SIAM J. CONTROL AND OPTIMIZATION
Vol. 30, No. 3, pp. 695–703, May 1992

MINIMAL LENGTH CURVES THAT ARE NOT EMBEDDABLE IN AN OPEN PLANAR SET: THE PROBLEM OF A LOST SWIMMER WITH A COMPASS*

R. HASSIN† AND A. TAMIR† ‡

Abstract. Given an open bounded set S in R^2, the problem of computing a path f of minimum size such that for every $x \in S$ the set $\{x\}+f$ intersects the boundary of S is considered. The existence of such paths is proved both when the path size is its length and when it is its (one-dimensional Hausdorff outer) measure. Some theorems characterizing optimal paths are proved and it is shown that when S is convex, the minimum width chords of $\mathrm{Cl}(S)$ are optimal with respect to both size definitions.

Key words. search theory, computational geometry, planar convex sets

AMS(MOS) subject classifications. 90B40, 52A10, 49A40

1. Introduction.

A fisherman on a small boat lost on a big lake in a very thick fog has no information regarding his location. He has zero visibility but possesses a compass and a map of the lake and its surroundings. The fisherman can do dead-reckoning navigation by selecting at each point in time an azimuth and by traveling along this direction for any distance d he wishes to cover. His objective is to minimize the distance he must travel to the shoreline.

Next consider a soldier lost in a mine field under zero visibility conditions. He has a compass, a map of the field (which does not indicate the mines), as well as a special spoke to search for the mines. The soldier wants to minimize the time he will need to reach a boundary of the field. If he traverses a certain segment of a path for the first time, he must search for mines, thus moving at a very low speed v. However, if his path repeats a segment for a second time, he can speed up, attaining a velocity that is practically infinite in comparison to v.

Suppose that both the fisherman and the soldier are conservative and that they wish to minimize the maximum travel distance (time) over all possible initial locations.

We use several examples to demonstrate the difference between the two models (see Fig. 1). The optimal path of the fisherman is given by the minimal length path, while that of the soldier is depicted by the minimal measure path in the examples below. For comparison purposes, we normalize v, the speed of the soldier, to one unit when he explores "new avenues," and we let him repeat a segment he has already traveled before with infinite velocity. With this assumption, the measure of a path does not exceed its length. (These two terms are properly defined in the next section.) In Fig. 1, examples (a), (c), (d), and (f), the optimal measure path is an arc; i.e., none of its points is visited more than once. Therefore, the length is equal to the measure. In example (b), the minimal length is 2, while the minimal measure is only $1 + \sqrt{3}/2$. In example (e) the difference between the measure and the length is ε.

*Received by the editors June 11, 1990; accepted for publication (in revised form) April 18, 1991.

†Beverly and Raymond Sackler Faculty of Exact Sciences, School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel.

‡Department of Statistics and Operations Research, New York University, New York, New York 10003.

