Pricing, replenishment, and timing of selling in a market with heterogeneous customers

Shoshana Anily [1] and Refael Hassin [2]

Abstract

We consider a deterministic pricing and replenishment model in which the retailer advertises a fixed price and the selling schedule, and customers can advance or delay their time of purchase incurring holding or shortage costs. We investigate the impact of heterogeneity in the customers' reservation prices. We show that the resulting optimal solution may be very different from that obtained when customers are homogeneous. We identify nine types of possible optimal sales strategies, and compute their profits. In particular, the solution may contain sales at several discrete points of time between consecutive replenishment epochs with no sales between them.

1 Introduction

Classic inventory models customarily assume that clients who do not find the product on the shelf upon their arrival, either quit (lost sales) or wait for the next reorder time (backlogging), see for example §3.3 in Zipkin (2003). Under backlogging, which holds especially for monopolists, customers, arriving when the product is not available, wait for the next replenishment. The retailer is then penalized by a shortage cost consisting of the administrative work involved in handling the shortage and the loss of good will. The shortage cost is often assumed to be proportional to the amount backlogged and possibly also to the backlogging duration.

In the more recent literature it has been recognized that customers are strategic, in particular they time their purchase to maximize their welfare, and profit maximizing sellers respond to the customers' strategy. Some of this literature is reviewed by Shen and Su (2007). In such models dynamic pricing, rationing of the amount of product on the shelves, stockpiling, and timing of sales play cardinal role. For example, short term price promotion may have several effects, like attracting customers of other brands, an increase of consumption, and stockpiling. The latter

[1]Faculty of Management, Tel Aviv University, Tel Aviv 69978, Israel. `anily@post.tau.ac.il`

[2]Department of Statistics and Operations Research, Tel Aviv University, 69987 Tel Aviv, Israel. `hassin@post.tau.ac.il`

means that customers may hold inventory at a cost, in order to consume it later when the price is raised to its regular level. An explanation for the potential benefits of this behavior was given by Eppen and Liebermann (1984): "Under certain conditions, price deals on nonperishable goods can benefit both retailer and customer by transferring part of the inventory holding cost from the former to the latter in return for an unusually low price."

Early papers on stockpiling are Salop and Stiglitz (1982) and Bucovetsky (1983), and more recent contributions include Hendel and Nevo (2004), Bell, Iyer and Padmanabhan (2002), and Lai, Debo and Sycara (2010). Other recent papers also consider forward-looking customers. Su (2007) considers customers who can buy at the current price or delay their purchase at a cost to purchase later. Su and Zhang (2008), Aviv and Pazgal (2008), and Mersereau and Zhang (2012) also consider forward-looking customers who can delay their purchase to enjoy future discounts, but are concerned about product availability. Liu and van Ryzin (2008), Zhang and Cooper (2009), and Cachon and Swinney (2009) consider firms that intentionally understock products to create rationing risk which induces customers to buy earlier than they would otherwise intend. Su and Zhang (2009) consider strategic sellers who use commitments to a particular quantity and compensations to customers during stockouts. Bansal and Maglaras (2009) show how sellers can manipulate the timing of purchase of their customers by dynamic pricing. Mesak, Zhang, and Pullis (2010) discuss demand manipulation by advance selling. Papanastasiou, Bakshi, and Savva (2012) show that strategic stock-outs can be used to influence social learning, leading to higher overall product adoption and increased firm profits. A different line of strategies where consumers' demand can be manipulated by an appropriate inventory holding strategy is described by Balakrishnan, Pangburn and Stavrulaki (2004), where consumers positively react when they observe high levels of inventory.

Our objective is to investigate the effect on the retailer's optimal strategy when customers are strategic, willing to pay for advancing or delaying their purchase due to an anticipated shortage. Examples for such behavior abound and we discuss some of them below. We investigate a basic deterministic problem in which the time horizon is continuous and infinite, the demand rate is constant (price dependent), and the information is assumed to be complete, i.e., the retailer knows the preferences of the customers, and the customers are aware of the price and selling periods. We assume that the price that the retailer chooses is constant over time. This assumption fits many real life situations, for example retailers that adopt the common 'everyday low price' policy (EDLP). Another case where this assumption is natural is described below, referring to the market

consisting of a discount price and customers who only buy at this price.

The model that we consider here is an extension of Glazer and Hassin (1986) where customers who are willing to buy the product can buy it earlier than the time they most desire it and incur inventory holding costs, buy the product later than needed and incur shortage costs, or give-up and leave the system without buying the product. This behavior is not a result of price fluctuations, but it is rather a strategic response to the retailer's policy on when to display the product on the shelf. In other words, the retailer may avoid selling the product in certain periods if this increases its profit. By doing so, it manipulates some of the customers to advance or postpone their purchase. We follow Glazer and Hassin (1986) by assuming a stationary and deterministic model where the seller is restricted to a fixed price. Thus, instead of altering the price dynamically, the seller optimizes profits by restricting sales to certain instants, and customers respond by timing their purchase. For the case of homogeneous customers (except for the time they need the good) Glazer and Hassin (1986) found that the solution may be one of three types: Continuous sales throughout the cycle, sales only at the time of inventory replenishment, or continuous sales through an interval followed by a *no-stock interval*, i.e., an interval in which the firm does not hold stock, that ends with the next replenishment. This finding adds to classic models by explaining real cases in which sellers do not hold inventory at all. Glazer and Hassin (1990) solve the same model but with the objective of maximizing social welfare rather than seller's profits. It is shown that planned shortages may be socially desired, and indeed, a profit maximizer generates *less shortage* than is socially desirable. In particular, the policy of continuous sales throughout the cycle is never socially optimal.

The assumption of identical customers greatly limits the applicability of these results. For example, when prices can be varied, Conlisk, Grestner, and Sobel (1984) and Sobel (1984) show that with heterogeneous customers, by periodically cutting the price sharply the firm can increase its profits by selling to customers with low reservation price. Our motivating question is: *Will the qualitative results obtained by Glazer and Hassin with homogeneous customers still hold when customers differ by their reservation prices?* We identify optimal integrated pricing, replenishment, and selling schedule policy for two types of customers, each associated with its own constant arrival rate and reservation price. We show that the optimal strategy may be significantly different when having heterogeneous customers. We identify nine possible outcomes, eight of them are variations of the possible strategies with homogeneous customers, but the last one contains in between two replenishment epochs, several *no-sale intervals*, i.e., intervals in which the retailer does not sell the

product, separated by sale points.

The latter type of policy may explain why in some cases firms limit their sales to discrete points of time and by doing so they manipulate their customers. We next describe a few examples where the product is sold at a pre-specified set of time epochs: (i) The Tuesday Morning chain of stores is known for its unique philosophy: "sell first-quality, famous designer and name-brand merchandise at extraordinarily discounted prices on an event basis... usually on the first Tuesday of the month". (ii) All over the world it is common to have market days, so that goods are not continuously available. Of course there might be various reasons for regular market days, like resource availability, low demand, and seasonal demand; our model provides another interesting view. (iii) Many firms offer periodical discounts which can be interpreted as discrete sales as mostly in between two discount periods, the sales are low (as the price is high). Such periodic discounting also has been discussed in the bullwhip effect literature. In between periods, the consumers typically forward buying their needs for the whole period. Our model could describe such price discount events. The product is sold continuously, but at certain times the price is vastly reduced. Many customers will try to wait for those discounts. Our model focuses on those customers who always wait for discounts. The other customers are outside the model. The special sales may then be repeated for a very short period (say, a day or a weekend). Since customers are aware of the special sales they will only buy on sales periods and will carry the inventory for future days when the price is high. (iv) The situation we are analyzing might occur in some of the home appliances and office equipment retail chains (e.g., Ace, Home Depot, Office Depot). It is quite common to find in these chains that they sell an item over a certain period, then it is taken off the shelf only to be returned a few weeks later. Sometimes this happens because they get bulk shipments in containers (typically from China) and sometimes there is discontinuity in the shipments of this product. Another explanation for these planned shortages is given by our results. (v) Many computer games (such as the popular Call of Duty series) have a new edition released once or twice a year, causing a period of sales of the original game followed by long no sales intervals. (vi) As our results show it is possible to have an optimal policy where the product is first sold continuously and then it is sold at a number discrete points until the next continuous sales interval. An example for such a mixed behavior is the real-estate industry, and in particular in Hong Kong. The replenishment is the time completion of a project, and the developers adopt various strategies to sell the apartments (see, Lai, Wang and Zhou (2004) for more examples and discussion): One-time clear strategy, is when a big real-estate project is finished, the developer

offers a reasonable price and consumers buy all apartments at the selling date. Usually it just takes one or several days to sell thousands of apartments. Alternatively, in a continuously selling strategy the developer offers a pretty high price and consumers come continuously and it may take one year to sell one thousand apartments. In yet another strategy that is employed, apartments are sold one year before they are used, resulting in consumers who need wait before moving in bearing a very high holding cost. (See the new regulation on selling price and quantity in the Hong Kong real estate market at http://www.info.gov.hk/gia/general/201005/26/P201005260094.htm and in particular Part (c).)

We note however, that the solution type where there are a number of discrete sales epochs in between two consecutive replenishments, is less frequently observed relative to the other eight cases, and one of our goals is to characterize the conditions under which this solution is obtained. In particular, we show that this solution requires that the consumers shortage costs be higher than their inventory holding cost, and that the firm's fixed replenishment cost is high.

