R = |\vec{r} - \vec{r'}|$. In Eq. 34 we have assumed, for simplicity, $z = z' = d/2$, i.e., the two oscillators are located at the mid-plane of the cavity at a distance R apart ($R \neq 0$). The parameters are

$$\begin{aligned}
&y_2 = \sqrt{\epsilon_2\left(\frac{R\xi}{c}\right)^2 + x^2} \qquad && y_1 = \sqrt{\epsilon_1\left(\frac{R\xi}{c}\right)^2 + x^2} \\
&\epsilon_2 = \epsilon_2(i\xi) && \epsilon_1 = \epsilon_1(i\xi)
\end{aligned} \tag{35}$$

The dielectric susceptibility $\epsilon(w)$ should be calculated from the observed absorption spectrum at real frequencies [14] by using the Kramers-Kronig relation $\epsilon(i\xi) = 1 + \frac{2}{\pi}\int_0^\infty \frac{w\mathrm{Im}\epsilon(w)}{w^2+\xi^2}dw$.

Therefore, the only macroscopic quantity characterizing the dispersion forces in a medium, within the continuum (Lifshitz) theory, is the imaginary part of its dielectric susceptibility. It can be shown that the dielectric susceptibility $\epsilon(i\xi)$ is a real monotonic decreasing function in ξ. Equation 34 is the starting point from which all of the following results derive.

3 Two oscillators between conducting plates

We begin with the idealized case where the bounding plates are conductors ($\epsilon_1 \to \infty$). This limit serves two purposes: (i) to confirm that our calculation properly converges to the results of Mahanty & Ninham [8]; (ii) to gain physical insight that will be helpful in the following sections. Taking $\epsilon_1 \to \infty$ in Eq. 34 we get

$$\begin{aligned}\frac{\xi^2}{c^2}\mathbf{G}_e^{(2)}(R) = \frac{1}{4\pi R^3 \epsilon_2}\int_0^\infty dx\{\{&\frac{x\left(\left(x^2+\frac{R^2\epsilon_2\xi^2}{c^2}\right)J_0(x) - xJ_1(x)\right)\tanh\left[\frac{d}{2R}\sqrt{x^2+\frac{R^2\epsilon_2\xi^2}{c^2}}\right]}{\sqrt{x^2+\frac{R^2\epsilon_2\xi^2}{c^2}}}, 0, 0\},\\ &\{0, \frac{x\left(\frac{R^2\epsilon_2\xi^2}{c^2}J_0(x) + xJ_1(x)\right)\tanh\left[\frac{d}{2R}\sqrt{x^2+\frac{R^2\epsilon2\xi^2}{c^2}}\right]}{\sqrt{x^2+\frac{R^2\epsilon_2\xi^2}{c^2}}}, 0\},\\ &\{0, 0, -\frac{x^3J_0(x)\coth\left[\frac{d}{2R}\sqrt{x^2+\frac{R^2\epsilon_2\xi^2}{c^2}}\right]}{\sqrt{x^2+\frac{R^2\epsilon_2\xi^2}{c^2}}}\}\}\end{aligned} \tag{36}$$

The following calculations coincide with the results of M&N [8] for $z = z' = \frac{d}{2}$, $\epsilon_2 = 1$ and assuming that the two oscillators are identical and the polarizability is determined by a single transition frequency, w_0, such that $\alpha_1(w) = \alpha_2(w) = \frac{e^2}{m(w_0^2 - w^2)}$.

3.1 Interaction in the non-retarded limit

The non-retarded limit, obtained for $c \to \infty$, corresponds to the situation when both d and R are much smaller than the characteristic wavelength $\lambda_0 = 2\pi c/w_0\sqrt{\epsilon_2(w_0)}$ $(R, d \ll \lambda_0)$.

The expression in the non-retarded limit, taking $\frac{R^2\xi^2\epsilon_2}{c^2} \to 0$ in Eq. 36, becomes

$$\lim_{c\to\infty}\frac{\xi^2}{c^2}\mathbf{G}_e^{(2)}(R) = \frac{1}{4\pi R^3\epsilon_2}\int_0^\infty dx\{\{x(xJ_0(x) - J_1(x))\tanh\left[\frac{dx}{2R}\right], 0, 0\},$$
$$\{0, xJ_1(x)\tanh\left[\frac{dx}{2R}\right], 0\}, \qquad (37)$$
$$\{0, 0, -x^2\coth\left[\frac{dx}{2R}\right]J_0(x)\}\}$$

Within the non-retarded limit we distinguish between two regimes: $d \ll R$ and $R \ll d$, in which the asymptotic dispersion interaction between two oscillators shall be calculated.

3.1.1 The unconfined case

For $R \ll d \ll \lambda_0$, the asymptotic expression for $\frac{R}{d} \to 0$, using Eq. 37, is

$$\lim_{c\to\infty}\frac{\xi^2}{c^2}\mathbf{G}_e^{(2)}(R) = \frac{1}{4\pi R^3\epsilon_2}\int_0^\infty dx\{\{x(xJ_0(x) - J_1(x)), 0, 0\}, \{0, xJ_1(x), 0\}, \{0, 0, -x^2J_0(x)\}\} \qquad (38)$$

Equation 21 then gives the non-retarded London interaction in a medium with ϵ_2 as obtained in Eq. 4,

$$U(R) = -\frac{3\hbar}{\pi R^6}\int_0^\infty \frac{\alpha_1(i\xi)\alpha_2(i\xi)}{\epsilon_2(i\xi)^2}d\xi \qquad (39)$$

Assuming that the distance between the particles is large compared with their dimensions, we may consider the polarizability of identical spherical particles of volume V with dielectric susceptibility ϵ_0,

$$\alpha_1(w) = \alpha_2(w) = \epsilon_2(w)\frac{\epsilon_0(w) - \epsilon_2(w)}{\epsilon_0(w) + 2\epsilon_2(w)}\frac{3V}{4\pi} \qquad (40)$$

and get the solution which was obtained by Pitaevskii in [6] for the non-retarded case,

$$U(R) = -\frac{27\hbar V^2}{16\pi^3}\frac{1}{R^7}\int_0^\infty \left[\frac{\epsilon_0(i\xi) - \epsilon_2(i\xi)}{\epsilon_0(i\xi) + 2\epsilon_2(i\xi)}\right]^2 d\xi \qquad (41)$$

To find the next order in this regime, we subtract Eq. 38 from Eq. 37 and changing x to $x = u\frac{R}{d}$, obtain

$$\lim_{c\to\infty}\frac{\xi^2}{c^2}\mathbf{G}_e^{(2)}(R) = \frac{1}{4\pi\epsilon_2}\int_0^\infty du\{\{\frac{2u\left(-RuJ_0\left(\frac{Ru}{d}\right)+dJ_1\left(\frac{Ru}{d}\right)\right)}{d^3(1+e^u)R},0,0\},
\\ \{0,-\frac{2uJ_1\left(\frac{Ru}{d}\right)}{d^2(1+e^u)R},0\},
\\ \{0,0,-\frac{2u^2J_0\left(\frac{Ru}{d}\right)}{d^3(-1+e^u)}\}\} \tag{42}$$

Since the main contribution to the integral comes from the neighborhood of $u \to 0$, the Bessel functions are expanded in power series of $\frac{R}{d}$, and can be integrated term by term to give

$$U(R) = -\frac{3\hbar}{\pi}\frac{1}{R^6}\left(1-\frac{5\zeta(3)}{6}\left(\frac{R}{d}\right)^3\right)\int_0^\infty\frac{\alpha_1(i\xi)\alpha_2(i\xi)}{\epsilon_2(i\xi)^2}d\xi \tag{43}$$

where ζ is a Riemann Zeta function. We see that, on the medial plane, the interaction is diminished from the unconfined result.

3.1.2 The confined case

For the region where $d \ll R \ll \lambda_0$, we use the identities (which can be proved using Poisson's summation formula, see e.g. [15]):

$$\coth\left[\frac{b}{2}\right] = \frac{2}{b}+\frac{b}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(\frac{\alpha}{2\pi})^2+n^2}
\\ \tanh\left[\frac{b}{2}\right] = \frac{4b}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(\frac{b}{\pi})^2+(2n-1)^2} \tag{44}$$

Setting $b = \frac{d}{R}x$, Eq. 37, after integration over x, becomes

$$\lim_{c\to\infty}\frac{\xi^2}{c^2}\mathbf{G}_e^{(2)}(R) = \sum_{n=1}^{\infty}\{\{-\frac{(2n-1)^2\pi K_0\left(\frac{\pi R(2n-1)}{d}\right)+\frac{d}{R}(2n-1)K_1\left(\frac{\pi R(2n-1)}{d}\right)}{d^3\epsilon_2},0,0\},
\\ \{0,\frac{(2n-1)K_1\left(\frac{\pi R(2n-1)}{d}\right)}{d^2R\epsilon_2},0\},
\\ \{0,0,\frac{4n^2\pi K_0\left(\frac{2n\pi R}{d}\right)}{d^3\epsilon_2}\}\} \tag{45}$$

where K_ν is a modified Bessel function in standard notation. For large $\frac{R}{d}$ it is sufficient for first-order approximation to retain only the term n=1 and use the asymptotic form of $K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}$ for large z. We obtain the following leading expression

$$U(R) = -\frac{4\hbar\pi^3}{d^5} \frac{e^{-\frac{2\pi R}{d}}}{R} \int_0^\infty \frac{\alpha_1(i\xi)\alpha_2(i\xi)}{\epsilon_2(i\xi)^2} d\xi \tag{46}$$

There is a cut-off at distances $R > d/2\pi$. Thus, we get that the non-retarded interaction energy between two oscillators confined between conducting plates is much weakened compared with the London result for free space.