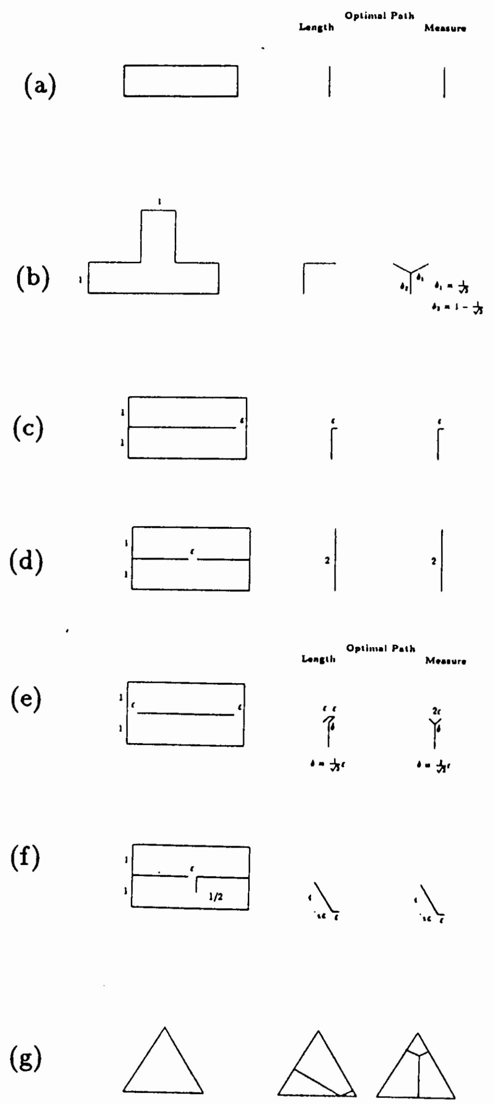

FIG. 1

Finally, example (d) demonstrates that the length and the measure are very sensitive to connectivity properties. In particular, they might both have discontinuity jumps when the boundary is slightly perturbed. The optimal length and measure are equal to 2 for any $\varepsilon > 0$. However, if $\varepsilon = 0$, i.e., the lake becomes disconnected, they both decrease to 1.

In this study, after we provide general existence theorems, we focus on convex open sets (lakes). We prove that the length and the measure are both equal to the width of the convex set, i.e., to the minimum distance between a pair of distinct parallel lines that bound the set. Therefore, an optimal path is a (shortest) line segment connecting this pair of lines. Example (g) demonstrates that the optimal path is not necessarily unique. It also shows that there may be nonlinear optimal paths even in the convex case. In particular, every path whose image is the three normals from a point in the equilateral triangle to its edges is a minimal measure path. A piecewise linear path with a single breakpoint on an edge of the equilateral triangle whose image is the two normals from that point to the other two edges is a minimal length path.

We do not know of any works dealing with optimal "navigation" with a compass. Works on navigation without a compass and related topics are described in [1], [3], and [5]–[14]. In the last section, we present a general framework unifying search problems of this nature.

2. The mathematical model. Let S be a bounded open set in R^2. Let ∂S denote the boundary of S and let $\mathrm{Cl}(S)$ denote its closure.

A *path* in R^2 is a continuous function f from the unit interval I in R^1 into R^2. A path f is called an *arc* (a simple Jordan curve) if f is one-to-one. We use the symbol $f(I)$ to indicate the *image* of I, i.e., the set of all elements $f(t)$ in R^2 where $t \in I$.

The *path length* of f, $\lambda(f)$, is defined as

$$\lambda(f) = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left\| f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right\| ,$$

where $\|\cdot\|$ is the Euclidean norm. When f is piecewise smooth (piecewise continuously differentiable), $\lambda(f) = \int_I \|f^{(1)}(t)\|dt$.

Let x, y be in R^2 and let A and B be subsets of R^2. Define $d(x,y) = \|x - y\|$, $d(x,A) = d(A,x) = \inf\{d(x,y) \mid y \in A\}$, and $\mathrm{diam}(A) = \sup\{d(x,y) \mid x, y \in A\}$. Let $d(A \triangle B)$ denote the Hausdorff distance between A and B, i.e.,

$$d(A \triangle B) = \max\{\sup\{d(x,B) \mid x \in A\}, \sup\{d(y,A) \mid y \in B\}\} .$$

Also, define $A + B = \{x + y | x \in A, y \in B\}$.

Let C be a subset of R^2. The *one-dimensional Hausdorff outer measure of C in R^2* is

$$\lambda_1(C) = \lim_{\varepsilon\to 0}\left(\inf\left\{\sum_{i=1}^{\infty} \mathrm{diam}(A_i) \,\middle|\, \bigcup_{i=1}^{\infty} A_i = C \ , \ \mathrm{diam}(A_i) \le \varepsilon \text{ for all } i\right\}\right) .$$

Consider a path f. Then f is *S-uncontained* if $f(0) = 0$ and for every $x \in S$ the set $\{x\} + f(I)$ intersects ∂S. The length of such a path f is given by $\lambda(f)$, defined above, and its *measure* is given by $\lambda_1(f(I))$.

Motivated by the examples presented above, we consider the following two optimization models.

Model 1. Find an S-uncontained path of minimum length.

Model 2. Find an S-uncontained path of minimum measure.

In Theorem 1, we prove the existence of optimal paths in either of the two models. It has been demonstrated above that the two models can have different solution paths. We show here that if the set S is convex, then both models are optimized by the same linear path. Specifically, an optimizer of both models is the *minimum width chord* of $\mathrm{Cl}(S)$. (The latter is defined as a line segment in R^2 of minimum length connecting any two distinct parallel lines that bound $\mathrm{Cl}(S)$ between them.)