The assumption that customers are charged for shortages and for holding inventory, endogenously and indirectly induces a shortage cost for the retailer without necessitating an assessment of unit shortage/backlogging cost by the firm. The strategic use of no-sale intervals may cause some of the customers to quit without buying. Indeed, the higher are the holding/shortage costs faced by the customers, more customers will quit in a no-sales interval, inducing a higher shortage penalty cost on the retailer. The assessment of the customer's inventory/shortage costs is simpler because these are more direct costs than the respective costs of the firm. For example, consider a car dealer who sells an imported model non-continuously. Customers who are aware that the product will not be available at the time they desire it can buy it earlier than needed (incurring financial charges), or wait until it is back in stock, paying the rental cost of an equivalent car. This example demonstrates that the assessment of shortage/holding cost that the customer incurs can be more straightforward than the assessment of the indirect shortage costs incurred by the seller because of loss of reputation.

We consider two customer types, denoted 1-customers and 2-customers, where 1-customers are associated with a higher reservation price. We identify nine possibilities for an optimal solution. Three consist of sales at replenishment instants only, to all 1-customers and to a proportion x of 2-customers, where $x = 0$, $x = 1$ or $0 < x < 1$. Two consist of continuous sales to either only 1-customers, or to all customers. We call policies that consist of sales at replenishment

instances or continuous sales policies - *simple.* Three more possibilities consist of a continuous sales interval followed by a no-stock interval. We call such policies *semi-continuous.* In the three possible optimal semi-continuous policies, sales are made to all 1-customers, to a proportion x of 2-customers appearing in the continuous sales interval, and to a proportion y of the 2-customers appearing in the no-sale interval, where $x = y = 0$, or $x = 1$ with $y = 0$ or $0 < y < 1$. Finally, we identify a possible optimal policy that consists of a continuous sales interval followed by at least two no-sale intervals. The sales at the continuous sales interval are to all customers, and at the no-sale intervals to 1-customers only. We derive the average rate of profits in each case. We also give conditions on the input parameters that restrict the possible optimal strategies.

Finally, we mention the extensive literature on inventory replenishment with capacity limits of the supplier that cause interruptions in the replenishment of the product, see for example, Parlar and Berkin (1991), Wang and Gerchak (1996), Güllü (1998), and Güllü, Önol and Erkip (1999). In these models there are periods when supply is not available or is partially available. Customers are usually uninformed about when these "dry periods" start and end. Closest to our model is the model of Atasoy, Güllü and Tan (2010) that considers a discrete time three-level supply chain where a manufacturer orders supply from an external supplier that may stop selling in certain periods. However, in order to help the manufacturer, the supplier provides him an accurate information about the availability of the supply in the next given number of periods. The manufacturer is facing deterministic periodic demands of customers. The paper considers the manufacturer's problem who needs to plan its own order quantities from the supplier in order to minimize his total expected costs that consist of his ordering costs, plus holding and backorder costs, given the available limited information about the dry periods. In this model, and in ours, the supply is not available at all times, and the manufacturer (in their model) or the customers (in ours) need to decide when and if to buy in order to minimize their costs. Though, there are several differences between this model and ours, and the most crucial one is that in their model the timing and length of supply availability is not strategically planned but result from external random forces.

The paper is organized as follows: In Section 2 we describe the model and present notation with preliminary results. In Section 3 we consider simple policies of selling either only at replenishment instants or continuously through the cycle. In Section 4 we consider semi-continuous policies, where sales are made continuously through the first part of the cycle, followed by a no-stock interval. In Section 5 we introduce a lemma that helps to restrict the search for an optimal solution to policies

with nondecreasing no-sale intervals. In Sections 6-8 we characterize the possible solutions which are neither simple nor semi-continuous depending on whether the customers' holding cost is greater or smaller than their shortage cost. For each policy type we compute the average rate of profit of the policy of this type which can be a candidate for being optimal if certain conditions on the input parameters are satisfied. Altogether we identify nine types of policies. The optimal solution for a given set of input parameters is then the policy with highest value among those whose necessary conditions are satisfied, provided that this value is positive. The results are summarized in the concluding section.

2   The model and preliminaries

We consider a deterministic model with a monopolistic firm that sells a single type of a product at a constant price to two types of customers, i.e., no price discrimination is allowed. The firm incurs a fixed replenishment cost of size K, and a variable cost c. It also pays a linear holding cost of h_f per unit of the product per unit of time. Customers differ by two parameters, the time when they most need the product, which we call their *demand time*, and their reservation price, which is their valuation of the product at that point of time. We assume that the first type of customers is characterized by a reservation price w_1, and by an arrival rate λ_1. The second type of customers is characterized by a reservation price w_2, and by an arrival rate λ_2, where $w_1 > w_2$ and $\lambda = \lambda_1 + \lambda_2$. We assume that each customer needs a single unit of the product. We use the following terminology: Customers that demand the product at t and whose reservation price is w_i are called (t, w_i)-customers. When the time of arrival is not important we simplify the notation and refer to the customer simply as an i-customer, where $i \in \{1, 2\}$.

Customers are ready to buy the product earlier or later than their demand time. Such a deviation comes at a cost. We assume that the costs are linear in the earliness or tardiness duration, similarly to the EOQ model with backlogging where the customer plays the role of the retailer. Specifically, for some positive parameters h_c and s, a (t, w)-customer is ready to pay for it at most $w - h_c \tau$ at time $t - \tau$ and at most $w - s\tau$ at time $t + \tau$ for any $\tau > 0$. This behavior can be interpreted as follows: By obtaining the product before t, the customer incurs an inventory holding cost of h_c per unit of time; by obtaining the product after t, the customer incurs shortage cost of

s per unit of time. We also use

$$H_c = \frac{2}{\frac{1}{s} + \frac{1}{h_c}}$$

for the harmonic mean of the two holding cost parameters of the customer. It is well known that the harmonic mean is bounded from above by the arithmetic mean, i.e., $H_c \leq \frac{h_c + s}{2}$. We are going to see in the analysis that H_c plays the role of an adjusted holding cost rate for the customer for both a positive and negative inventory level. Note that $H_c = h_c$ when $h_c = s$. In addition let,

$$\rho = \frac{s}{h_c + s}.$$

The firm wants to maximize its average rate of profits by choosing a price p, a sales schedule, and a replenishment policy. Like in the EOQ model, there exists an optimal cyclic stationary policy, and without loss of generality we focus on the first cycle $[0, T]$. The Zero-Inventory-Ordering property holds here and therefore a new order is placed after the stock is depleted. However, unlike in the EOQ model, the firm is allowed not to sell the product continuously during the cycle if this increases its average rate of profit. We assume that the policy of the firm is known to the customers and therefore, a customer whose demand-time is at a no-sale point may decide to buy the product earlier or later when it is sold, or alternatively, to quit and not buy it at all. As a result, the optimal policy structure may be such that the stock depletes earlier than at T.

We fully characterize an optimal cyclic solution of our model for two types of customers. We show that there exists an optimal solution where for some $T_I \in [0, T]$, the firm sells continuously up to T_I. If the stock is depleted at T_I, then the interval (T_I, T) is a *no-stock interval*, i.e. an interval in which the firm does not hold any stock. In such a case, $T_I = 0$ means that the firm never holds stock and it sells only at replenishment epochs, and $T_I = T$ means that the firm sells continuously through the cycle. We call these two types of extreme policies *simple policies*:

Definition 1 A policy is *simple* if the sales are continuous or only at replenishment instants.

A semi-continuous policy is obtained if $0 < T_I < T$ and the stock is depleted at T_I, meaning that (T_I, T) is a no-stock interval:

Definition 2 A policy is *semi-continuous-sales*, or for short semi-continuous, if sales are continuous until the stock is depleted.

The interval $(a, a + \Delta)$ is said to be a *no-sale interval* if the firm sells the product only at points a and $a + \Delta$, and nowhere else in the interval. In particular, the no-stock interval (T_I, T) in simple and semi-continuous policies is a no-sale interval. Apart from simple and semi-continuous policies defined above, other possible candidates for optimal cyclic policies exist. Such policies consist of an interval $[0, T_I]$ of continuous sales, $0 \le T_I < T$, and thereafter the stock at T_I is sold at a number of discrete points before T. In other words, such a policy consists of a (possibly empty) interval $[0, T_I]$ of continuous sales, followed by at least two no-sale intervals, where the last one that ends at T is also a no-stock interval.

We distinguish several cases and solve each case separately. The optimal policy is obtained by solving all cases and picking up the best one. We denote the average cost per unit of time of a given policy by V. A policy is *profitable* if its average rate of profit is positive. If no profitable policy exists, it is optimal for the firm to do nothing. In such a case the optimal average profit is 0. We compute nine candidate policies and mark their values as $V^{(1)}, \ldots, V^{(9)}$. These values depend on two input parameters, namely, KH_c and Kh_f. It turns out that, as in Glazer and Hassin (1986), ρ plays a central role in the analysis. If sales are made at the two ends of an interval but not within it, and all customers who arrive within the interval decide to buy the product, ρ is the fraction of customers arriving during the interval and buying the product, who prefer advancing their purchase to its beginning, while the other fraction of $1 - \rho$ defer their purchase to the end of the interval. Note that some of the customers arriving during such an interval may quit without buying the product, however, as we prove, it is never optimal for a profitable policy to contain a sub-interval such that all customers born in it are lost.