3.2 Interaction in the retarded region

In this region we examine two cases; $\lambda_0 \ll R \ll d$ and $\lambda_0, d \ll R$

3.2.1 The unconfined case

For $\lambda_0 \ll R \ll d$, we take $\frac{d}{R} \to \infty$ in Eq. 36 to get the leading approximation. After integration over x we get

$$\begin{aligned} \frac{\xi^2}{c^2}\mathbf{G}_e^{(2)}(R) = \{\{ & -\frac{e^{-\frac{R\xi\sqrt{\epsilon_2}}{c}}\left(1+\frac{R\xi\sqrt{\epsilon_2}}{c}\right)}{2\pi R^3\epsilon_2}, 0, 0\}, \\ & \{0, \frac{e^{-\frac{R\xi\sqrt{\epsilon_2}}{c}}\left(1+\frac{R\xi\sqrt{\epsilon_2}}{c}+\frac{R^2\xi^2\epsilon_2}{c^2}\right)}{4\pi R^3\epsilon_2}, 0\}, \\ & \{0, 0, \frac{e^{-\frac{R\xi\sqrt{\epsilon_2}}{c}}\left(1+\frac{R\xi\sqrt{\epsilon_2}}{c}+\frac{R^2\xi^2\epsilon_2}{c^2}\right)}{4\pi R^3\epsilon_2}\}\} \end{aligned} \tag{47}$$

The expression for the interaction energy becomes

$$\begin{aligned} U(R) = -\frac{\hbar}{\pi R^6} \int_0^\infty & e^{-\frac{2R\xi\sqrt{\epsilon_2(i\xi)}}{c}} \frac{\alpha_1(i\xi)\alpha_2(i\xi)}{\epsilon_2(i\xi)^2} \\ & \times \left(3 + \frac{6R\xi\sqrt{\epsilon_2(i\xi)}}{c} + \frac{5R^2\xi^2\epsilon_2(i\xi)}{c^2} + \frac{2R^3\xi^3\epsilon_2(i\xi)^{3/2}}{c^3} + \frac{R^4\xi^4\epsilon_2(i\xi)^2}{c^4}\right) d\xi \end{aligned} \tag{48}$$

Since $R \gg c/w_0\sqrt{\epsilon_2(w_0)}$, the main contribution comes from the neighborhood of $\xi \to 0$,

$$U(R) = -\frac{\hbar}{\pi R^6}\frac{\alpha_1(0)\alpha_2(0)}{\epsilon_2(0)^2}\int_0^\infty e^{-\frac{2R\xi\sqrt{\epsilon_2(0)}}{c}} \times \left(3 + \frac{6R\xi\sqrt{\epsilon_2(0)}}{c} + \frac{5R^2\xi^2\epsilon_2(0)}{c^2} + \frac{2R^3\xi^3\epsilon_2(0)^{3/2}}{c^3} + \frac{R^4\xi^4\epsilon_2(0)^2}{c^4}\right) d\xi \tag{49}$$

Substituting $\frac{\xi R\sqrt{\epsilon_2(0)}}{c} = u$ the expression for the interaction energy becomes

$$U(R) = -\frac{\hbar c\alpha_1(0)\alpha_2(0)}{\pi R^7\epsilon_2^{5/2}(0)}\int_0^\infty due^{-2u}(3 + 6u + 5u^2 + 2u^3 + u^4) \tag{50}$$

which yields

$$U(R) = -\frac{23}{4\pi}\hbar c\frac{\alpha_1(0)\alpha_2(0)}{\epsilon_2^{5/2}(0)}\frac{1}{R^7} \tag{51}$$

This is, as expected, the Casimir-Polder interaction energy in a medium with ϵ_2 as obtained in Eq. 5. Substituting the polarizations as defined by Eq. 40 we get the expression obtained by Pitaevskii [6] for an unconfined system in the retarded case,

$$U(R) = -\frac{207V^2}{64\pi^3}\frac{\hbar c}{\sqrt{\epsilon_2}}\left[\frac{\epsilon_0(0) - \epsilon_2(0)}{\epsilon_0(0) + 2\epsilon_2(0)}\right]^2\frac{1}{R^7} \tag{52}$$

3.2.2 The confined case

For $\lambda_0, d \ll R$, taking the series expansion of Eq. 36 in small $\frac{d}{R}$, the zeroth-order term is

$$\begin{aligned}\frac{\xi^2}{c^2}\mathbf{G}_e^{(2)}(R) &= \int_0^\infty dx\{\{0,0,0\},\{0,0,0\},\{0,0,-\frac{x^3 J_0(x)}{2d\pi R^2\epsilon_2(x^2 + \frac{R^2\epsilon_2\xi^2}{c^2})}\}\} \\ &= \{\{0,0,0\},\{0,0,0\},\{0,0,\frac{\xi^2 K_0\left(\frac{R\xi\sqrt{\epsilon_2}}{c}\right)}{2c^2 d\pi}\}\}\end{aligned} \tag{53}$$

the expression for the interaction energy from Eq. 20 becomes

$$U(R) = -\frac{2h}{d^2\pi}\int_0^\infty \frac{\xi^4}{c^4}K_0\left(\frac{R\xi\sqrt{\epsilon_2(i\xi)}}{c}\right)^2\alpha_1(i\xi)\alpha_2(i\xi)d\xi \tag{54}$$

again, since $R \gg c/w_0\sqrt{\epsilon_2(w_0)}$ the main contribution comes from the neighborhood of $\xi \to 0$,

$$U(R) = -\frac{2h}{d^2\pi}\int_0^\infty \frac{\xi^4}{c^4} K_0\left(\frac{R\xi\sqrt{\epsilon_2(0)}}{c}\right)^2 \alpha_1(0)\alpha_2(0)d\xi \tag{55}$$

Substituting $\frac{\xi R\sqrt{\epsilon_2(0)}}{c} = u$, the expression for the interaction energy becomes

$$\begin{aligned} U(R) &= -\frac{2\hbar c\alpha_1(0)\alpha_2(0)}{d^2\pi R^5\epsilon_2^{5/2}(0)}\int_0^\infty u^4 K_0(u)^2 du \\ &= -\frac{27\pi\hbar c}{256 d^2}\frac{\alpha_1(0)\alpha_2(0)}{\epsilon_2^{5/2}(0)}\frac{1}{R^5} \end{aligned} \tag{56}$$

This result represents a strong enhancement of the interaction energy compared with the Casimir-Polder result for the unconfined case and unites with M&N result when $\epsilon_2 = 1$[2]. The confinement makes the interaction decay only as $1/R^5$, instead of $1/R^7$.

[2]notice that there is a mistake in the prefactor in [8], which has been corrected in [16]. The result obtained in [16] coincides with Eq. 56 when we use the asymptotic expression for the modified Bessel function prior to integrating.

4 Two oscillators between dielectric plates

In this section we find asymptotic expressions for the dispersion interaction between two oscillators confined by dielectric plates. We will use a similar approach as that applied for conducting plates.

4.1 Interaction in the non-retarded limit

As mentioned before in Sec. 3, the non-retarded limit, obtained for $c \to \infty$, corresponds to the situation when both d and R are much smaller than the characteristic wavelength $\lambda_0 = 2\pi c/w_0\sqrt{\epsilon_2(w_0)}$ $(R, d \ll \lambda_0)$. Taking the limit $\frac{R^2\xi^2\epsilon_2(w_0)}{c^2} \to 0$, Eq. 34 becomes

$$\begin{aligned}\lim_{c\to\infty}\frac{\xi^2}{c^2}\mathbf{G}_e^{(2)}(R) = \frac{1}{4\pi R^3\epsilon_2}\int_0^\infty dx\{&\{\frac{x(1+e^{-x\frac{d}{R}}\gamma)(xJ_0(x)-J_1(x))}{1-e^{-x\frac{d}{R}}\gamma},0,0\},\\ &\{0,\frac{x(1+e^{-x\frac{d}{R}}\gamma)J_1(x)}{1-e^{-x\frac{d}{R}}\gamma},0\},\\ &\{0,0,-\frac{x^2(1-e^{-x\frac{d}{R}}\gamma)J_0(x)}{1+e^{-x\frac{d}{R}}\gamma}\}\}\end{aligned} \tag{57}$$

where $\gamma = \frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}$. The dispersion interaction is then given by substituting $\lim\limits_{c\to\infty}\frac{\xi^2}{c^2}\mathbf{G}_e^{(2)}(R)$ in Eq. 21. Using $\frac{1}{1-q} = \sum\limits_{n=0}^{\infty} q^n$ for $|q| < 1$ and integrating over x give the image series

$$\begin{aligned}\lim_{c\to\infty}\frac{\xi^2}{c^2}\mathbf{G}_e^{(2)}(R) = \frac{1}{4\pi R^3\epsilon_2}\{&\{-2+2\sum_{n=1}^{\infty}\frac{(-2+\frac{d^2n^2}{R^2})\gamma^n}{(1+\frac{d^2n^2}{R^2})^{5/2}},0,0\},\\ &\{0,1+2\sum_{n=1}^{\infty}\frac{\gamma^n}{(1+\frac{d^2n^2}{R^2})^{3/2}},0\},\\ &\{0,0,1+2\sum_{n=1}^{\infty}\frac{(1-\frac{2d^2n^2}{R^2})(-\gamma)^n}{(1+\frac{d^2n^2}{R^2})^{5/2}}\}\}\end{aligned} \tag{58}$$

In the non-retarded limit we distinguish again between two regimes: $d \ll R$ and $R \ll d$, in which the asymptotic dispersion interaction shall be calculated.

4.1.1 The unconfined case

For $R \ll d \ll \lambda_0$, the leading order obtained from taking the limit $\frac{d}{R} \to \infty$ in Eq. 57 is

$$U(R) = -\frac{3\hbar}{\pi}\frac{1}{R^6}\int_0^\infty \frac{\alpha_1(i\xi)\alpha_2(i\xi)}{\epsilon_2^2(i\xi)} \tag{59}$$

For $\epsilon_2 = 1$ we get, as expected, the non-retarded London interaction between two oscillators in free space.