THEOREM 1. *Let S be an open bounded set in R^2.*

a) *There exists an S-uncontained path of minimum length.*

b) *There exists an S-uncontained path of minimum measure.*

Proof. Denote $M = \mathrm{diam}(\mathrm{Cl}(S))$. To prove the theorem, it is certainly sufficient to consider only those paths whose length (and measure) is bounded by M.

(a) Let β denote the infimum of the lengths of all S-uncontained paths. Consider an S-uncontained path f with length $\lambda(f)$. We represent f by a standard parametrization $\widehat{f}$ on its length: $\widehat{f} : I \to R^2$, where $\widehat{f}$ is continuous and $\widehat{f}(0) = 0$. $\widehat{f}$ is M-Lipschitz. Using the Arzelá–Ascoli theorem [4, p. 266], we conclude that the set of all (parametrized) S-uncontained and M-Lipschitz paths is nonempty and compact in the uniform convergence topology. If $\{\widehat{f}_n\}$ is a sequence of (parametrized) S-uncontained paths such that $\lambda(\widehat{f}_n) \to \beta$, there exists a subsequence $\{\widehat{f}_{n_i}\}$ converging uniformly to an S-uncontained path g. Since path length is lower semicontinuous, we obtain $\lambda(g) \leq \lim_i \lambda(\widehat{f}_{n_i}) = \beta$.

(b) Let γ denote the infimum of the measures of all S-uncontained paths, and let $\{\widehat{f}_n\}$ be a sequence of (parametrized) S-uncontained paths such that $\lambda_1(\widehat{f}_n(I)) \to \gamma$. As in part (a), let $\{\widehat{f}_{n_i}\}$ be a subsequence converging uniformly to an S-uncontained path h. It follows that the sets $\{\widehat{f}_{n_i}(I)\}$ converge in Hausdorff distance to $h(I)$. Therefore, using [6, Thm. 3] we conclude that

$$\lambda_1(h(I)) \leq \lim_i \lambda_1(\widehat{f}_{n_i}(I)) = \gamma \, . \qquad \square$$

Remark. Instead of using the Arzelá-Ascoli theorem in the above proof, one might prefer a more elementary argument as exhibited in [3]. It is based on repeatedly using the Bolzano–Weierstrass property in the usual plane metric.

We will need the following definition: Let $\alpha \in I$. The *subpath* of f defined by α, f^α, is

$$f^\alpha(t) = \begin{cases} f(t), & 0 \leq t \leq \alpha \\ f(\alpha), & \alpha \leq t \leq 1 \, . \end{cases}$$

THEOREM 2. *Let f be an S-uncontained path. Then there exist $\overline{x} \in \mathrm{Cl}(S)$ and $\alpha \in I$ such that f^α is S-uncontained and $\{\overline{x}\} + f^\alpha(I) \subseteq \mathrm{Cl}(S)$.*

Proof. For each $x \in S$ define

$$t(x) = \sup\{t \in I \mid x + f(s) \in S \quad \text{for all} \quad 0 \leq s < t\} \, .$$

Since f is S-uncontained, $x + f(t(x))$ is in ∂S.

Define $\alpha = \sup\{t(x) \mid x \in S\}$.

For each $x \in S$, $\{x\} + f^\alpha(I)$ intersects ∂S. Thus, f^α is S-uncontained. From the definition of α and the compactness of $\mathrm{Cl}(S)$, there exists a sequence $\{x^n\}$ of points in S that converges to some $\overline{x} \in \mathrm{Cl}(S)$, and $\{t(x^n)\}$ converges to α. We claim that $\{\overline{x}\} + f^\alpha(I) \subseteq \mathrm{Cl}(S)$.

Suppose, by contradiction, that there exists some s, $0 < s < \alpha$, and $\overline{x} + f^\alpha(s) \notin \mathrm{Cl}(S)$. Let $\varepsilon = d(\overline{x} + f^\alpha(s), \mathrm{Cl}(S)) > 0$. Let n be such that $d(x^n, \overline{x}) < \varepsilon/2$, and $s < t(x^n) \leq \alpha$. Then for any $y \in \mathrm{Cl}(S)$, $d(y, x^n + f^\alpha(s)) \geq d(y, \overline{x} + f^\alpha(s)) - d(x^n, \overline{x}) \geq \varepsilon/2$. Therefore, $d(x^n + f^\alpha(s), \mathrm{Cl}(S)) \geq \varepsilon/2$ for some $s < t(x^n)$. This contradicts the definition of $t(x^n)$. $\square$

Next we prove that if S is a bounded open convex set in R^2, then there exists an S-uncontained path of minimum length and minimum measure that is a linear function. In particular, we show that the length of such a linear path is the width of $\mathrm{Cl}(S)$.