Lemma 3 *A profitable optimal policy does not contain a time interval such that all customers born in it are lost.*

Proof: Suppose that the there exists an optimal cyclic profitable policy Π, with price p and cycle $[0, T)$ that contains sub-intervals of total length $0 < \tau < T$ in which all customers are lost. Let $V(\Pi) > 0$ denote the average profit of Π. Consider an alternative policy Π', with a cycle length of $T - \tau$, that is exactly as Π except that all intervals in which all customers are lost are removed from the cycle. The number of customers in a cycle that buy the product in Π and Π' is exactly the same, and thus the revenue per cycle and the fixed cost per cycle are not affected by this change. Moreover, the holding cost per cycle of Π' is bounded from above by the holding cost per cycle of

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Π. Thus, the total profit in a cycle in Π' is at least as large as that of Π, and as the cycle length of Π' is smaller than that of Π, Π' is a strictly better policy, contradicting the optimality of Π. ∎

As we show, it is most common that the optimal policy is either simple or semi-continuous, i.e., it is most likely that inserting no-sale intervals between T_I and the no-stock interval is sub-optimal. However, it turns out that for particular sets of input data this is possible.

We first analyze the behavior of customers who are arriving at a no-sale interval $(a, a+\Delta)$. For a fixed price p, define

$$w(p, \Delta) = p + 0.5 H_c \Delta, \tag{1}$$

and

$$\theta_{a,\Delta} = a + \rho\Delta.$$

Figure 1 illustrates $\theta_{a,\Delta}$ and $w(p, \Delta)$, and a fixed value of w such that $p < w < w(p, \Delta)$. The indifference curve of a (t, w)-customer describes how much such a customer is willing to pay for the product at every instant τ. It raises with slope $h_c = 0.5\frac{H_c}{\rho}$ for $\tau < t$ and decreases with slope $s = 0.5\frac{H_c}{1-\rho}$ for $\tau > t$. Given a no-sale interval $(a, a + \Delta)$, sales to (t, w)-customers with $a + 2\rho\frac{w-p}{H_c} < t < a + \Delta - 2(1-\rho)\frac{w-p}{H_c}$ are lost, (t, w)-customers with $a \leq t \leq \min\{a + 2\rho\frac{w-p}{H_c}, \theta_{a,\Delta}\}$ buy at a, and those with $\max\{a + \Delta - 2(1-\rho)\frac{w-p}{H_c}, \theta_{a,\Delta}\} \leq t \leq a + \Delta$ buy at $a + \Delta$. Observe that for a reservation price $w = w(p, \Delta)$, there is no loss of $w(p, \Delta)$-customers in a no-sale interval $(a, a + \Delta)$. Moreover, since $w(p, \Delta) - h_c \rho \Delta = w(p, \Delta) - s(1 - \rho)\Delta = p$, a $(\theta_{a,\Delta}, w(p, \Delta))$-customer is indifferent among buying at a, buying at $a + \Delta$, and not buying at all. If $\rho > 0.5$ then the sales at a are higher than at $a + \Delta$, and when $\rho \leq 0.5$, the sales at $a + \Delta$ are higher than at a.

From now on we focus on two customer types. Let $\Delta_i = \frac{w_i - p}{\sigma}$ denote the maximum length of a no-sale interval without loss of any i-customers, $i = 1, 2$. In the following we compute the sales volume at the two extreme points of a no-sale interval $(a, a + x)$ from those customers arriving in the interval. The sales volume at a from those customers is

$$\lambda_1 \rho \min(x, \Delta_1) + \lambda_2 \rho \min(x, \Delta_2),$$

The sales volume at $a + x$ from those customers is

$$\lambda_1 (1 - \rho) \min(x, \Delta_1) + \lambda_2 (1 - \rho) \min(x, \Delta_2).$$

In the following sections we derive nine types of policies that may be candidates for the optimal solution.
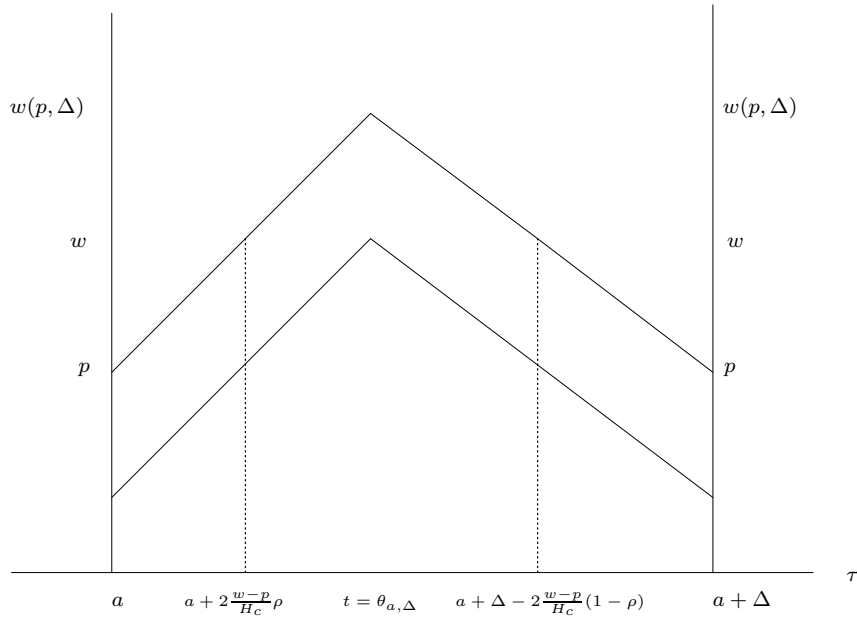
Figure 1: Indifference curves of a $(\theta_{a,\Delta}, w)$-customer and a $(\theta_{a,\Delta}, w(p, \Delta))$-customer

3   Simple solutions

Each of the two kinds of simple policies is considered separately.

3.1   Sales at replenishment instants

There are four cases with sales made only at replenishment instants that need to be considered. Observe that in this case $T \in [\Delta_2, \Delta_1]$, because if $T > \Delta_1$ there is an interval where both types of customers are lost, which is impossible by Lemma 3, and if $T < \Delta_2$ then p can be increased without losing sales.

- Sales only to 1-customers

  This may happen if the seller chooses $p \geq w_2$. In this case, $T = \Delta_1 = \frac{2(w_1 - p)}{H_c}$, or $p = w_1 - 0.5 H_c T$. Thus, $V(T) = \lambda_1[(w_1 - c) - 0.5 H_c T] - \frac{K}{T}$. It is optimized at $T = \sqrt{\frac{2K}{\lambda_1 H_c}}$. If $\sqrt{\frac{0.5 K H_c}{\lambda_1}} > w_1 - w_2$, then $p < w_2$, implying that this solution can be ignored as selling to the two types of customers at the replenishment epochs gives a better solution. The resulting solution is given in Figure 2.

  We next proceed to the cases of sales to both types of customers, i.e. $p < w_2$.

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$$\text{Let } p = w_1 - \sqrt{\frac{KH_c}{2\lambda_1}};$$
$$\text{If } p \geq w_2 \text{ then}$$
$$T = \sqrt{\frac{2K}{\lambda_1 H_c}};$$
$$V^{(1)} = \lambda_1(w_1 - c) - \sqrt{2KH_c\lambda_1}.$$

Figure 2: Sales to 1-customers at replenishment instants

- No loss of customers

  There is no loss of customers if $p < w_2$ and $T \leq \Delta_2$, but clearly $T < \Delta_2$ is suboptimal since in this case p can be increased without losing sales. Hence, $T = \Delta_2$, and $V(T) = \lambda[(w_2 - c) - 0.5H_cT] - \frac{K}{T}$ is optimized at the solution given in Figure 3.

$$p = w_2 - \sqrt{\frac{KH_c}{2\lambda}};$$
$$\text{If } p > c \text{ then}$$
$$T = \sqrt{\frac{2K}{H_c\lambda}};$$
$$V^{(2)} = \lambda(w_2 - c) - \sqrt{2KH_c\lambda}.$$

Figure 3: Sales to all customers at replenishment instants

It is interesting to note the similarity to the EOQ formula: though $V^{(1)}$ and $V^{(2)}$ return the retailer's revenue in the case that the retailer does not hold any inventory, yet its revenue is the same as the revenue of a retailer that sells continuously at price w_1 (w_2) to a rate of λ_1 (λ_2) of customers, and whose holding cost is the adjusted holding cost of the customer, namely H_c.