4.1.2 The confined case

For $d \ll R \ll \lambda_0$ the asymptotic form is obtained by taking $\frac{d}{R} \to 0$ in Eq. 57. This yields

$$U(R) = -\frac{3\hbar}{\pi R^6}\int_0^\infty \frac{5 + \left(\epsilon_1(i\xi)/\epsilon_2(i\xi)\right)^4}{6\epsilon_1^2(i\xi)}\alpha_1(i\xi)\alpha_2(i\xi)d\xi \tag{60}$$

We get a corrected prefactor to the London interaction depending on the dielectric susceptibilities. When $\epsilon_1 = \epsilon_2$ the prefactor reduce, as expected, back to that of the unconfined space, Eq. 4.

An interesting asymptotic case is obtained when taking the limit of $\epsilon_1 \ll \epsilon_2$ and $d \ll R$. After taking $\epsilon_1 \ll \epsilon_2$ we get from Eq. 57

$$\begin{aligned}\lim_{c\to\infty}\frac{\xi^2}{c^2}\mathbf{G}_e^{(2)}(R) = \frac{1}{4\pi R^3\epsilon_2}\int_0^\infty & dx\{\{x^2\coth[\frac{dx}{2R}]J_0(x) - x\coth[\frac{dx}{2R}]J_1(x), 0, 0\}, \\ & \{0, x\coth[\frac{dx}{2R}]J_1(x), 0\}, \\ & \{0, 0, -x^2\tanh[\frac{dx}{2R}]J_0(x)\}\}\end{aligned} \tag{61}$$

We now use the identities presented in Eq. 44 and integrate over x to get

$$\lim_{c\to\infty}\frac{\xi^2}{c^2}\mathbf{G}_e^{(2)}(R) = \frac{1}{4\pi R^3\epsilon_2}\int_0^\infty dx\{\{-\frac{2R}{d}-\sum_{n=1}^{\infty}\frac{8n\pi R^3(2n\pi K_0(\frac{2n\pi R}{d})+\frac{d}{R}K_1(\frac{2n\pi R}{d}))}{d^3},0,0\},$$
$$\{0,\frac{2R}{d}+\sum_{n=1}^{\infty}\frac{8n\pi R^2K_1(\frac{2n\pi R}{d})}{d^2},0\},$$
$$\{0,0,\sum_{n=1}^{\infty}\frac{4(2n-1)^2\pi^2R^3K_0(\frac{(2n-1)\pi R}{d})}{d^3}\}\} \tag{62}$$

For $d \ll R$ the main contribution comes from the "free" terms ($n = 0$) and we get the interaction energy

$$U(R) = -\frac{4\hbar}{d^2\pi}\frac{1}{R^4}\int_0^\infty \frac{\alpha_1(i\xi)\alpha_2(i\xi)}{\epsilon_2(i\xi)^2}d\xi \tag{63}$$

This result shows that, under the condition $\epsilon_1 \ll \epsilon_2$, the confinement makes the non-retarded interaction decay only as $1/R^4$ instead of $1/R^6$.

4.2 Interaction in the retarded region

In this region we look again at two limiting cases; $\lambda_0 \ll R \ll d$ and $\lambda_0, d \ll R$

4.2.1 The unconfined case

For $\lambda_0 \ll R \ll d$, when taking $d/R \to \infty$ in Eq. 34 the dyadic Green function becomes

$$\frac{\xi^2}{c^2}\mathbf{G}_e^{(2)}(R) = \frac{1}{4\pi R^3\epsilon_2}\int_0^\infty dx\{\{\frac{x(-xJ_1(x)+J_0(x)(x^2+\frac{R^2\xi^2\epsilon_2}{c^2}))}{\sqrt{x^2+\frac{R^2\xi^2\epsilon_2}{c^2}}},0,0\},$$
$$\{0,\frac{x(xJ_1(x)+\frac{R^2\xi^2\epsilon_2}{c^2}J_0(x))}{\sqrt{x^2+\frac{R^2\xi^2\epsilon_2}{c^2}}},0\}, \tag{64}$$
$$\{0,0,-\frac{x^3J_0(x)}{\sqrt{x^2+\frac{R^2\xi^2\epsilon_2}{c^2}}}\}\}$$

Integration over x yields

$$\frac{\xi^2}{c^2}\mathbf{G}_e^{(2)}(R) = \{\{-\frac{e^{-\frac{R\xi\sqrt{\epsilon_2}}{c}}(1+\frac{R\xi\sqrt{\epsilon_2}}{c})}{2\pi R^3\epsilon_2}, 0, 0\},$$
$$\{0, \frac{e^{-\frac{R\xi\sqrt{\epsilon_2}}{c}}(1+\frac{R\xi\sqrt{\epsilon_2}}{c}+\frac{R^2\xi^2\epsilon_2}{c^2})}{4\pi R^3\epsilon_2}, 0\},$$
$$\{0, 0, \frac{e^{-\frac{R\xi\sqrt{\epsilon_2}}{c}}(1+\frac{R\xi\sqrt{\epsilon_2}}{c}+\frac{R^2\xi^2\epsilon_2}{c^2})}{4\pi R^3\epsilon_2}\}\} \tag{65}$$

which is similar to Eq. 47. The same derivation gives the following leading term

$$U(R) = -\frac{23}{4\pi}\hbar c\frac{\alpha_1(0)\alpha_2(0)}{\epsilon_2^{5/2}(0)}\frac{1}{R^7} \tag{66}$$

This is the Casimir-Polder result [3, 11] for free space when $\epsilon_2 = 1$.

4.2.2 The confined case

For $\lambda_0, d \ll R$, the asymptotic result obtained by taking $\frac{d}{R} \to 0$ in Eq. 34 is

$$\frac{\xi^2}{c^2}\mathbf{G}_e^{(2)}(R) = \frac{1}{4\pi R^3}\int_0^\infty dx\{\{\frac{x(-xJ_1(x)+J_0(x)(x^2+R^2\frac{\xi^2}{c^2}\epsilon_1))}{\epsilon_1\sqrt{x^2+R^2\frac{\xi^2}{c^2}\epsilon_1}}, 0, 0\},$$
$$\{0, \frac{x(xJ_1(x)+R^2\frac{\xi^2}{c^2}J_0(x)\epsilon_1)}{\epsilon_1\sqrt{x^2+R^2\frac{\xi^2}{c^2}\epsilon_1}}, 0\},$$
$$\{0, 0, -\frac{x^3 J_0(x)\epsilon_1}{\sqrt{x^2+R^2\frac{\xi^2}{c^2}\epsilon_1}\epsilon_2^2}\}\} \tag{67}$$

After integration over x, the result is

$$\frac{\xi^2}{c^2}\mathbf{G}_e^{(2)}(R) = \{\{-\frac{e^{-\frac{R\xi\sqrt{\epsilon_1}}{c}}(1+\frac{R\xi\sqrt{\epsilon_1}}{c})}{2\pi R^3\epsilon_1}, 0, 0\},$$
$$\{0, \frac{e^{-\frac{R\xi\sqrt{\epsilon_1}}{c}}(1+\frac{R\xi\sqrt{\epsilon_1}}{c}+\frac{R^2\xi^2\epsilon_1}{c^2})}{4\pi R^3\epsilon_1}, 0\},$$
$$\{0, 0, \frac{e^{-\frac{R\xi\sqrt{\epsilon_1}}{c}}\epsilon_1(1+\frac{R\xi\sqrt{\epsilon_1}}{c}+R^2\frac{\xi^2}{c^2}\epsilon_1)}{4\pi R^3\epsilon_2^2}\}\} \tag{68}$$

and the resulting expression for the interaction energy is

$$U(R) = -\frac{\hbar}{2\pi R^6} \int_0^\infty d\xi e^{-\frac{2R\xi\sqrt{\epsilon_1(i\xi)}}{c}} \frac{\alpha_1(i\xi)\alpha_2(i\xi)}{\epsilon_1(i\xi)^2}$$
$$\times \left(5 + \frac{10R\xi\sqrt{\epsilon_1(i\xi)}}{c} + \frac{7R^2\xi^2\epsilon_1(i\xi)}{c^2} + \frac{2R^3\xi^3\epsilon_1(i\xi)^{3/2}}{c^3} + \frac{R^4\xi^4\epsilon_1(i\xi)^2}{c^4}\right.$$
$$\left. + \left(\frac{\epsilon_1(i\xi)}{\epsilon_2(i\xi)}\right)^4 \left(1 + \frac{R\xi\sqrt{\epsilon_1(i\xi)}}{c} + \frac{R^2\xi^2\epsilon_1(i\xi)}{c^2}\right)^2\right) \tag{69}$$

The main contribution to the integral comes from the neighborhood of $\xi \to 0$. Substituting $\frac{\xi R\sqrt{\epsilon_1(0)}}{c} = u$ the expression for the interaction energy becomes

$$U(R) = -\frac{\hbar c\alpha_1(0)\alpha_2(0)}{2\pi\epsilon_1^{5/2}(0)} \frac{1}{R^7} \int_0^\infty e^{-2u}(5+\beta^4+2u^3(1+\beta^4)+u^4(1+\beta^4)+2u(5+\beta^4)+u^2(7+3\beta^4))du \tag{70}$$

where $\beta = \frac{\epsilon_1(0)}{\epsilon_2(0)}$, which yields

$$U(R) = -\frac{23}{4\pi}\hbar c \frac{\alpha_1(0)\alpha_2(0)}{\epsilon_2^{5/2}(0)} \frac{33+13\beta^4}{46\beta^{5/2}} \frac{1}{R^7} \tag{71}$$

We get a correction prefactor to the free Casimir-Polder interaction which properly reduce to the unconfined case of Eq. 5 when $\epsilon_1 = \epsilon_2$.

The asymptotic expression for the case where $\epsilon_1 \ll \epsilon_2$, in the retarded limit, is absent since it is more complicated analytically.