We now need the following definition.

DEFINITION. Let u be a point in $\mathrm{Cl}(S)$. Then d in R^2 is a *feasible direction* of S at u if there exists $\varepsilon > 0$ such that $u + \varepsilon d$ is in S. Otherwise d is called *infeasible*.

THEOREM 3. *Let f be an S-uncontained path for a bounded open convex set $S \subseteq R^2$, and let $x \in \mathrm{Cl}(S)$ satisfy $\{x\} + f(I) \subseteq \mathrm{Cl}(S)$. Define the tangency set $\widehat{R}(f,x)$*

$$\widehat{R}(f,x) = \partial S \cap \left(\{x\} + f(I)\right) .$$

Then one of the following holds:

(1) *There exist two distinct and parallel supporting (subgradient) lines to $\mathrm{Cl}(S)$ at some pair of points in $\widehat{R}(f,x)$.*

(2) *There exist three distinct supporting lines to $\mathrm{Cl}(S)$ at some points in $\widehat{R}(f,x)$, such that the triangle generated by these lines contains $\mathrm{Cl}(S)$.*

Proof. Since f is S-uncontained, it follows that there is no direction d at x and an $\varepsilon > 0$ such that the set $\{x + \varepsilon d\} + f(I)$ is contained in S. The points in $\widehat{R}(f,x)$ block any translation of the set $\{x\} + f(I)$ into S. Formally, it follows that the set of infeasible directions at all the points in $\widehat{R}(f,x)$ exhausts all directions in R^2.

Consider a supporting line ℓ to $\mathrm{Cl}(S)$ at some point u in $\widehat{R}(f,x)$. With each infeasible direction at u, we associate a point on the unit circle corresponding to its angle from the horizontal x_1-axis. Thus the supporting line ℓ is associated with a closed subarc I_ℓ of the unit circle of length π. (I_ℓ captures all infeasible directions defined by the half plane not containing $\mathrm{Cl}(S)$.)

Consider next the collection of subarcs obtained by looking at all points in $\widehat{R}(f,x)$ and their supporting lines. From the above, it follows that the union of all the subarcs is the unit circle. If there exists a pair of subarcs whose union is the unit circle, then (1) holds. Otherwise, the union of the interiors of the subarcs is again the unit circle. Due to compactness of the unit circle, there is a finite subcollection of subarcs whose union is the unit circle. To summarize, there is a finite number of at least three supporting lines at points in $\widehat{R}(f,x)$ that define a convex bounded polygon containing $\mathrm{Cl}(S)$. Since no pair of these lines is parallel, it is a simple matter to verify inductively that any such polygon can be bounded by a triangle formed by three of its supporting lines. This completes the proof. □

LEMMA 1. *Let $\{\ell_1, \ell_2, \ell_3\}$ be a collection of three distinct pairwise nonparallel lines in R^2. Then the minimum over R^2 of the sum of (Euclidean) distances from the three lines is attained at a point where two of these lines intersect.*

Proof. For x in R^2, let $g_i(x) = d(x, \ell_i)$, $i = 1,2,3$, and $g(x) = g_1(x)+g_2(x)+g_3(x)$.

Consider an arbitrary line L in R^2. If L and ℓ_i are parallel, the restriction of $g_i(x)$ to L is a constant function. Otherwise, this restriction is piecewise linear with one breakpoint at the intersection point of L and ℓ_i. Therefore, the restriction of $g(x)$ to L is a (convex) piecewise linear function having at most three breakpoints (the intersection points of L with the three given lines). The minimum of $g(x)$ over L is attained at an intersection point of L with some line ℓ_i, $i = 1,2,3$.