- Sales to all 1-customers and some 2-customers with $T = \Delta_1$

  Here $T = \Delta_1 = \frac{2(w_1 - p)}{H_c}$ or $p = w_1 - 0.5H_cT$. Note that the maximum price at which the product should be sold in order for all 1-customers to buy it is at most the difference between w_1 and the maximum cost a 1-customer is charged for buying earlier or later than desired. Thus, $\Delta_2 = \frac{2(w_2 - p)}{H_c} = T - \frac{2(w_1 - w_2)}{H_c}$. Hence,

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$$
\begin{aligned}
V(T) &= \frac{1}{T}\{[\lambda_1 T + \lambda_2 \Delta_2][(w_1 - c) - 0.5 H_c T] - K\} \\
&= -0.5\lambda H_c T + [\lambda(w_1 - c) + \lambda_2(w_1 - w_2)] - \frac{1}{T}\left(\lambda_2(w_1 - c)\frac{2(w_1 - w_2)}{H_c} + K\right) \\
&= \alpha_1 T + \alpha_0 + \frac{\alpha_{-1}}{T},
\end{aligned}
$$

where $\alpha_{-1} = -\left(2\lambda_2(w_1 - c)\frac{(w_1 - w_2)}{H_c} + K\right)$, $\alpha_0 = \lambda(w_1 - c) + \lambda_2(w_1 - w_2)$, and $\alpha_1 = -0.5\lambda H_c$.
The optimal solution has $T = \sqrt{\frac{\alpha_{-1}}{\alpha_1}}$ and the profit rate is $\alpha_0 - 2\sqrt{\alpha_{-1}\alpha_1}$, as both α_{-1}
and α_1 are negative. This solution is feasible if $p < w_2$, or equivalently $\lambda(w_1 - w_2)^2 <$
$\lambda_2(w_1 - c)(w_1 - w_2) + 0.5 H_c K$. The solution is given in Figure 4. We do not attempt to
provide an intuitive explanation to the expressing of $V^{(3)}$, but we note that also here the
adjusted holding cost of the customer is induced on the retailer. In fact, if K and H_c are
interchanged then neither the price p nor the average cost $V^{(3)}$ are affected. It is just the
cycle length T that may change.

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$p = w_1 - \sqrt{\frac{\lambda_2(w_1-c)(w_1-w_2)+0.5H_cK}{\lambda}}$;

If $c < p < w_2$ then

$T = \frac{2(w_1-p)}{H_c}$;

$V^{(3)} = \lambda(w_1 - c) + \lambda_2(w_1 - w_2) - 2\sqrt{\lambda[0.5KH_c + \lambda_2(w_1 - c)(w_1 - w_2)]}$.

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Figure 4: Sales to all 1-customers and some 2-customers at replenishment instants

- Sales to all 1-customers and some 2-customers with $\Delta_2 < T < \Delta_1$

Clearly, in this case $p < w_2$. The analysis is as in the previous case but here we also need to
compute the optimal p:

$$
V(T, p) = \frac{1}{T}\{(\lambda_1 T + \lambda_2 \Delta_2)(p - c) - K\} = \frac{1}{T}\left\{\left(\lambda_1 T + \lambda_2 \frac{2(w_2 - p)}{H_c}\right)(p - c) - K\right\}.
$$

Equating to 0 the partial derivative with respect to p gives the optimal value $p = \frac{1}{2}\left(0.5\frac{\lambda_1}{\lambda_2}H_c T + w_2 + c\right)$
and $\Delta_2 = \frac{1}{2}\left(\frac{2(w_2-c)}{H_c} - \frac{\lambda_1}{\lambda_2}T\right)$. $p < w_2$ implies that $T < \frac{\lambda_2}{\lambda_1}\frac{2(w_2-c)}{H_c}$. Let $\gamma = \frac{\lambda_2}{\lambda_1}\frac{2(w_2-c)}{H_c}$.
Thus, we consider the range $T < \gamma$, where $V(T) = \alpha_1 T + \alpha_0 + \frac{\alpha_{-1}}{T}$, with $\alpha_1 = \frac{\lambda_1^2 H_c}{8\lambda_2}$,
$\alpha_0 = 0.5\lambda_1(w_2 - c)$, and $\alpha_{-1} = \frac{\lambda_2(w_2-c)^2}{2H_c} - K$.

If $\alpha_{-1} \le 0$ then $V'(T) > 0$, and T cannot be an internal point in the range $(\Delta_2, \min\{\Delta_1, \gamma\})$.

If $\alpha_{-1} > 0$ then the function $V(T)$ is convex and the maximum again cannot be obtained at an internal point in the range $(\Delta_2, \min\{\Delta_1, \gamma\})$. Thus, there is no internal maximum of $V(T)$, and this case cannot hold.

3.2 Continuous sales

In a continuous sales policy, customers either buy the product at the time they want it, or they don't buy the product. Thus, the customers in this case are not charged for holding stock or shortage. We observe that continuous sales with price $w_2 < p < w_1$ is never optimal because 2-customers do not buy and an increase in p does not cause loss of 1-customers.

- $p = w_1$: This is the maximum value that the price can assume. An optimal policy with $p = w_1$ must be a continuous-sales policy. In such a case only 1-customers buy and $V(T) = \lambda_1(w_1 - c) - 0.5h_f\lambda_1 T - \frac{K}{T}$ is optimized at

$$T = \sqrt{\frac{2K}{\lambda_1 h_f}},$$

giving the solution described in Figure 5.

$$
\boxed{
\begin{array}{l}
p = w_1; \\
T = \sqrt{\frac{2K}{\lambda_1 h_f}}; \\
V^{(4)} = \lambda_1(w_1 - c) - \sqrt{2\lambda_1 K h_f}.
\end{array}
}
$$

Figure 5: Continuous sales to 1-customers

- $p = w_2$: Here both customer types buy the product and $V(T) = \lambda(w_2 - c) - 0.5h_f\lambda T - \frac{K}{T}$ implying that $T = \sqrt{\frac{2K}{\lambda h_f}}$. The solution is given in Figure 6.

In $V^{(4)}$ and $V^{(5)}$ the revenue of the retailer consists of its gross earnings minus the optimal EOQ cost where the holding cost rate is the retailer's holding cost h_f.

$$p = w_2;$$
$$T = \sqrt{\frac{2K}{\lambda h_f}};$$
$$V^{(5)} = \lambda(w_2 - c) - \sqrt{2\lambda K h_f}.$$

Figure 6: Continuous sales to all customers

4 Semi-continuous solutions

In this section we characterize cases in which the optimal solution is semi-continuous (recall Definition 2. We deal separately with three cases according to the position of p relative to the interval $[w_2, w_1)$. We note that $p = w_1$ implies the continuous sales policy to 1-customers, as in Figure 5, and if $p > w_1$ no customers buy the product. Thus it is sufficient to consider $p < w_1$.

4.1 $p < w_2$

Note that in an optimal solution $T - T_I \in [\Delta_2, \Delta_1]$, since a longer no-stock interval causes loss of 1-customers and a shorter one means that it is possible to raise p without affecting the sales pattern. The rate of profit is

$$V(T, T_I, p) = \frac{1}{T} \left\{ (p - c)[\lambda_1 T + \lambda_2 (T_I + \Delta_2)] - h_f \left[0.5 \lambda T_I^2 + \rho T_I [\lambda_1 (T - T_I) + \lambda_2 \Delta_2] \right] - K \right\}.$$

Looking for an internal solution with respect to p, we equate to zero the partial derivative with respect to p, giving

$$p = \frac{1}{2} \left\{ 0.5 H_c \left(\frac{\lambda_1}{\lambda_2} T + T_I \right) + (w_2 + c) + h_f \rho T_I \right\},$$

and therefore, $\Delta_2 = \frac{1}{H_c}[(w_2 - c - 0.5\frac{\lambda_1}{\lambda_2}T H_c) - (0.5 H_c + h_f \rho) T_I]$. These relations imply that $V(T, T_I) = \frac{1}{T} \left[0.5 a H_c T_I^2 + b(T) T_I + d(T) \right]$, where

$$
\begin{aligned}
a &= \frac{1}{4}\lambda_2 + \frac{1}{H_c} h_f \rho \lambda_2 + \frac{1}{(H_c)^2} h_f^2 \rho^2 \lambda_2 - \frac{1}{H_c} h_f \lambda + \frac{0.5}{H_c} h_f \lambda_1 \rho \\
&= \frac{\lambda_2}{(H_c)^2} (0.5 H_c + h_f \rho)^2 - \frac{0.5 h_f}{H_c} \left(\frac{\lambda}{2} - \lambda_1 \rho \right),
\end{aligned}
$$

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$$b(T) = \frac{1}{2}(0.5H_c - h_f\rho)\left(\lambda_1 T + \frac{2\lambda_2(w_2 - c)}{H_c}\right),$$

$$d(T) = \frac{H_c}{8\lambda_2}\left(\lambda_1 T + \frac{2\lambda_2(w_2 - c)}{H_c}\right)^2 - K.$$

Equating to zero the partial derivative of $V(T, T_I)$ with respect to T_I gives $T_I = -\frac{b(T)}{aH_c}$ and for an internal solution we need $a \cdot b(T) < 0$, $T - T_I \in [\Delta_2, \Delta_1]$, and $c < p < w_2$.

Substituting T_I into $V(T, T_I)$ gives

$$V(T) = \frac{1}{T}\left(-\frac{b^2(T)}{2aH_c} + d(T)\right) = 0.5\alpha_1 H_c T + \alpha_0 + \frac{\alpha_{-1}}{T}$$

where

$$\alpha_1 = \frac{\lambda_1^2}{4}\left(-\frac{(0.5H_c - h_f\rho)^2}{a(H_c)^2} + \frac{1}{\lambda_2}\right),$$

$$\alpha_0 = 2\alpha_1\lambda_2\frac{w_2 - c}{\lambda_1},$$

$$\alpha_{-1} = 0.5\alpha_1 H_c\left(\frac{2\lambda_2(w_2 - c)}{\lambda_1 H_c}\right)^2 - K.$$

The sign of $b(T)$ is determined by the sign of $(0.5H_c - h_f\rho) = \rho(h_c - h_f)$.