5 The intermediate region

In this section we would like to get further insight into the effect of the boundaries on the dispersion interactions between particles. To examine the qualitative modification of the distance dependence we allow ourselves a strong assumption regarding the dielectric susceptibilities $\epsilon_i(i\xi), (i = 1, 2)$. In the non-retarded limit we assume that the dielectric susceptibilities can be taken as constant in the vicinity of the typical radiation frequency of the particles (Recall that the requirement actually refers to $\epsilon(i\xi)$, which is a real, monotonic function of ξ). In the retarded limit the dielectric susceptibilities are taken as the static dielectric constants. We examine the interaction over wide ranges of dielectric mismatch, which will be very hard to obtain in practice, aiming, again, at a qualitative understanding of the confinement effects.

5.1 Interaction in the non-retarded limit

In order to get the behavior induced by confinement we did a numerical integration of Eq. 57 while assuming that the dielectrics susceptibilities are constant. Fig. 2 shows a diagram of the interaction energy in the non-retarded region as a function of the dielectric mismatch and d/R in a $\log_{10} - \log_{10}$ scale. The color code is also in $\log_{10}$ scale where blue represents suppression and red enhancement.

The green area represents the unconfined-space behavior which, occurs either when the plates are far and the particles not "see" the boundaries, or when the dielectric mismatch equals one. The blue area represents the "metal-out-like" regime where there is an exponential decay with distance, corresponding to the asymptotic function obtained in Eq. 46. This behavior appears as an intermediate region when the dielectric mismatch is larger than one. Notice that the intermediate region becomes wider the larger the dielectric mismatch.

The red area appearing for large d/R represents a "dielectric regime", corresponding to the asymptotic function obtained in Eq. 60, where there is an enhancement prefactor (or small suppression) depending on the dielectric mismatch, with a $1/R^6$ dependence.

The behavior of suppression and then enhancement was also observed in Ref. [9] where the inner medium was taken as vacuum, the particles as isotropic identical molecules with a single transition frequency, and the dielectric susceptibility for the plates as that of a Thomas-Fermi plasma $\epsilon(i\xi) = 1 + \frac{w_p^2}{\xi^2}$.

For a dielectric mismatch smaller than one we get an intermediate region corresponding to Eq. 63, which gives a $1/R^4$ dependence ,i.e., enhancement relative to the $1/R^6$ dependence obtained in the unconfined case.

Fig. 3 shows a plot for small dielectric mismatches in the non-retarded region. It demonstrates again the rich behavior of the dispersion interaction due to confinement.

5.2 Interaction in the retarded region

For the retarded region we take the static limit of the polarizability and dielectric susceptibilities. A numerical integration is performed for Eq. 34 to get the interaction energy for various inter-particle distance and dielectric mismatches. The results are shown in Fig. 4.

For either small distances or when the dielectric mismatch is one we get, as expected, the unconfined energy (green area). In the intermediate region (for dielectric mismatch higher than one) we get an enhancement due to the metal-like behavior represented by Eq. 56, yielding a power law of $1/R^5$ instead of $1/R^7$. At large separations we get the asymptotic

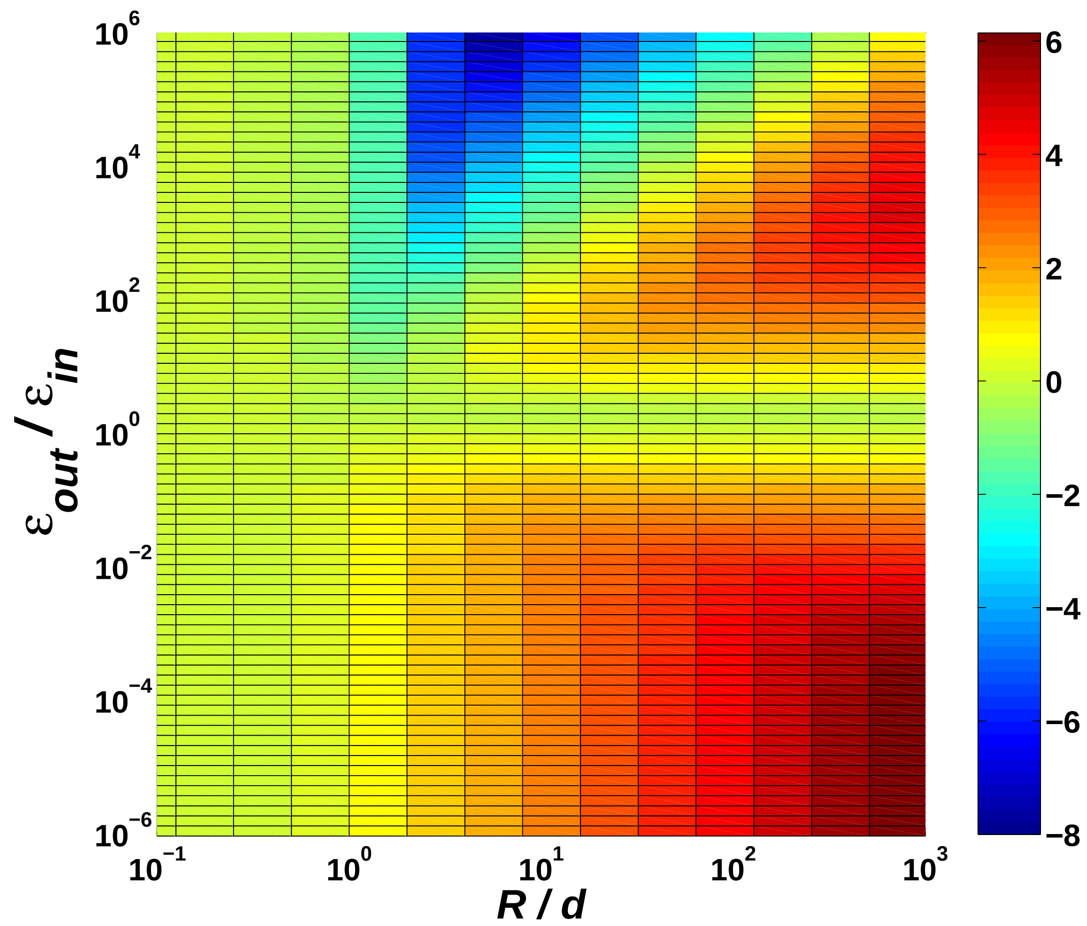

Figure 2: The non-retarded interaction energy as obtained from numerical integration of Eq. 57 for constant $\epsilon(i\xi)$. The colors represent a $\log_{10}$ scale of the interaction energy normalized by its unconfined counterpart (Eq. 4).

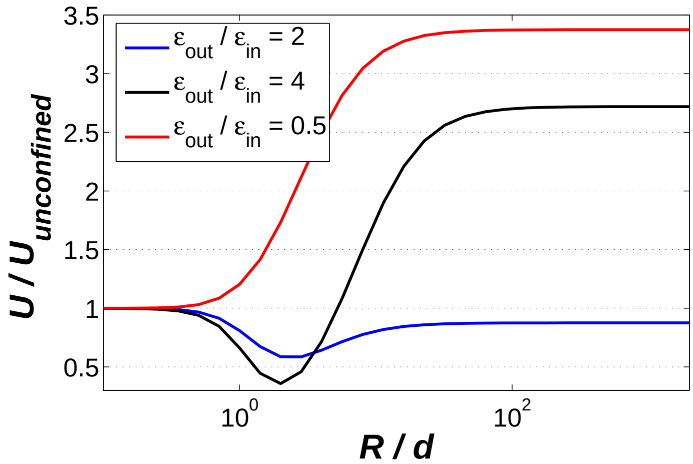

Figure 3: The non-retarded interaction energy as obtained from numerical integration of Eq. 57 for constant $\epsilon(i\xi)$. The interaction energy is normalized by its unconfined counterpart (Eq. 4).

formula of Eq. 71, which has the $1/R^7$ dependence as in the unconfined case with a prefactor depending on the dielectric mismatch. Figure 5 shows a plot for small dielectric mismatches

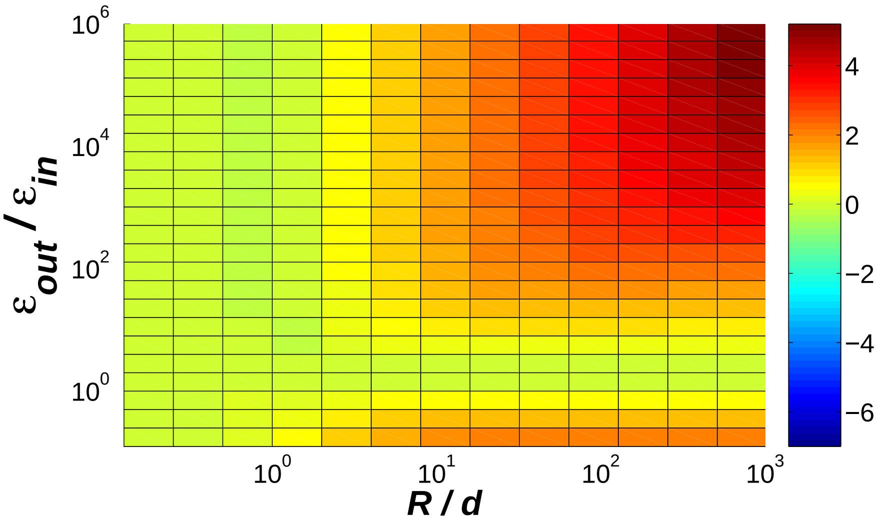

Figure 4: The retarded interaction energy as obtained from numerical integration of Eq. 34 for constant $\epsilon(i\xi)$. The colors represent a $\log_{10}$ scale of the interaction energy normalized by its unconfined counterpart (Eq. 5).

in the retarded region, which demonstrating the rich behavior of the dispersion interaction due to confinement.