Let x^* be a minimum point of $g(x)$ over R^2, and consider some line L containing x^*. Then there exists some line ℓ_i and the point z^*, $\{z^*\} = L \cap \ell_i$, such that $g(z^*) = g(x^*)$. Consider next the minimization of $g(x)$ over ℓ_i. From the above, the minimum is attained at an intersection point of ℓ_i with some other line ℓ_j, $j = 1,2,3$, $j \neq i$. □

LEMMA 2. *Let S be the interior of a triangle, and let X be a closed connected set in R^2 such that there is no x in R^2 with $\{x\} + X \subseteq S$. Then there is $\overline{x}$ in R^2 such that $\{\overline{x}\} + X$ intersects the three edges of the triangle.*

Proof. Let e_1, e_2, and e_3 denote the three edges of the triangle, and let ℓ_1, ℓ_2, and ℓ_3 denote the three lines containing the three edges, respectively. Also, for $i = 1,2,3$, let ℓ_i^+ be the half plane, determined by ℓ_i, that contains the given triangle. Since X

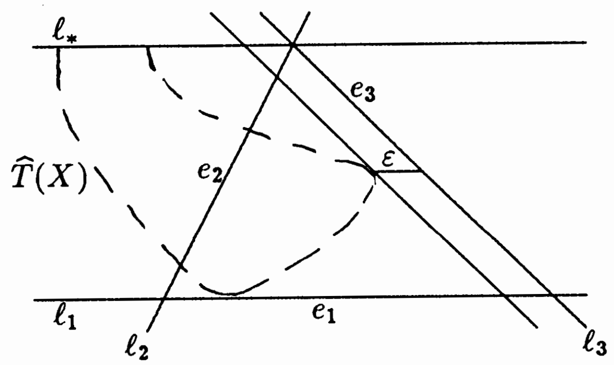

FIG. 2

is closed, there exists a translation of X, say $T(X) = \{z\} + X$ for some $z \in R^2$, that intersects ℓ_1 at some (relative) interior point of e_1 and is contained in ℓ_1^+.

We first show that $T(X)$ must also intersect either e_2 or e_3. Indeed, if it does not, then, since $T(X)$ is closed, there is a positive distance between $T(X)$ and $e_2 \cup e_3$. Also, $T(X)$ is included in the triangle since $T(X)$ is connected. Therefore, there is a perturbed translation of $T(X)$ along the normal to ℓ_1 that yields a translation, say $T^1(X) = \{u\} + X$ for some $u \in R^2$, which is included in S.

Thus, suppose that $T(X)$ intersects e_2 but not e_3. (If it intersects both, the result holds.) Let ℓ_* be a line parallel to ℓ_1 and containing the vertex of the triangle opposing e_1. Also let ℓ_*^+ denote the half plane, defined by ℓ_*, which contains the triangle.

Without loss of generality, suppose that ℓ_1 is the (horizontal) x_1-axis in the plane. Define

$$\widehat{T}(X) = T(X) \cap \ell_1^+ \cap \ell_3^+ \cap \ell_*^+ \ .$$

Let $\varepsilon = Minimum\{y - x_1 \mid (y, x_2) \in e_3,\ (x_1, x_2) \in \widehat{T}(X)\}$. Due to the fact that $T(X)$ is closed and does not intersect e_3, ε is well defined and positive. (See Fig. 2.)

Next, we translate $T(X)$ along ℓ_1 toward ℓ_3 by a distance of $\varepsilon > 0$. Let $T^2(X)$ denote the translated set obtained by this move. If $T^2(X)$ intersects e_2, the result holds. Otherwise, there are two cases to consider. First, suppose that $T^2(X)$ contains a point outside the triangle. Then a contradiction to the connectivity of $T^2(X)$ is easily obtained. Therefore, suppose that $T^2(X)$ is contained in the triangle and does not intersect e_2.

A perturbed translation of $T^2(X)$ along the bisector of the triangle angle, defined by e_1 and e_3, moves $T^2(X)$ into S. This completes the proof. □

THEOREM 4. *Let S be the interior of a triangle. Then any minimum width chord of* Cl(S) *is an S-uncontained path of minimum length and minimum measure.*