We consider three cases: $h_c = h_f$; $h_c > h_f$; and $h_c < h_f$.

- $h_c = h_f$: In this case $b(T) = T_I = 0$, implying that an optimal semi-continuous sales policy does not exist.

- $h_c > h_f$: In this case $b(T) > 0$, and we must have $a < 0$ to have an internal optimal solution. This also implies $\alpha_1 > 0$. Now, $T_I = -\frac{b(T)}{aH_c}$ and p are linear increasing functions of T, while Δ_2 and Δ_1 are linear decreasing functions of T.

  $V(T)$ is convex if $\alpha_{-1} \geq 0$ and otherwise it is concave. If it is convex, its maximum is obtained at an extreme point. In this case either T is as large as possible, namely the T that gives $p = w_2$, which is not the case considered here, or T is as small as possible, i.e., when either $T = T_I$ or $T_I = 0$, resulting in a simple solution considered in Section 3. If $\alpha_{-1} < 0$ then $V(T)$ is monotone increasing and concave, meaning again that there is no internal solution.

- $h_c < h_f$: In this case $b(T) < 0$. For an internal solution for T_I to exist, we must have $a > 0$. Also in this case T_I and p are linear increasing functions of T, while Δ_2 is linear decreasing in T.

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- If $\alpha_{-1} \geq 0$, $V(T)$ is convex and therefore, the optimal T is at an extreme value and either $p = w_2$, which is not the case considered here, or $T \in \{0, T_I\}$, which gives a simple solution.

- If $\alpha_{-1} < 0$, $V(T)$ is concave.

  * If $\alpha_1 \geq 0$, $V(T)$ is an increasing function, attaining its maximum at the extreme value where $p = w_2$.

  * If $\alpha_1 < 0$ then the maximum of $V(T)$ is obtained at a value that satisfies $T = \sqrt{\frac{2\alpha_{-1}}{\alpha_1 H_c}}$. This case requires further investigation. Note that $\alpha_1 < 0$ is equivalent to $0 < a < \frac{\lambda_2(0.5H_c - h_f\rho)^2}{(H_c)^2}$. Substituting a in this inequality, $\alpha_1 < 0$ is equivalent to $\rho < 0.5$. Thus, $\rho < 0.5$ implies that both $\alpha_1 < 0$ and $\alpha_{-1} < 0$.

This internal semi-continuous solution is given in Figure 7.

If $\rho < 0.5$, $h_c < h_f$, and if $a > 0$, $c < p < w_2$, and $T - T_I \in [\frac{2(w_2-p)}{H_c}, \frac{2(w_1-p)}{H_c}]$, where

$a = \frac{\lambda_2}{4}\left(1 + \frac{2h_f}{H_c}\rho\right)^2 - \frac{2h_f}{H_c}\left(\frac{\lambda}{2} - \lambda_1\rho\right)$;

$\alpha_1 = \frac{\lambda_1^2}{4}\left(-\frac{1}{4a}\left(1 - \frac{2h_f}{H_c}\rho\right)^2 + \frac{1}{\lambda_2}\right)$;

$T = \sqrt{\left(\frac{2\lambda_2(w_2-c)}{\lambda_1 H_c}\right)^2 - \frac{2K}{\alpha_1 H_c}}$;

$T_I = -\frac{1}{4a}(0.5H_c - h_f\rho)\left(\lambda_1 T + 2\lambda_2\frac{w_2-c}{H_c}\right)$;

$p = \frac{1}{2}\left\{0.5\left(\frac{\lambda_1}{\lambda_2}T + T_I\right)H_c + (w_2 + c) + h_f\rho T_I\right\}$;

then,

$V^{(6)} = 2\alpha_1\left(\lambda_2\frac{w_2-c}{\lambda_1} + \sqrt{\left(\lambda_2\frac{w_2-c}{\lambda_1}\right)^2 - \frac{0.5KH_c}{\alpha_1}}\right)$.

Figure 7: Semi-continuous sales: $p < w_2$

4.2 $p = w_2$

In this case both customer types buy the product during $[0, T_I]$, and only 1-customers buy it during (T_I, T). As we consider here semi-continuous policies, we restrict ourselves to $0 < T_I < T$. Clearly, $T - T_I \leq \Delta_1$ in order to avoid loss of 1-customers.

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$$V(T, T_I) = \lambda_1(w_2 - c) + \frac{1}{T}\left\{\lambda_2(w_2 - c)T_I - h_f\left[\frac{1}{2}\lambda T_I^2 + T_I\rho\lambda_1(T - T_I)\right] - K\right\}$$

$$= \lambda_1(w_2 - c) + \frac{1}{T}\left\{T_I^2 h_f\left(\rho\lambda_1 - \frac{1}{2}\lambda\right) + T_I(\lambda_2(w_2 - c) - h_f\lambda_1\rho T) - K\right\}.$$

Fix T.

- If $2\rho\lambda_1 \leq \lambda$, the function $V(T_I)$ is concave, implying a candidate for an internal maximum, namely $T_I = \frac{\lambda_2(w_2 - c) - \rho\lambda_1 h_f T}{h_f(\lambda - 2\rho\lambda_1)}$. This candidate is relevant (internal) only if $0 < T_I < T$, and $T - T_I < \Delta_1$.

  Substituting T_I in $V(T, T_I)$ gives

  $$V(T) = \lambda_1(w_2 - c) + \frac{1}{T}\left\{\frac{2(\lambda_2(w_2 - c) - h_f\rho\lambda_1 T)^2}{(\lambda - 2\rho\lambda_1)h_f} - K\right\}$$

  $$= \lambda_1(w_2 - c) + \frac{2}{(\lambda - 2\rho\lambda_1)h_f}\left\{\alpha_1 T + \alpha_0 + \frac{\alpha_{-1}}{T}\right\},$$

  where $\alpha_1 = h_f^2\rho^2\lambda_1^2 > 0$, and $\alpha_{-1} = \lambda_2^2(w_2 - c)^2 - 0.5(\lambda - 2\rho\lambda_1)h_f K$. If $\alpha_{-1} \geq 0$ then $V(T)$ is convex, and if $\alpha_{-1} < 0$ then it is monotone increasing. In both cases the maximum is obtained at an extreme value of T where $T_I \in \{0, T, T - \Delta_1\}$. The solutions $T_I \in \{0, T\}$ are simple, and were considered in Section 3. Thus, only $T_I = T - \Delta_1$ is relevant.

- If $2\rho\lambda_1 > \lambda$, the function $V(T_I)$ is convex and its maximum is obtained at a boundary value, $T_I \in \{0, T, T - \Delta_1\}$. As only semi-continuous policies are considered here, we get also here that only $T_I = T - \Delta_1$ is relevant.

Therefore, we continue by substituting $T_I = T - \Delta_1$ into $V(T, T_I)$:

$$V(T) = \lambda_1(w_2 - c) + \frac{1}{T}\left\{\lambda_2(w_2 - c)(T - \Delta_1) - h_f\left[\frac{\lambda}{2}(T - \Delta_1)^2 + (T - \Delta_1)\Delta_1\rho\lambda_1\right] - K\right\}$$

$$\equiv \frac{\alpha_{-1}}{T} + \alpha_0 + \alpha_1 T,$$

where $\alpha_{-1} = \Delta_1^2 h_f(\rho\lambda_1 - 0.5\lambda) - \Delta_1\lambda_2(w_2 - c) - K \leq 0$, $\alpha_0 = \lambda(w_2 - c) + h_f\Delta_1(\lambda - \rho\lambda_1)$, and $\alpha_1 = -0.5h_f\lambda < 0$. If $\alpha_{-1} = 0$, $V(T)$ is decreasing in T, and therefore the maximum is obtained at $T = \Delta_1$, resulting in a simple solution, see Section 3. If $\alpha_{-1} < 0$, then $V(T)$ is concave having an internal maximum with $T = \sqrt{\frac{\alpha_{-1}}{\alpha_1}}$ and profit $\alpha_0 - 2\sqrt{\alpha_1\alpha_{-1}}$, as described in Figure 8.

$$p = w_2;$$

$$\Delta_1 = \frac{2(w_1 - w_2)}{H_c};$$

$$T = \sqrt{\frac{\lambda_2(w_2 - c)\Delta_1 + 0.5h_f\Delta_1^2(\lambda - 2\rho\lambda_1) + A}{0.5h_f\lambda}};$$

$$T_I = T - \Delta_1;$$

If $T_I > 0$ then

$$V^{(7)} = \lambda(w_2 - c) - h_f \frac{2(w_1 - w_2)}{H_c}(-\lambda + \rho\lambda_1) - \sqrt{2\lambda h_f \left[\frac{2(w_1 - w_2)}{H_c}\left(\lambda_2(w_2 - c) + h_f \frac{2(w_1 - w_2)}{H_c}\left(\frac{\lambda}{2} - \rho\lambda_1\right)\right) + K\right]}.$$