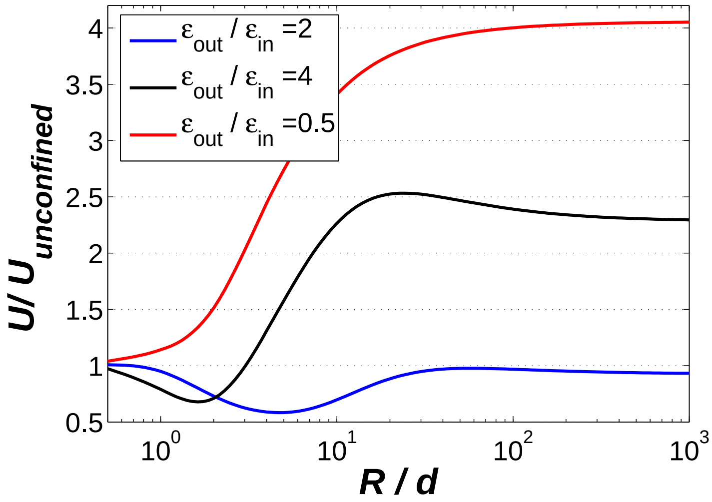

Figure 5: The retarded interaction energy as obtained from numerical integration of Eq. 34 for constant $\epsilon(i\xi)$. The interaction energy is normalized by its unconfined counterpart (Eq. 5).

6 Concluding remarks

In this work we have investigated the zero temperature dispersion interaction between two particles embedded in a medium confined by two dielectric plates. The particles are neutral, non-polar and positioned at the mid-plane of the cavity. We have worked within a continuum assumption, representing the media by their dielectric susceptibilities. Although this assumption is always to be questioned, Dzyaloshinskii, Lifshitz and Pitaevskii provided a convincing theory, valid for inter-particle distances much larger than the molecular distances in the media in which the tensor Green function for imaginary frequency using the dielectric susceptibilities is the key quantity.

It has been shown that confinement leads to qualitatively different behavior of dispersion forces. There is a rich behavior as a function of the inter-particle distance: unconfined behavior at short distances, dramatic changes at intermediate distances, and at very large distances unconfined behavior with a correction (usually enhancement) factor depending on the dielectric mismatch. The intermediate region corresponds to the asymptotic behavior where the dielectric mismatch (between the outer and inner media) goes to zero (infinity) leading to a metal-in-like (metal-out-like). In the non-retarded limit (short distances), when the dielectric mismatch between the outer and inner media is smaller (larger) than one, an enhancement (suppression) in the power law, $1/R^4$ (exponential decay) is obtained in the intermediate region. These effects in the non-retarded limit can be visualized as arising from the interaction of the dipoles not only with each other, but also with their images. In the retarded limit, for intermediate distances there is mostly an increase from the unconfined space value a $1/R^5$ dependence. The enhancement effect in the retarded interaction can be envisaged as arising from guided electromagnetic waves propagating along the surfaces.

These qualitatively new effects should be relevant to a broad range of experimental and computational systems containing confined particles, for example, particles embedded in porous media or biological constrictions. Yet, an extension of this work for finite temperatures is still required (see Appendix A).

A Temperature dependence of dispersion interactions

An important aspect of the dispersion interaction between particles which we have not dealt with in this work, is the effect of temperature. We remark that, in order to include temperature dependence, we should consider the energy associated with each mode which is not just the zero-point energy $\hbar w/2$, but rather the Helmholtz free energy, which for an oscillator of frequency w_j is

$$g(w_j) = k_{\mathrm{B}}T \ln\left(2\sinh\left(\frac{\hbar w_j}{2k_{\mathrm{B}}T}\right)\right) \tag{72}$$

By using the identity

$$\sum_{w_j} g(w_j) = \frac{1}{2\pi i}\oint g(w)\frac{1}{D(w)}\frac{dD(w)}{dw}dw \tag{73}$$

where $g(w_j)$ and $D(w)$ are analytic functions, it has been shown [1] that the expression for the free energy can be rewritten as a Matsubara summation over discrete frequencies

$$F(T) = k_B T {\sum_{n=0}^{\infty}}' \ln D(i\xi_n), \qquad \xi_n = 2\pi k_B T n/\hbar \tag{74}$$

The prime on the summation indicates that a weight of 1/2 is applied to the $n = 0$ term. When we are interested in changes in the free energy brought about by having both oscillators interacting with the field, we get

$$F(T) = k_B T {\sum_{n=0}^{\infty}}' \ln\left(\frac{D_{12}(i\xi_n)}{D_1(i\xi_n)D_2(i\xi_n)}\right) \tag{75}$$

where $D_1(w)$ and $D_2(w)$ are the secular determinants related to the energy of the isolated oscillators and $D_{12}(w)$ is related to the self energy when both are coupled to the field. By using this expression, a natural extension of the theory presented in this thesis can be developed. For the effect of temperature on two oscillators between conducting plates see [16].

B The structure of the Green functions

B.1 The method of scattering superposition

To understand the method of scattering superposition, let us examine a simple example for one dimensional Green function that satisfies the equation

$$\begin{aligned} &\frac{d^2}{dx^2}g^{(1)}(x,x') + k_1^2 g^{(1)}(x,x') = -\delta(x-x'), \qquad && x \geq 0 \\ &\frac{d^2}{dx^2}g^{(2)}(x,x') + k_2^2 g^{(2)}(x,x') = 0, && x \leq 0 \end{aligned} \tag{76}$$

with the proper boundary conditions. The solution has the form

$$\begin{aligned} g^{(1)}(x,x') &= g_0(x,x') + g_s^{(1)}(x,x') \\ g^{(2)}(x,x') &= g_s^{(2)}(x,x') \end{aligned} \tag{77}$$

where $g_0(x,x')$ is the free space Green function and $g_s^i(x,x')$ is the scattered wave function which is a solution of the homogeneous differential equation ($i = 1,2$). The free space Green function can be found by Fourier transform and using the residue theorem

$$g_0(x,x') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ih(x-x')}}{h^2-k_1^2}dh = \frac{i}{2k_1}e^{ik_1|x-x'|} \tag{78}$$

In this example, Fourier transform is equivalent to the method of using eigenfunctions expansion since e^{ihx} is an eigenfunction of the differential equation. The method of eigenfunctions expansion is used in the forthcoming treatment of the dyadic Green function. Hence, the solution has the form

$$\begin{aligned} g^{(1)}(x,x') &= \frac{i}{2k_1}\begin{cases} e^{ik_1(x-x')} + Re^{ik_1(x+x')} & x \geq x' \\ e^{-ik_1(x-x')} + Re^{ik_1(x+x')} & 0 \leq x \leq x' \end{cases} \\ g^{(2)}(x,x') &= \frac{i}{2k_1}Te^{-i(k_2x-k1x')} \qquad\qquad\qquad x \geq 0 \end{aligned} \tag{79}$$

where R and T are unknown coefficients (the reflection and transmission coefficients of propagating wave) to be determined by applying the boundary conditions.

B.2 Eigenfunction expansion of the free-space dyadic Green functions

In order to calculate the dyadic Green functions we saw that we first need to calculate the free space dyadic Green function. The free-space magnetic dyadic Green function for the magnetic field introduced previously satisfies the equation

$$\begin{aligned} &\nabla \times \nabla \times \mathbf{G}_{m0}(\vec{r}, \vec{r'}) - k^2 \mathbf{G}_{m0}(\vec{r}, \vec{r'}) = \nabla \times [\mathbf{I}\delta(\vec{r} - \vec{r'})] \\ &\text{where} \quad k^2 = \epsilon(w)\frac{w^2}{c^2} \end{aligned} \tag{80}$$

and the radiation condition at infinity $\lim_{r\to\infty}[\nabla \times \mathbf{G_{m0}}(\vec{r}, \vec{r'}) - ik\hat{r} \times \mathbf{G_{m0}}(\vec{r}, \vec{r'})] = 0$. Since $\nabla \cdot \mathbf{G_{m0}} = 0$, by using the identity $\nabla \times \nabla \times \vec{A} = -\nabla^2 \vec{A} + \nabla(\nabla \cdot \vec{A})$ the equation becomes:

$$\nabla^2 \mathbf{G}_{m0} + k^2 \mathbf{G}_{m0} = -\nabla \times [\mathbf{I}\delta(\vec{r} - \vec{r'})] \tag{81}$$

It can be shown that its explicit expression is

$$\mathbf{G}_{m0}(\vec{r}, \vec{r'}) = \nabla \times [\mathbf{I}G_0(\vec{r}, \vec{r'})] = \nabla G_0(\vec{r}, \vec{r'})) \times \mathbf{I} \tag{82}$$

where $G_0(\vec{r}, \vec{r'}) = \frac{e^{ik|\vec{r}-\vec{r'}|}}{4\pi|\vec{r}-\vec{r'}|}$ is the free space Green function satisfying the three-dimensional scalar wave equation

$$\nabla^2 G_0(\vec{r}, \vec{r'}) + k^2 G_0(\vec{r}, \vec{r'}) = -\delta(\vec{r} - \vec{r'}) \tag{83}$$

For problems with cylindrical symmetry, we need the eigenfunction expansion of this function in order to construct the functions by the method of scattering superposition. The dyadic

Green function $\mathbf{G_{m0}}$ is obtained from the divergence-free vector solutions of the equation

$$\nabla^2 \vec{F}_\kappa + \kappa^2 \vec{F}_\kappa = 0 \tag{84}$$

The two independent divergence-free solutions of Eq. 84 can be written in the form

$$\begin{aligned} \vec{M}_\kappa(\vec{r}) &= \nabla \times (\mathbf{\hat{a}}\psi_\kappa) \\ \vec{N}_\kappa(\vec{r}) &= \frac{1}{\kappa}\nabla \times \nabla \times (\mathbf{\hat{a}}\psi'_\kappa) \end{aligned} \tag{85}$$

where ψ_κ and ψ'_κ satisfy the equation $\nabla^2\psi_\kappa + \kappa^2\psi_\kappa = 0$, $\hat{a}$ is a unit vector[3] (which is taken in the direction of z) and κ are suitably adjusted to make the function $\vec{F}_\kappa = \vec{M}_\kappa + \vec{N}_\kappa$ satisfy the right boundary conditions. Both $\vec{M}$ and $\vec{N}$ can be normalized as

$$\int \vec{M}^*_\kappa \cdot \vec{M}_{\kappa'} d^3r = \int \vec{N}^*_\kappa \cdot \vec{N}_{\kappa'} d^3r = \Lambda_\kappa \delta_{\kappa\kappa'} \tag{86}$$