Proof. It is clear that any minimum width chord of Cl(S) is an S-uncontained path. Consider an S-uncontained path f. Let $X = f(I)$. From Lemma 2 there exists a translation, say $T(X) = \{z\} + X$ for some $z \in R^2$, that intersects the three edges of Cl(S), e_1, e_2, and e_3 at points x^1, x^2, and x^3, respectively. (The points are not necessarily distinct.) $\lambda_1(X)$, the measure of X (with respect to the one-dimensional Hausdorff measure) is bounded below by the measure of a minimum measure set that (arcwise) connects x^1, x^2, and x^3. It is known [2] that a Euclidean Steiner tree connecting this triplet of points is a minimum measure set. Furthermore, the measure of such a tree is the sum of the (Euclidean) distances from some point in R^2 to x^1, x^2, and x^3. Using Lemma 1, we note that the measure of the connecting Steiner tree is greater than or equal to the width of $\mathrm{Cl}(S)$. Thus we conclude that the measure of X is not smaller then the width of $\mathrm{Cl}(S)$. Finally, the length of f, $\lambda(f)$ is bounded below by the length of any minimal length path connecting x^1, x^2, and x^3. Thus, $\lambda(f)$ is bounded below by the measure of the above Steiner tree. This completes the proof. □

THEOREM 5. *Let S be an open bounded convex set in R^2. Then any minimum width chord of* $\mathrm{Cl}(S)$ *is an S-uncontained path of minimum length and minimum measure.*

Proof. Let f be an S-uncontained path. To prove that $\lambda(f)$ and $\lambda_1(f(I))$ are both bounded below by the width of $\mathrm{Cl}(S)$, we may suppose that the assumptions of Theorems 2 and 3 are satisfied. Thus we refer to the two cases stated in Theorem 3. If (1) holds, then clearly both $\lambda(f)$ and $\lambda_1(f(I))$ are bounded below by the distance between the two parallel supporting lines. If (2) holds, we apply Theorem 4 to the triangle defined in this case. Again, both $\lambda(f)$ and $\lambda_1(f(I))$ are bounded below by the width of that triangle, which, in turn, is bounded by the width of $\mathrm{Cl}(S)$. This completes the proof. □

3. Concluding remarks and open problems. We show above that if S is bounded and convex, then an optimal S-uncontained curve is linear and its length is the width of S. Linearity might be lost if S is not convex. In fact, we might have no linear minimal length curve even for the case when S is the union of seven pairwise disjoint open rectangles of the same width. Consider the case when S is a planar polygon, given by an ordered sequence of its vertices. If S is convex, its width can be computed in time that is linear in the number of vertices [15]. When S is not convex, the complexity of determining minimal length or minimal measure curves is still unknown. We suspect that it is NP-hard. We conjecture that minimal curves are piecewise linear and that the number of pieces is polynomial in the number of vertices. If some optimal curve is indeed piecewise linear and a bound on the number of pieces is known a priori, then we can construct a finite scheme to compute minimal curves. The existence of such a scheme follows directly from the theory of Tarski on solvability over real closed fields [16] since the model can be formulated as an algebraic sentence.

We demonstrate above that minimal measure curves are not necessarily simple, i.e., one-to-one. However, we conjecture that there exists a minimal length curve that is simple.

Finally, we mention several extensions and generalizations of the above models. First, we can consider the extension to R^n for $n \geq 3$. We suspect that Theorem 5 holds for this general case as well. Second, we can consider disconnected solution sets. Let X be a closed set in R^2 that contains the origin. Call X an *S-uncontained set* if for any x in S, the set $\{x\} + X$ intersects the boundary of S. The extended optimization model seeks an S-uncontained compact set of minimal measure with respect to the Hausdorff measure defined above. Since we do not require connectedness of X, we might possibly obtain a solution whose measure is smaller than the solution to Model 2. Indeed, the example in Fig. 3, due to Gal, demonstrates this possibility.

Our model deals with optimal navigation with a compass. We cite in the Introduction several works that discuss navigation models without a compass. To give some mathematical precision to the distinction between a lost swimmer with a compass and a lost swimmer without a compass, consider the following unifying model.

Let $\mathcal{R}$ be a set of transformations of R^2, and let S be an open set in R^2. A path

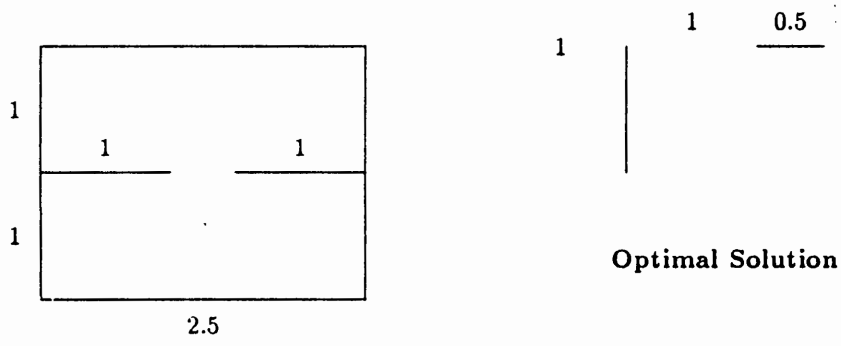

FIG. 3

$f : I \to R^2$ with $f(0) = 0$ will be called $(\mathcal{R}, S)$-uncontained if, for every $x \in S$ and a transformation $r \in \mathcal{R}$, the set $\{x\} + r(f(I))$ intersects ∂S.