Figure 8: Semi-continuous sales: $p = w_2$

4.3 $w_2 < p < w_1$

Here only 1-customers buy. Again, as in this section we deal with semi-continuous policies, we look for solutions with $0 < T_I < T$. In addition, $T - T_I \leq \Delta_1$ in order to avoid loss of 1-customers in some intervals. Thus,

$$V(T, T_I, p) = \lambda_1(p - c) - \frac{1}{T}\left(0.5\lambda_1 h_f T_I^2 + \lambda_1 h_f \rho(T - T_I)T_I + K\right).$$

In an optimal solution $T - T_I = \Delta_1 = \frac{2(w_1 - p)}{H_c}$, otherwise p can be increased to increase profits. Substituting $p = w_1 - 0.5H_c(T - T_I)$ into the cost function we get:

$$V(T, T_I) = \lambda_1\left(w_1 - c - 0.5H_c(T - T_I)\right) - \frac{1}{T}\left(0.5\lambda_1 h_f T_I^2 + \lambda_1 h_f \rho(T - T_I)T_I + K\right).$$

By fixing T we get $V(T_I) = a(T) + bT_I + c(T)T_I^2$, where

$$a(T) = \lambda_1\left(w_1 - c - 0.5H_c T\right) - \frac{K}{T},$$

$$b = \lambda_1(0.5H_c - h_f\rho) > 0,$$

$$d(T) = \frac{\lambda_1 h_f}{T}(\rho - 0.5).$$

The sign of $d(T)$ is determined by the sign of $\rho - 0.5$. If $d(T) \geq 0$ then $V(T_I)$ is convex in T_I, meaning that its maximum is obtained at an extreme value of T_I, namely $T_I \in \{0, T\}$, both are simple solutions that were considered in Section 3. Thus, suppose that $d(T) < 0$, or equivalently

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$\rho < 0.5$. In this case $V(T_I)$ is concave in T_I and a possible internal maximum is

$$T_I = -\frac{b}{2c(T)} = \frac{0.5H_c - h_f\rho}{h_f(1-2\rho)}T.$$

In order for T_I to be positive, and because $\rho < 0.5$, it must hold that $0.5H_c - h_f\rho > 0$, which is equivalent to $h_c > h_f$. In addition we need $T_I < T$, which holds only if $h_f > s$. The two conditions $h_f < h_c$ and $h_f > s$ imply that $\rho < 0.5$. Note also that the condition $p > w_2$ implies that $T - T_I < \frac{2(w_1 - w_2)}{H_c}$.

Substituting the expression for T_I into $V(T, T_I)$ gives

$$V(T) = \lambda_1(w_1 - c) - \frac{K}{T} + \frac{(0.5H_c + h_f\rho)^2 - H_c h_f}{2h_f(1-2\rho)}\lambda_1 T,$$

which is a concave function of T. If the coefficient of T is nonnegative, $V(T)$ is increasing meaning that its maximum is obtained at an extreme point which occurs when $p = w_2$, a case which is not the case considered here. Otherwise, if the coefficient of T is negative, (that is, $(0.5H_c + h_f\rho)^2 < H_c h_f$) $V(T)$ is maximized at $T = \sqrt{\frac{2Kh_f}{\lambda_1}\frac{1-2\rho}{H_c h_f - (0.5H_c + h_f\rho)^2}}$. The resulting solution is given in Figure 9.

$$T = \sqrt{\frac{2Kh_f(1-2\rho)}{\lambda_1\left[H_c h_f - (0.5H_c + h_f\rho)^2\right]}} \ ;$$

$$T_I = \frac{(0.5H_c - h_f\rho)}{h_f(1-2\rho)}T;$$

$$p = w_1 - 0.5H_c(T - T_I);$$

If $s < h_f < h_c$, $(0.5H_c + h_f\rho)^2 < H_c h_f$, $p > w_2$, and $T - T_I \leq \frac{2(w_1-p)}{H_c}$ then

$$V^{(8)} = \lambda_1(w_1 - c) - 2\sqrt{\frac{\lambda_1}{2h_f}\frac{H_c h_f K - (0.5H_c + h_f\rho)^2}{1-2\rho}}.$$

Figure 9: Semi-continuous sales: $w_2 < p < w_1$

5   Solutions which are neither simple nor semi-continuous

In the next lemma we prove that optimal policies which are neither simple nor semi-continuous consist of a single, possibly empty, continuous-sales interval that starts at the replenishment epoch, followed by at least two no-sale intervals. The next lemma provides some further properties of such an optimal solution.

Lemma 4 *There exists an optimal solution where the no-sale intervals are ordered in nondecreasing length. Moreover, a no-sale interval is not followed by an interval of continuous sales (hence there may be at most one continuous-sales interval and it must start at 0).*

Proof: We prove the first part of the lemma. The second part can be considered as a limit case and be proved similarly. Consider consecutive sales at $\tau_0, \tau_0 + x, \tau_0 + x + y$, and suppose that $x > y$. We will show that selling at $\tau_0 + y$ instead of at $\tau_0 + x$ does not decrease profits. W.l.o.g. let $\tau_0 = 0$ and $h_f = 1$. The change does not affect the total sales and therefore we only consider inventory holding costs. We assume in the analysis below that $x + y < T$, since if $x + y = T$ (i.e., $[x, x + y]$ is a no-stock interval) it is clear that the savings associated with selling earlier and thus postponing more sales to T are greater, so that the claim in this case also follows.

Let $C_i(x)$ $(C_i(y))$ be the holding cost associated with i-customers if the product is sold at x $(y,$ respectively). We distinguish three cases:

- $x, y \leq \Delta_i$. In this case

$$
\begin{aligned}
C_i(x) &= \lambda_i\{x[(1-\rho)x + \rho y] + [(x+y)y(1-\rho)]\} \\
&= \lambda_i[(x^2 + y^2)(1-\rho) + xy].
\end{aligned}
$$

  and $C_i(x) = C_i(y)$.

- $y \leq \Delta_i < x$. In this case

$$
C_i(x) = \lambda_i\{x[(1-\rho)\Delta_i + \rho y] + [(x+y)y(1-\rho)]\},
$$

  and

$$
C_i(y) = \lambda_i\{y[(1-\rho)y + \rho\Delta_i] + [(x+y)\Delta_i(1-\rho)]\},
$$

  giving $C_i(x) - C_i(y) = \lambda_i y(x - \Delta_i) > 0$ by our assumption that $x > \Delta_i$.

- $y > \Delta_i$. Also here $C_i(x) > C_i(y)$. The sales to i-customers at $x + y$ are of size $(1-\rho)\Delta_i$, and at the middle sales point (x or y) they are of size Δ_i. But selling at y rather than at x saves in inventory costs.

■

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In view of the lemma, a general cyclic policy for the problem can be represented by a continuous-sales interval $[0, T_I]$, $T_I \geq 0$ followed by $k \geq 0$ no-sale intervals. In Sections 3 and 4 we considered the case $T_I = 0$ and $k = 1$, which is the simple policy with sales only at replenishment instants, the case $T_I = T$ and $k = 0$, which is the simple continuous-sales policy, and the case $0 < T_I < T$ and $k = 1$, which is the semi-continuous policy. Let x_i for $i = 1, \ldots, k$ denote the length of the $i-$th no-sale interval. Thus, $T = T_I + \sum_{i=1}^{k} x_i$. By Lemma 4, w.l.o.g. $x_1 \leq x_2 \leq \cdots \leq x_k$. Thus, the product is sold continuously in $[0, T_I]$, and thereafter positive quantities are sold in discrete points: $T_I + \sum_{i=1}^{\ell} x_i$ for $\ell = 0, 1, \ldots, k$.

6 High customer holding cost: $\rho \leq 0.5$

In this section we characterize the solution assuming $\rho \leq 0.5$, or equivalently, the customers' holding cost rate, h_c, is higher than their backlogging cost rate, s. We prove that in this case there exists an optimal solution which is either simple or semi-continuous. Then we analyze the case $\rho = 0.5$ separately, as we use it in the next section to characterize the optimal solutions when $\rho > 0.5$.

Theorem 5 *If $\rho \leq 0.5$, there exists an optimal policy which is either simple or semi-continuous.*

Proof: Consider an optimal policy such that $T_I > 0$. Suppose also that in the given policy there exists a no-sale interval $[a, a + \Delta]$, such that $0 \leq a < a + \Delta \leq T_I$. Denote by D_a and $D_{a+\Delta}$ the amounts sold at the respective ends of the no-sale interval. According to the analysis of Section 2 and because $\rho \leq 0.5$, $D_a \leq D_{a+\Delta}$. Thus, the average holding cost paid by the firm for these units would decrease if it sold continuously to each of these customers at time they were born. Moreover, continuous sales may cause reneging customers born during the interval to buy as well, resulting in more profits to the firm. ∎

Indeed, in view of Theorem 5, the optimal solution for $\rho \leq 0.5$ can be obtained by enumerating all the solutions $V^{(\ell)}$ for $\ell = 1, \ldots, 8$ and picking up max $\left\{ 0, V^{(1)}, V^{(2)}, \ldots, V^{(8)} \right\}$.