To satisfy equation of the form $\nabla \times \nabla \times \mathbf{G_0}(\vec{r}, \vec{r'}) - k^2 \mathbf{G_0}(\vec{r}, \vec{r'}) = \mathbf{I}\delta(\vec{r} - \vec{r'})$ we get

$$\mathbf{G}_0(\vec{r}, \vec{r'}) = \sum_\kappa \frac{\vec{M}_\kappa(\vec{r})\vec{M}^*_\kappa(\vec{r'}) + \vec{N}_\kappa(\vec{r})\vec{N}^*_\kappa(\vec{r'})}{\Lambda_\kappa(k^2 - \kappa^2)} \tag{87}$$

Therefore to satisfy Eq. 81 $\mathbf{G_{m0}}$ has the form

$$\mathbf{G}_{m0}(\vec{r}, \vec{r'}) = \sum_\kappa \kappa \frac{\vec{N}_\kappa(\vec{r})\vec{M}^*_\kappa(\vec{r'}) + \vec{M}_\kappa(\vec{r})\vec{N}^*_\kappa(\vec{r'})}{\Lambda_\kappa(k^2 - \kappa^2)} \tag{88}$$

where we use the following symmetrical relations between $\vec{M}_\kappa$ and $\vec{N}_\kappa$

$$\begin{aligned} \vec{N}_\kappa &= \frac{1}{\kappa}\nabla \times \vec{M}_\kappa \\ \vec{M}_\kappa &= \frac{1}{\kappa}\nabla \times \vec{N}_\kappa \end{aligned} \tag{89}$$

which are valid when an identical generating function ψ is used for both $\vec{M}_\kappa$ and $\vec{N}_\kappa$. Therefore, in cylindrical coordinates ψ has the form:

$$\psi(r, \phi, z) = \begin{bmatrix} J_n(\lambda r) \\ N_n(\lambda r) \end{bmatrix} \begin{bmatrix} cos(n\phi) \\ sin(n\phi) \end{bmatrix} \begin{bmatrix} e^{ihz} \\ e^{-ihz} \end{bmatrix} \tag{90}$$

[3]The conditions for the unit vector $\hat{a}$ can be found, for example, in [17]

where $\lambda^2 + h^2 = \kappa^2$, n is integer, J_n is Bessel functions and N_n is Neuman functions. Since Neuman functions diverge at the origin, the solution contains Bessel functions only. Hence two sets of functions may be defined:

$$\begin{aligned}
\mathbf{M}_{\substack{e\\o}}(h) &= \mathbf{M}_{\substack{even\\odd}}(n, \lambda, h) = \nabla \times [J_n(\lambda r) \begin{matrix} cos(n\phi) \\ \\ sin(n\phi) \end{matrix} e^{ihz} \hat{z}] \\
\mathbf{N}_{\substack{e\\o}}(h) &= \mathbf{N}_{\substack{even\\odd}}(n, \lambda, h) = \frac{1}{\kappa} \nabla \times \nabla \times [J_n(\lambda r) \begin{matrix} cos(n\phi) \\ \\ sin(n\phi) \end{matrix} e^{ihz} \hat{z}]
\end{aligned} \tag{91}$$

The orthogonal properties of these functions can be stated as follows :

$$\begin{aligned}
&\int \mathbf{M}_{\substack{e\\o}}(n, \lambda, h) \cdot \mathbf{N}_{\substack{e\\o}}(n', \lambda', -h') d^3 r = 0 \\
&\int \mathbf{M}_{\substack{e\\o}}(n, \lambda, h) \cdot \mathbf{M}_{\substack{e\\o}}(n', \lambda', -h') d^3 r = 2(1 + \delta_0)\pi^2 \lambda \delta(\lambda - \lambda')\delta(h - h')\delta_{nn'} \\
&\int \mathbf{N}_{\substack{e\\o}}(n, \lambda, h) \cdot \mathbf{N}_{\substack{e\\o}}(n', \lambda', -h') d^3 r = 2(1 + \delta_0)\pi^2 \lambda \delta(\lambda - \lambda')\delta(h - h')\delta_{nn'}
\end{aligned} \tag{92}$$

where $\delta_{nn'}$ is Kronecker delta and δ_0 is equal to one when $n = 0$ and zero otherwise. In view of Eq. 80 and Eq. 88, the expansion of $\mathbf{G}_{m0}$ can be written now in the form

$$\begin{aligned}
\mathbf{G}_{m0}(\vec{r}, \vec{r'}) = &\int_0^\infty d\lambda \int_{-\infty}^\infty dh \sum_{n=0}^\infty \frac{(2 - \delta_0)\kappa}{4\pi^2 \lambda(\kappa^2 - k^2)} \\
&\cdot [\mathbf{N}_e(h)\mathbf{M}'_e(-h) + \mathbf{N}_o(h)\mathbf{M}'_o(-h) + \mathbf{M}_e(h)\mathbf{N}'_e(-h) + \mathbf{M}_o(h)\mathbf{N}'_o(-h)]
\end{aligned} \tag{93}$$

where the primed functions $\mathbf{M}'$ and $\mathbf{N}'$ are defined with respect to (r', ϕ', z') and λ and h are two continuous eigenvalues. Only positive values of λ are included because $J_n(\lambda r)$ and $J_n(-\lambda r)$ are not independent functions. The above Fourier integral can be evaluated with the aid of the residue theorem in the h-plane. The poles of the integrand are located at $h = \pm h_2$ where $h_2 = \sqrt{k_2^2 - \lambda^2}$ (where the wave number k is now replaced by k_2 and

$\kappa^2 = h_2^2 + \lambda^2$). This yields

$$\mathbf{G}^{\pm}_{m0}(\vec{r}, \vec{r'}) = k_2 \int_0^\infty d\lambda \sum_{n=0}^{\infty} \frac{i(2-\delta_0)}{4\pi\lambda h_2} \cdot [\mathbf{N}_e(\pm h_2)\mathbf{M}'_e(\mp h_2) + \mathbf{N}_o(\pm h_2)\mathbf{M}'_o(\mp h_2) + \mathbf{M}_e(\pm h_2)\mathbf{N}'_e(\mp h_2) + \mathbf{M}_o(\pm h_2)\mathbf{N}'_o(\mp h_2)] \qquad z \gtrless z' \tag{94}$$

Since $\mathbf{G_{m0}}(\vec{r}, \vec{r'})$ is discontinuous at $z = z'$, we can write

$$\mathbf{G_{m0}}(\vec{r}, \vec{r'}) = \mathbf{G_{m0}}^{+}(\vec{r}, \vec{r'})U(z - z') + \mathbf{G_{m0}}^{-}(\vec{r}, \vec{r'})U(z' - z) \tag{95}$$

where $U(z - z')$ is a unit step. Thus

$$\begin{aligned} \nabla \times \mathbf{G_{m0}}(\vec{r}, \vec{r'}) =& [\nabla \times \mathbf{G_{m0}}^{+}(\vec{r}, \vec{r'})]U(z - z') + \nabla U(z - z') \times \mathbf{G_{m0}}^{+}(\vec{r}, \vec{r'}) \\ &+ [\nabla \times \mathbf{G_{m0}}^{-}(\vec{r}, \vec{r'})]U(z' - z) + \nabla U(z' - z) \times \mathbf{G_{m0}}^{-}(\vec{r}, \vec{r'}) \end{aligned} \tag{96}$$

Using that

$$\begin{aligned} \nabla U(z - z') &= \hat{z}\delta(z - z') \\ \nabla U(z' - z) &= -\hat{z}\delta(z' - z) \end{aligned} \tag{97}$$

we get

$$\begin{aligned} \nabla \times \mathbf{G_{m0}}(\vec{r}, \vec{r'}) =& [\nabla \times \mathbf{G_{m0}}^{+}(\vec{r}, \vec{r'})]U(z - z') + [\nabla \times \mathbf{G_{m0}}^{-}(\vec{r}, \vec{r'})]U(z' - z) \\ &+ \hat{z}\delta(z - z') \times [\mathbf{G_{m0}}^{+}(\vec{r}, \vec{r'}) - \mathbf{G_{m0}}^{-}(\vec{r}, \vec{r'})] \end{aligned} \tag{98}$$

In view of Eq. 30, where $\hat{n} = \hat{z}$, the above equation can be written in the form

$$\begin{aligned} \nabla \times \mathbf{G_{m0}}(\vec{r}, \vec{r'}) =& [\nabla \times \mathbf{G_{m0}}^{+}(\vec{r}, \vec{r'})]U(z - z') + [\nabla \times \mathbf{G_{m0}}^{-}(\vec{r}, \vec{r'})]U(z' - z) \\ &+ (\mathbf{I} - \hat{z}\hat{z})\delta(\vec{r} - \vec{r'}) \end{aligned} \tag{99}$$

By using Eq. 25, the expression for $\mathbf{G_{e0}}^{(2)}(\vec{r},\vec{r'})$ can now be written in the form

$$\begin{aligned}\mathbf{G_{e0}}^{(2)}(\vec{r},\vec{r'}) &= \frac{1}{k_2^2}[-\hat{z}\hat{z}\delta(\vec{r}-\vec{r'}) + (\nabla\times\mathbf{G}^{+}_{\mathbf{m0}})U(z-z') + (\nabla\times\mathbf{G}^{-}_{\mathbf{m0}})U(z'-z)] \\ &= -\frac{1}{k_2^2}\hat{z}\hat{z}\delta(\vec{r}-\vec{r'}) \\ &+ \int_0^\infty d\lambda \sum_{n=0}^{\infty}\frac{(2-\delta_0)}{4\pi\lambda h_2}\cdot[\mathbf{M}_e(\pm h_2)\mathbf{M}'_e(\mp h_2) + \mathbf{M}_o(\pm h_2)\mathbf{M}'_o(\mp h_2) \\ &\qquad + \mathbf{N}_e(\pm h_2)\mathbf{N}'_e(\mp h_2) + \mathbf{N}_o(\pm h_2)\mathbf{N}'_o(\mp h_2)] \quad z \gtrless z'\end{aligned} \tag{100}$$