In our model of the swimmer with the compass, $\mathcal{R}$ consists only of the identity transformation. The problem of the swimmer without a compass is modeled by letting $\mathcal{R}$ be the group of all rotations. Other interesting cases are when $\mathcal{R}$ is the group of all isometries and when $\mathcal{R} = SL_2(R)$, the group of all linear transformations with a determinant being equal to +1 or -1. The existential result of Theorem 1 can easily be generalized to the above examples of $\mathcal{R}$. Unlike the general result stated in Theorem 5 for convex sets S in our model, finding and verifying an optimal path for a specific set (e.g., a rectangle or a half plane) is fairly involved even while focusing on the swimmer-without-compass model.

We have assumed in all the above models that there is no information about the initial location of the swimmer within the set S. These models must be modified when such information becomes available. For example, if in our original model of navigation with a compass, the swimmer is known to be within a subset S' of S, we require from a path f that only for each $x \in S'$ the set $\{x\} + f(I)$ intersects the boundary of S. The result of Theorem 1 can be extended to this case as well. It is interesting to find sufficient conditions on S and S' that will yield results similar to those stated in Theorem 5.

We have also assumed throughout that the objective is to minimize the maximum path size. In other situations, different objectives may exist, such as minimizing the expected size of the path. In such cases, it is also meaningful to consider probabilistic information on the initial location.

REFERENCES

[1] R. BELLMAN, *Minimization problem*, Bull. Amer. Math. Soc., 62 (1956), p. 270.

[2] F. R. CHUNG AND R. L. GRAHAM, *Steiner trees for ladders*, Ann. Discrete Math., 2 (1978), pp. 173–200.

[3] H. T. CROFT, *Curves intersecting certain sets of great circles on the sphere*, J. London Math. Soc., (2), 1 (1969), pp. 461–469.

[4] N. DUNFORD AND J. T. SCHWARTZ, *Linear Operators, Part* I: *General Theory*, Wiley–Interscience, New York, 1967.

[5] H. G. EGGLESTON, *The maximal inradius of the convex cover of a plane connected set of given length*, Proc. London Math. Soc. (3), 45 (1982), pp. 456–478.

[6] V. FABER, J. MYCIELSKI, AND P. PEDERSEN, *On the shortest curve which meets all the lines which meet a circle*, Ann. Polon. Math., 44 (1984), pp. 249–266.

[7] V. FABER AND J. MYCIELSKI, *The shortest curve that meets all the lines that meet a convex body*, Amer. Math. Monthly, 93 (1986), pp. 796–801.

[8] S. GAL, *Search Games*, Academic Press, New York, 1980.

[9] B. GLUSS, *An alternative solution to the "lost in the sea" problem*, Naval Res. Logist., 8 (1961), pp. 117–121.

[10] ———, *The minimax path in a search for a circle in a plane*, Naval Res. Logist., 10 (1963), pp. 357–360.

[11] O. GROSS, *A search problem due to Bellman*, Rand research memorandum RM–1603 Rand Institute, 1955.

[12] J. R. ISBELL, *An optimal search pattern*, Naval Res. Logist. 4 (1957), pp. 357–359.

[13] H. JORIS, *Le chasseur perdu dans la forêt: un probléme de géométrie plane*, Elem. Math., 35 (1980), pp. 1–14.

[14] P. A. P. MORAN, *On a problem of S. Ulam*, J. London Math. Soc., 21 (1946), pp. 175–179.

[15] F. P. PREPARATA AND M. I. SHAMOS, *Computational Geometry—An Introduction*, Springer-Verlag, New York, 1985.

[16] A. TARSKI, *A Decision Method for Elementary Algebra and Geometry*, 2nd edition, revised, University of California Press, Berkeley, Los Angeles, CA 1951.