Theorem 6 *If $\rho = 0.5$ and $p \neq w_2$ then the optimal solution is simple.*

Proof: In view of Theorem 5 the optimal solution for $\rho = 0.5$ is either simple or semi-continuous. By Lemma 3, a necessary condition for a semi-continuous policy to be optimal is $p < w_1$. It remains

to show that if $p < w_1$, $\rho = 0.5$ and $p \neq w_2$, there does not exist an optimal semi-continuous policy. If $p < w_2$, the only possible optimal semi-continuous policy obtained requires $\rho < 0.5$, see Figure 7. If $w_2 < p < w_1$, the only possible optimal semi-continuous policy obtained requires $s < h_c$ which is equivalent to $\rho < 0.5$, see Figure 9, concluding the proof. ∎

7 High shortage cost: $\rho > 0.5$ and $p \notin \{w_1, w_2\}$

We now analyze the case where the customer's shortage cost is higher than the customer's holding cost, i.e., $\rho > 0.5$. Theorem 7 states that when $\rho > 0.5$ it is suboptimal to have intervals of continuous sales, unless the optimal price satisfies $p \in \{w_2, w_1\}$.

Theorem 7 *Suppose $\rho > 0.5$. If the optimal price is not in $\{w_2, w_1\}$ then the optimal solution does not contain continuous-sales intervals.*

Proof: First we exclude semi-continuous policies. By Lemma 3 semi-continuous policies may be optimal only for $p < w_1$. According to Figures 7-9, if $p \neq w_2$, there does not exist a semi-continuous optimal policy if $\rho > 0.5$. Moreover, from Figures 5 and 6, a simple policy with continuous sales cannot be optimal. ∎

Theorem 7 implies that if $\rho > 0.5$, the optimal price p satisfies $p < w_1$. In the next two theorems we prove that for $\rho > 0.5$ and $p \neq w_2$ the optimal policy is sales at replenishment instants only.

Theorem 8 *Consider an optimal solution* sol *for an instance I defined by $\rho > 0.5$, c, h_f, h_c, s and K, with monotone non-decreasing no-sale intervals of lengths $x_1 \leq \cdots \leq x_k$. Define the associated instance I' with $c' = c$, $h'_f = h_f$, $K' = K$, and $h'_c = s' = \frac{2h_c s}{h_c + s}$. Note that $\rho' = \frac{1}{2}$ and $h_c^{H'} = H_c$. Denote the profits for these instances by $V(\text{sol})$ and $V'(\text{sol})$. Then $V(\text{sol}) \leq V'(\text{sol})$.*

Proof: Denote by D_i the total sales to customers arriving during the ith no-sale interval. Clearly $D_1 \leq \cdots \leq D_k$, and the sequence $D_1, \ldots, D_k$ depends on H_c but not on ρ. However, ρ does affect the timing of the sales in the sales points. The sales at time $T_I + x_1 + \cdots + x_l$ for $\ell = 1, \ldots, k-1$ are $(1 - \rho)D_l + \rho D_{l+1}$. Since $D_l \leq D_{l+1}$, the coefficients of ρ are nonnegative in all of these cases. It follows that the holding cost with $\rho > \frac{1}{2}$ is at least as large as the holding cost of the firm with $\rho = \frac{1}{2}$ *and the same value of H_c*. Since the total revenue is the same, the profit decreases with ρ, for $\rho > 0.5$, while H_c is kept constant. Therefore $V(\text{sol}) \leq V'(\text{sol})$. ∎

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Theorem 9 *Suppose $\rho > 0.5$ and that the optimal price is not in $\{w_2, w_1\}$. Then optimal solution is sales at replenishment instants only.*

Proof: Consider an instance I with $\rho > 0.5$ and its associated instance I' as in Theorem 8. Consider any non-simple solution sol with monotone nondecreasing no-sale intervals as in Lemma 4. By Theorem 6, if $\rho = 0.5$ and $p \neq w_2$ then there is an optimal simple solution sol$'$ to I'. Then,

$$V(\text{sol}) \leq V'(\text{sol}) \leq V'(\text{sol}') = V(\text{sol}').$$

The first inequality follows from Theorem 8, the second by optimality of sol$'$ to I', and the equality since the value of a simple solution depends on H_c but not on ρ. Therefore sol$'$ is a better solution for I than sol. According to Lemma 7 an optimal solution for I does not contain a continuous-sales interval, implying that the optimal policy for I is simple with sales only at replenishment instants.
∎

Corollary 10 *When $\rho > \frac{1}{2}$ the optimal solution is the best among the simple solutions with discrete or continuous sales, see Figures 2-4, and the best solution obtained under the assumption $p = w_2$.*

8 High shortage cost: $\rho > 0.5$, and $p = w_2$

In this section we consider instances in which the optimal price is $p = w_2$.

Lemma 11 *The optimal solution consists of an initial interval of length $T_I \geq 0$ with continuous sales to both types, and sales to 1-customers only after this. These sales consist of $k \geq 0$ no-sale intervals all of length Δ_1, and possibly another no-sale interval of length $\alpha\Delta_1$ with $0 \leq \alpha < 1$ which starts at T_I.*

Remark 12 This section assumes $p = w_2$, and therefore only 1-customers buy in no-sale intervals. In view of Lemma 3, all no-sale intervals have a length of at most Δ_1. Considering the first case in the proof of Lemma 4, which is the case relevant here as there is no loss of 1-customers, the profit is not affected by the order of the no-sale intervals, except for the last one which must be the longest one.

Proof: By Lemma 4 there exists an optimal solution with nondecreasing no-sale intervals. If the claim does not hold then there exists an index $1 < i \leq k$ such that $x_{i-1} \leq x_i < \Delta_1$. We claim that increasing x_i while decreasing x_{i-1} by the same amount increases the profit. First note that the change does not affect sales since only 1-customers are involved and all no-sale intervals remain bounded by Δ_1. For comparing holding costs assume w.l.o.g. that x_{i-1} starts at 0, and we mark $x = x_1$. Let $x_1 + x_2 = \tau$ and by assumption $x \leq \tau/2$. It is sufficient to show that the holding cost is increasing in x. The inventory holding costs associated with customers born in $[0, \tau]$, for $\tau < T$, in the given solution amount to

$$C_1(x) = \lambda_1 h_f \{x[(1 - \rho)x + \rho(\tau - x)] + \tau(\tau - x)(1 - \rho)\}.$$

If $\tau = T$ the last term in the curly brackets should be removed. The derivative with respect to x for $\tau < T$ is proportional to

$$2(1 - 2\rho)x + \tau[2\rho - 1] \geq 2(1 - 2\rho)\frac{\tau}{2} + (2\rho - 1)\tau = 0,$$

where the inequality follows since $\rho \geq 0.5$ and $x \leq \tau/2$. If $\tau = T$ the derivative with respect to x is proportional to $2x(1 - 2\rho) + \rho\tau \geq \tau(1 - 2\rho) + \rho\tau = \tau(1 - r) \geq 0$ for the same reasons as above. That means that the holding cost is increasing in x. Performing a sequence of changes of this type we end up with a solution as claimed and its cost is not greater than that of the original solution. ∎

Therefore, for $\rho > 0.5$ and $p = w_2$, if the optimal solution is not simple then it falls in one of the following two options:

- A semi-continuous solution with $T_I = T - \Delta_1$ and of profit $V^{(7)}$, see Figure 8.

- A solution which is neither simple nor semi-continuous. Such a solution consists of a continuous-sales interval $[0, T_I]$, $0 \leq T_I < T$, followed by $k+1$ no-sale intervals. The first of these intervals, i.e., the one starting at T_I, may be empty, and in any case its length is strictly less than Δ_1. All the other k no-sale intervals are of length Δ_1. In the sequel of this section we consider this type of policies that are neither simple nor semi-continuous, namely, policies with $k + \lceil \alpha \rceil \geq 2$.

The following observation states that the possibility of $0 < \alpha < 1$ and $k = 0$ can be excluded:

Observation 13 *Consider a problem with $\rho > 0.5$. If there exists an optimal policy with $p = w_2$*

whose cycle contains a single no-sale interval (which is also a no-stock interval), then its length is Δ_1.

The proof follows directly from Figures 2 and 8.

8.1 $0 < \alpha < 1$ **and** $k \geq 1$

In this subsection we prove that the possibility of $0 < \alpha < 1$ can be excluded from consideration also when $k > 0$.

Using Remark 12, we assume without loss of generality that the no-sale interval of length $\alpha\Delta_1$ starts at T_I. Fix T and k, then $T_I + \alpha\Delta_1$ is also fixed at value $T - k\Delta_1$, and $\frac{d\alpha}{dT_I} = -\frac{1}{\Delta_1}$. The terms in the profit function that are affected by the choice of α are: The profit from sales to 2-customers, $\lambda_2(w_2 - c)T_I$; the holding costs on sales in $[0, T_I)$, $0.5h_f(\lambda_1 + \lambda_2)T_I^2$; the holding costs on sales at T_I, $h_f\lambda_1\rho\alpha\Delta_1 T_I$; and the holding costs on sales at $T_I + \alpha\Delta_1$, $h_f\lambda_1(1 - \rho)\alpha\Delta_1(T - k\Delta_1)$. We use $\frac{d\alpha}{dT_I} = -\frac{1}{\Delta_1}$ to obtain that the derivative of the profit V with respect to T_I is proportional to $\lambda_2(w_2 - c) - h_f\left[T_I\lambda_2 + \alpha\lambda_1\Delta_1(2\rho - 1)\right]$.

The second derivative is proportional to $2\rho\lambda_1 - \lambda$.