When the simplified notations

$\mathbf{M}(h)\mathbf{M}'(h) = \mathbf{M}_e(h)\mathbf{M}'_e(h) + \mathbf{M}_o(h)\mathbf{M}'_o(h)$ and $\mathbf{M}(h)\mathbf{M}'(h) = \mathbf{N}_e(h)\mathbf{N}'_e(h) + \mathbf{N}_o(h)\mathbf{N}'_o(h)$

are being used the expression for $\mathbf{G_{e0}}^{(2)}(\vec{r},\vec{r'})$ is

$$\begin{aligned}\mathbf{G_{e0}}^{(2)}(\vec{r},\vec{r'}) &= -\frac{1}{k_2^2}\hat{z}\hat{z}\delta(\vec{r}-\vec{r'}) \\ &+ \int_0^\infty d\lambda \sum_{n=0}^{\infty}\frac{(2-\delta_0)}{4\pi\lambda h_2}[\mathbf{M}(\pm h_2)\mathbf{M}'(\mp h_2) + \mathbf{N}(\pm h_2)\mathbf{N}'(\mp h_2)] \quad z \gtrless z'\end{aligned} \tag{101}$$

B.3 Eigenfunction expansions of the scattered terms

In analogy to the simple case described in section B.1 we build the expressions for the scattered terms. Using the residue theorem in the h-plane the scattered terms must have the form .

$$\mathbf{G_{es}}^{(1)}(\vec{r},\vec{r'}) = \int_0^{\infty} d\lambda \sum_{n=0}^{\infty} \frac{i(2-\delta_0)}{4\pi\lambda h_2}$$
$$\cdot \{\mathbf{M}(h_2)[a_1^+\mathbf{M}'(h_1) + a_1^-\mathbf{M}'(-h_1)] + \mathbf{N}(h_2)[c_1^+\mathbf{N}'(h_1) + c_1^-\mathbf{N}(-h_1)]\}$$
$$\mathbf{G_{es}}^{(2)}(\vec{r},\vec{r'}) = \int_0^{\infty} d\lambda \sum_{n=0}^{\infty} \frac{i(2-\delta_0)}{4\pi\lambda h_2}$$
$$\cdot \{\mathbf{M}(h_2)[a_2^+\mathbf{M}'(h_2) + a_2^-\mathbf{M}'(-h_2)] + \mathbf{M}(-h_2)[b_2^+\mathbf{M}'(h_2) + b_2^-\mathbf{M}'(-h_2)]$$
$$+ \mathbf{N}(h_2)[c_2^+\mathbf{N}'(h_2) + c_2^-\mathbf{N}(-h_2)] + \mathbf{N}(-h_2)[d_2^+\mathbf{N}'(h_2) + d_2^-\mathbf{N}(-h_2)]\}$$
$$\mathbf{G_{es}}^{(3)}(\vec{r},\vec{r'}) = \int_0^{\infty} d\lambda \sum_{n=0}^{\infty} \frac{i(2-\delta_0)}{4\pi\lambda h_2}$$
$$\cdot \{\mathbf{M}(-h_2)[b_3^+\mathbf{M}'(h_1) + b_3^-\mathbf{M}'(-h_1)] + \mathbf{N}(-h_2)[d_3^+\mathbf{N}'(h_1) + d_3^-\mathbf{N}(-h_1)]\} \tag{102}$$

By applying the boundary conditions from Eq. 33 at the interfaces $z = d, z = 0$, the sixteen unknown coefficients can be determined by

$$\begin{aligned}
a_1^+ &= \frac{\rho(1+\rho)e^{id(h_2-h_1)}}{\Gamma} & a_1^- &= \frac{(1+\rho)e^{id(h_2-h_1)}}{\Gamma} \\
c_1^+ &= \frac{k_2\rho(1+\rho)e^{id(h_2-h_1)}}{k_1\Gamma'} & c_1^- &= \frac{k_2\rho(1+\rho)e^{id(h_2-h_1)}}{k_1\Gamma'} \\
a_2^+ &= \frac{\rho}{\Gamma} & a_2^- &= \frac{\rho^2 e^{i2dh_2}}{\Gamma} \\
b_2^+ &= \frac{\rho^2 e^{i2dh_2}}{\Gamma} & b_2^- &= \frac{\rho e^{i2dh_2}}{\Gamma} \\
c_2^+ &= \frac{\rho'}{\Gamma'} & c_2^- &= \frac{\rho'^2 e^{i2dh_2}}{\Gamma'} \\
d_2^+ &= \frac{\rho'^2 e^{i2dh_2}}{\Gamma'} & d_2^- &= \frac{\rho' e^{i2dh_2}}{\Gamma'} \\
b_3^+ &= \frac{1+\rho}{\Gamma} & b_3^- &= \frac{1+\rho e^{i2dh_2}}{\Gamma} \\
d_3^+ &= \frac{k_2(1+\rho')}{k_1\Gamma'} & d_3^- &= \frac{k_2\rho'(1+\rho')e^{i2dh_2}}{k_1\Gamma'}
\end{aligned} \tag{103}$$

where the parameters are defined by

$$\begin{aligned}
&\Gamma = 1-\rho^2 e^{idh_2}, \quad \Gamma' = 1-\rho'^2 e^{idh_2}\\
&\rho = \frac{h_2-h_1}{h_2+h_1}, \quad \rho' = \frac{k_1^2 h_2 - k_2^2 h_1}{k_1^2 h_2 + k_2^2 h_1}\\
&h_2 = \sqrt{k_2^2-\lambda^2}, \quad h_1 = \sqrt{k_1^2-\lambda^2}\\
&k_2^2 = \epsilon_2(w)\frac{w^2}{c^2} \qquad k_1^2 = \epsilon_1(w)\frac{w^2}{c^2}
\end{aligned} \tag{104}$$

B.4 Analyzing the dyadic Green function

We simplified the dyadic Green function in the following manner. Without loss of generality, we can take $\phi = \phi'$ to get the following expression for $\mathbf{G}_e^{+(2)}$

$$\begin{aligned}
&\mathbf{G}_e^{+(2)}(r,r',z,z') = -\frac{1}{k_2^2}\hat{z}\hat{z}\delta(r-r')\delta(z-z') + \int_0^\infty d\lambda \sum_{n=0}^{\infty} \frac{i(2-\delta_0)}{4\pi\lambda h_2}\\
&\{\{\frac{1}{4}e^{-ih_2(z+z')}(\frac{4n^2(1+e^{2izh_2}\rho)(1-\Gamma+e^{2ih_2z'}\rho)J_n(r\lambda)J_n(r'\lambda)}{r\Gamma\rho r'} -\\
&\frac{\lambda^2(J_{n-1}(r\lambda)-J_{n+1}(r\lambda))(J_{n-1}(r'\lambda)-J_{1+n}(r'\lambda))h_2^2(-1+e^{2izh_2}\rho')(-1+\Gamma'+e^{2ih_2z'}\rho')}{k_2^2\Gamma'\rho'})\\
&,0,\frac{ie^{-ih_2(z+z')}\lambda^3 J_n(r'\lambda)(J_{n-1}(r\lambda)-J_{n+1}(r\lambda))h_2(-1+e^{2izh_2}\rho')(1-\Gamma'+e^{2ih_2z'}\rho')}{2k_2^2\Gamma'\rho'}\},\{0,\\
&\frac{1}{4}e^{-ih_2(z+z')}(\frac{\lambda^2(1+e^{2izh_2}\rho)(1-\Gamma+e^{2ih_2z'}\rho)(J_{n-1}(r\lambda)-J_{n+1}(r\lambda))(J_{n-1}(r'\lambda)-J_{1+n}(r'\lambda))}{\Gamma\rho}\\
&-\frac{4n^2 J_n(r\lambda)J_n(r'\lambda)h_2^2(-1+e^{2izh_2}\rho')(-1+\Gamma'+e^{2ih_2z'}\rho')}{rk_2^2 r'\Gamma'\rho'}),0\},\\
&\{\frac{ie^{-ih_2(z+z')}\lambda^3 J_n(r\lambda)(J_{n-1}(r'\lambda)-J_{1+n}(r'\lambda))h_2(1+e^{2izh_2}\rho')(-1+\Gamma'+e^{2ih_2z'}\rho')}{2k_2^2\Gamma'\rho'},\\
&0,\frac{e^{-ih_2(z+z')}\lambda^4 J_n(r\lambda)J_n(r'\lambda)(1+e^{2izh_2}\rho')(1-\Gamma'+e^{2ih_2z'}\rho')}{k_2^2\Gamma'\rho'}\}\}
\end{aligned} \tag{105}$$

Another simplification is taking the limit $r' \to 0$ (without loss of generality). By doing so only the zero and first order Bessel function remain after the summation yielding

$$\begin{aligned}
\mathbf{G}_e^{+(2)}(r,z,z') = &-\frac{1}{k_2^2}\hat{z}\hat{z}\delta(r)\delta(z-z')+ \\
&\int_0^\infty d\lambda\{\{\frac{ie^{-ih_2(z+z')}}{8\pi h_2}(\frac{2(1+e^{2izh_2}\rho)(1-\Gamma+e^{2ih_2z'}\rho)J_1(r\lambda)}{r\Gamma\rho} \\
&-\frac{\lambda(J_0(r\lambda)-J_2(r\lambda))h_2^2(-1+e^{2izh_2}\rho')(-1+\Gamma'+e^{2ih_2z'}\rho')}{k_2^2\Gamma'\rho'}), \\
&0,\frac{e^{-ih_2(z+z')}\lambda^2J_1(r\lambda)(-1+e^{2izh_2}\rho')(1-\Gamma'+e^{2ih_2z'}\rho')}{4\pi k_2^2\Gamma'\rho'}\}, \\
&\{0,\frac{ie^{-ih_2(z+z')}}{8\pi h_2}(\frac{\lambda(1+e^{2izh_2}\rho)(1-\Gamma+e^{2ih_2z'}\rho)(J_0(r\lambda)-J_2(r\lambda))}{\Gamma\rho} \\
&-\frac{2J_1(r\lambda)h_2^2(-1+e^{2izh_2}\rho')(-1+\Gamma'+e^{2ih_2z'}\rho')}{rk_2^2\Gamma'\rho'}),0\}, \\
&\{-\frac{e^{-ih_2(z+z')}\lambda^2J_1(r\lambda)(1+e^{2izh_2}\rho')(-1+\Gamma'+e^{2ih_2z'}\rho')}{4\pi k_2^2\Gamma'\rho'},0, \\
&\frac{ie^{-ih_2(z+z')}\lambda^3J_0(r\lambda)(1+e^{2izh_2}\rho')(1-\Gamma'+e^{2ih_2z'}\rho')}{4\pi h_2k_2^2\Gamma'\rho'}\}\}
\end{aligned} \tag{106}$$

Substituting $w=i\xi$, $\mathbf{G}_e^2(r)$ can be written in the form

$$\begin{aligned}
&\mathbf{G}_e^{+(2)}(r,z,z') = \frac{1}{k_2^2}\hat{z}\hat{z}\delta(r)\delta(z-z')+ \\
&\int_0^\infty d\lambda\{\{-\frac{e^{h_2(z+z')}}{8\pi r\Gamma\rho h_2k_2^2\Gamma'\rho'}(2r\Gamma\lambda\rho J_0(r\lambda)h_2^2(-1+e^{-2zh_2}\rho')(-1+\Gamma'+e^{-2h_2z'}\rho') \\
&+2J_1(r\lambda)((-1-e^{-2zh_2}\rho)(1-\Gamma+e^{-2h_2z'}\rho)k_2^2\Gamma'\rho' \\
&-\Gamma\rho h_2^2(-1+e^{-2zh_2}\rho')(-1+\Gamma'+e^{-2h_2z'}\rho'))), \\
&0,-\frac{e^{h_2(z+z')}\lambda^2J_1(r\lambda)(-1+e^{-2zh_2}\rho')(1-\Gamma'+e^{-2h_2z'}\rho')}{4\pi k_2^2\Gamma'\rho'}\},\{0, \\
&-\frac{e^{h_2(z+z')}}{8\pi r\Gamma\rho h_2k_2^2\Gamma'\rho'}(-r\lambda(1+e^{-2zh_2}\rho)(1-\Gamma+e^{-2h_2z'}\rho)(J_0(r\lambda)-J_2(r\lambda))k_2^2\Gamma'\rho' \\
&+2\Gamma\rho J_1(r\lambda)h_2^2(-1+e^{-2zh_2}\rho')(-1+\Gamma'+e^{-2h_2z'}\rho')),0\}, \\
&\{\frac{e^{-h_2(z+z')}\lambda^2J_1(r\lambda)(e^{2zh_2}+\rho')(e^{2h_2z'}(-1+\Gamma')+\rho')}{4\pi k_2^2\Gamma'\rho'},0, \\
&\frac{e^{-h_2(z+z')}\lambda^3J_0(r\lambda)(e^{2h_2z'}(-1+\Gamma')-\rho')(e^{2zh_2}+\rho')}{4\pi h_2k_2^2\Gamma'\rho'}\}\}
\end{aligned} \tag{107}$$

where the parameters from Eq. 104 are now redefined by

$$\begin{aligned}
&\Gamma = 1-\rho^2 e^{-dh_2}, \quad \Gamma' = 1-\rho'^2 e^{-dh_2} \\
&\rho = \frac{h_2-h_1}{h_2+h_1}, \quad \rho' = \frac{k_1^2 h_2 - k_2^2 h_1}{k_1^2 h_2 + k_2^2 h_1} \\
&h_2 = \sqrt{k_2^2+\lambda^2}, \quad h_1 = \sqrt{k_1^2+\lambda^2} \\
&k_2^2 = -\epsilon_2(i\xi)\frac{\xi^2}{c^2} \qquad k_1^2 = -\epsilon_1(i\xi)\frac{\xi^2}{c^2}
\end{aligned} \tag{108}$$

For simplifying the problem, we look at a specific case in which $z = z' = \frac{d}{2}$ and assuming that $r \neq 0$ in order to obtain the dispersion interaction between the oscillators positioned in the middle of the two planes. Thus we get a diagonal tensor

$$\begin{aligned}
\mathbf{G}_e^{(2)}(r) = \int_0^\infty & d\lambda\{\{(h_1\sinh[dh_2]h_2(k_1^2+k_2^2)(r\lambda J_0(r\lambda)h_2^2+J_1(r\lambda)(-h_2^2+k_2^2))+ \\
&\cosh[dh_2](h_2^2k_1^2+h_1^2k_2^2)(r\lambda J_0(r\lambda)h_2^2+J_1(r\lambda)(-h_2^2+k_2^2))- \\
&(h_2^2k_1^2-h_1^2k_2^2)(r\lambda J_0(r\lambda)h_2^2-J_1(r\lambda)(h_2^2+k_2^2)))/ \\
&(8\pi r h_2(h_1\cosh[\frac{dh_2}{2}]+\sinh[\frac{dh_2}{2}]h_2)k_2^2(\cosh[\frac{dh_2}{2}]h_2k_1^2+h_1\sinh[\frac{dh_2}{2}]k_2^2)) \\
&,0,0\},\{0,\frac{1}{8\pi}(\frac{\lambda(J_0(r\lambda)-J_2(r\lambda))(h_1\sinh[\frac{dh_2}{2}]+\cosh[\frac{dh_2}{2}]h_2)}{h_2(h_1\cosh[\frac{dh_2}{2}]+\sinh[\frac{dh_2}{2}]h_2)}+ \\
&\frac{4h_1J_1(r\lambda)h_2}{r(1+\cosh[dh_2])h_2k_1^2+h_1r\sinh[dh_2]k_2^2}+\frac{2J_1(r\lambda)h_2\tanh[\frac{dh_2}{2}]}{rk_2^2}),0\}, \\
&\{0,0,-\frac{\lambda^3J_0(r\lambda)(\cosh[\frac{dh_2}{2}]h_2k_1^2+h_1\sinh[\frac{dh_2}{2}]k_2^2)}{4\pi h_2k_2^2(\sinh[\frac{dh_2}{2}]h_2k_1^2+h_1\cosh[\frac{dh_2}{2}]k_2^2)}\}\}
\end{aligned} \tag{109}$$

Defining $\lambda = \frac{x}{r}$, $h_i = \frac{y_i}{r}$ where $y_i = \sqrt{\epsilon_i(\frac{r\xi}{c})^2+x^2}$ and multiplying the Green function by

$\frac{\xi^2}{c^2}$, we have

$$
\begin{aligned}
\frac{\xi^2}{c^2}\mathbf{G}_e^{(2)}(r) &= \frac{1}{4\pi r^3 \epsilon_2}\int_0^\infty dx \begin{pmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \end{pmatrix} \\
G_1(r) =& (\sinh\left[\frac{dy_2}{r}\right] y_1 y_2(\epsilon_1+\epsilon_2)((xJ_0(x)-J_1(x))y_2^2+\frac{r^2\xi^2\epsilon_2}{c^2}J_1(x)) \\
&+\cosh\left[\frac{dy_2}{r}\right]((xJ_0(x)-J_1(x))y_2^2+\frac{r^2\xi^2\epsilon_2}{c^2}J_1(x))(y_2^2\epsilon_1+y_1^2\epsilon_2) \\
&-(y_2^2\epsilon_1-y_1^2\epsilon_2)(xJ_0(x)y_2^2-J_1(x)(y_2^2+\frac{r^2\xi^2\epsilon_2}{c^2}))) \\
&/(2y_2(\cosh\left[\frac{dy_2}{2r}\right]y_1+\sinh\left[\frac{dy_2}{2r}\right]y_2)(\cosh\left[\frac{dy_2}{2r}\right]y_2\epsilon_1+\sinh\left[\frac{dy_2}{2r}\right]y_1\epsilon_2)) \\
G_2(r) =& (\sinh\left[\frac{dy_2}{r}\right] y_1 y_2(\epsilon_1+\epsilon_2)(J_1(x)y_2^2+\frac{r^2\xi^2\epsilon_2}{c^2}(xJ_0(x)-J_1(x)))+ \\
&(J_1(x)y_2^2+\frac{r^2\xi^2\epsilon_2}{c^2}(-xJ_0(x)+J_1(x)))(-y_2^2\epsilon_1+y_1^2\epsilon_2)+ \\
&\cosh\left[\frac{dy_2}{r}\right](J_1(x)y_2^2+\frac{r^2\xi^2\epsilon_2}{c^2}(xJ_0(x)-J_1(x)))(y_2^2\epsilon_1+y_1^2\epsilon_2)) \\
&/(2y_2(\cosh\left[\frac{dy_2}{2r}\right]y_1+\sinh\left[\frac{dy_2}{2r}\right]y_2)(\cosh\left[\frac{dy_2}{2r}\right]y_2\epsilon_1+\sinh\left[\frac{dy_2}{2r}\right]y_1\epsilon_2)) \\
G_3(r) =& -\frac{x^3 J_0(x)(\cosh\left[\frac{dy_2}{2r}\right]y_2\epsilon_1+\sinh\left[\frac{dy_2}{2r}\right]y_1\epsilon_2)}{y_2(\sinh\left[\frac{dy_2}{2r}\right]y_2\epsilon_1+\cosh\left[\frac{dy_2}{2r}\right]y_1\epsilon_2)}
\end{aligned}
\tag{110}
$$

Eq. 110 is in the form used in the derivation.

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