- Suppose $2\rho\lambda_1 \geq \lambda$. In this case, for any given T and $k \geq 1$, the profit function V is a convex function of T_I and therefore it is maximized at one of the extreme values $\alpha = 0$, $\alpha = 1$.

- Suppose $2\rho\lambda_1 < \lambda$. In this case, for any given T and $k \geq 1$, the profit function V is a concave function of T_I, and therefore the there is another candidate which satisfies the first-order optimality conditions. We will show that this solution cannot be optimal.

  Equating the derivative of V with respect to T_I to 0 gives,
  $$T_I = \frac{w_2 - c}{h_f} - \alpha\Delta_1\lambda_1\frac{2\rho - 1}{\lambda_2}. \tag{2}$$

Thus,
$$T = T_I + (\alpha + k)\Delta_1 = \frac{w_2 - c}{h_f} + k\Delta_1 - \alpha\Delta_1\frac{2\rho\lambda_1 - \lambda}{\lambda_2}. \tag{3}$$

For this solution, with at least two no-sale intervals, to be optimal, the cost of carrying inventory to $T_I + \alpha\Delta_1$ must be smaller than the profit from selling there:
$$T_I + \alpha\Delta_1 = \frac{w_2 - c}{h_f} - \alpha\Delta_1\frac{2\rho\lambda_1 - \lambda}{\lambda_2} \leq \frac{w_2 - c}{h_f},$$

or equivalently $2\rho\lambda_1 \geq \lambda$, contradicting the assumption of the claim. Therefore, an improved solution can be obtained by canceling the sale at $T_I + \alpha\Delta_1$. The effect of this cancellation is that some customers who previously bought there will buy instead at T_I and by that save the firm inventory holding costs. Others who previously bought there will not buy at all, and by assumption this also increases the firm's profit.

8.2 $\alpha = 0$

Denote $V(T) = V_k(T)$ if $T = T_I + k\Delta_1$, where $k \geq 1$. Thus,

$$V_k(T) = \lambda(w_2 - c) - \frac{1}{T}\left\{ \lambda_2(w_2 - c)k\Delta_1 + \frac{1}{2}h_f\lambda T_I^2 + \lambda_1 h_f \left[T_I(k - 1 + \rho)\Delta_1 + k\frac{k-1}{2}\Delta_1^2 \right] + K \right\}$$

$$= \lambda(w_2 - c)\frac{1}{T}\left\{ \lambda_2(w_2 - c)k\Delta_1 + \frac{1}{2}h_f\lambda(T - k\Delta_1)^2 + \lambda_1 h_f\Delta_1 \left[(T - k\Delta_1)(k - 1 + \rho) + k\frac{k-1}{2}\Delta_1 \right] + K \right\}.$$

Substituting $T_I = T - k\Delta_1$ gives $V_k(T) = E - \left(BT + \frac{C}{T} \right)$, where

$E = \lambda(w_2 - c) + (w_1 - w_2)\frac{2h_f}{H_c}[\lambda_2 k + \lambda_1(1 - \rho)]$;

$B = \frac{1}{2}h_f\lambda$;

$C = K + k\Delta_1 \left[\lambda_2(w_2 - c) + \frac{1}{2}h_f\Delta_1(\lambda_2 k - \lambda_1(2\rho - 1)) \right]$;

$BC = \frac{1}{2}\lambda h_f \left(K + \frac{2k(w_1 - w_2)}{H_c} \left[\lambda_2 w_2 + \frac{(w_1 - w_2)h_f}{H_c}(\lambda_2 k - \lambda_1(2\rho - 1)) \right] \right)$.

If $C < 0$ then $V_k(T)$ is decreasing in T and obtains its maximum at the lower boundary, $T = k\Delta_1$ and $T_I = 0$, so that there are sales to 1-customers only. Using the results of Glazer and Hassin (1986) for a single type of customers, such a structure with $k > 1$ is not possible. Therefore, we assume that $C > 0$. In this case $V_k(T)$ is concave with maximum at $T = \sqrt{C/B}$. The solution is given in Figure 10. **Note that $V_1^{(9)} = V^{(7)}$, i.e., the case $k = 1$ is the semi-continuous case of Figure 8.**

9 Summary

The nine possibilities for an optimal solution are:

- Sales at replenishment instants to 1-customers only, with value $V^{(1)}$.

- Sales at replenishment instants without loss of any customer, with value $V^{(2)}$.

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Let

$E = \lambda(w_2 - c) + (w_1 - w_2)\frac{2h_f}{H_c}[\lambda_2 k + \lambda_1(1 - \rho)];$

$B = \frac{1}{2}h_f\lambda;$

$C = K + k\Delta_1\left[\lambda_2(w_2 - c) + \frac{1}{2}h_f\Delta_1(\lambda_2 k - \lambda_1(2\rho - 1))\right].$

If $C > 0$, and $\sqrt{C/B} \geq k\Delta_1$, then

$V_k^{(9)} = E - 2\sqrt{BC}.$

Figure 10: Solution with k no-sale intervals

- Sales at replenishment instants to all 1-customers but only a fraction of the 2-customers, with value $V^{(3)}$.

- Continuous sales to 1-customers only at price w_1 and value $V^{(4)}$.

- Continuous sales to all customers at price w_2, with value $V^{(5)}$.

- Continuous sales to all customers followed by a no-stock interval with sales to all 1-customers but only a fraction of the 2-customers. This is a semi-continuous policy at price lower than w_2, and value $V^{(6)}$.

- Continuous sales to all customers followed by a no-stock interval with sales to 1-customers only. This is a semi-continuous policy at price w_2, and value $V^{(7)}$.

- Continuous sales to 1-customers followed by a no-stock interval with sales to 1-customers only. This is a semi-continuous policy at price higher than w_2 but lower than w_1, and value $V^{(8)}$.

- Continuous sales to all customers at price w_2 during an initial interval followed by k no-sale intervals in which 2-customers are lost, with value $V_k^{(9)}$.

We note that with a single type of customers, Glazer and Hassin (1986) proved that the optimal solution is either simple or semi-continuous. Indeed, multiplicity of no-sale intervals in our generalized model comes with $p = w_2$, meaning that if $w_1 = w_2$ the solution requires continuous sales, without a no-sale interval.
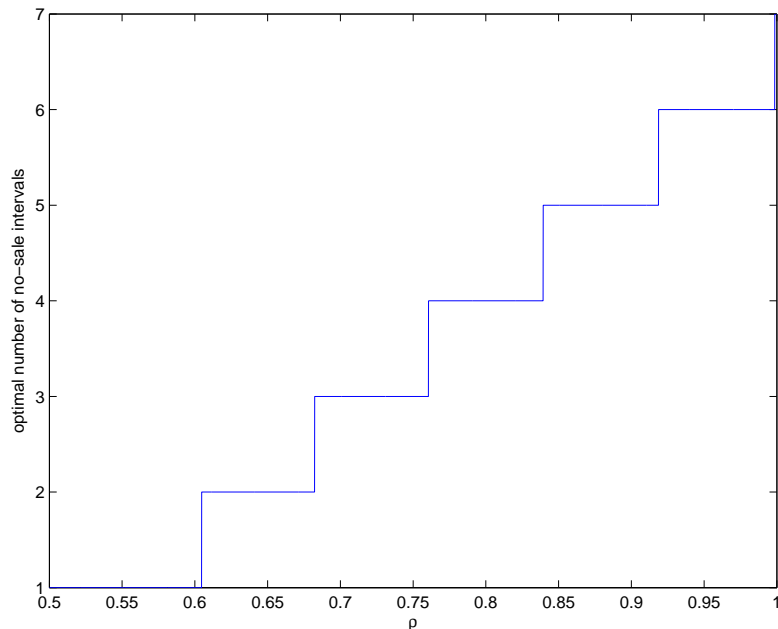
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Figure 11: Optimal number of intervals (above) and the value function (below)

We conducted an intensive computational study, but since there are seven parameters, there doesn't seem to be a meaningful way to describe the results. We concentrate here on the interesting case, which is not possible with homogeneous customers, where $V_k^{(9)}$ is optimal for $k \geq 2$, is obtained only with high values of Kh_f and KH_c, for example when the fixed cost K is high. Figure 11 shows the optimal solution value as a function of ρ, for $(w_1, w_2, \lambda_1, \lambda_2, Kh_f, KH_c, c) = (8.1, 7.7, 19.3, 1.6, 635, 345, 0)$. (For these values of the parameters we have $V^{(i)} \leq 0$ for $i = 1, \ldots, 5$.) The values of $V_k^{(9)}$ are given in Table 1 for $k = 1, \ldots, 7$.

Future research should address the possible structures of optimal policies for a general number of customer types. In particular, is the optimal policy for a discrete number of customer types greater than 2 is still either simple, semi-continuous, or consisting of a continuous sales interval followed by a number of no-sale intervals? A challenging extension assumes a continuous probability distribution $F(w)$ on the proportion of customers with a reservation price no greater than w.

Another line of research that may be of interest is to check the benefit of dynamic pricing and price promotions within the framework of our model. An interesting future question is how customers will react if the next period selling price is uncertain.

k	ρ	$V_1^{(9)}$	$V_2^{(9)}$	$V_3^{(9)}$	$V_4^{(9)}$	$V_5^{(9)}$	$V_6^{(9)}$	$V_7^{(9)}$
1	0.500	5.077	4.940	4.705	4.373	3.948	3.433	2.830
2	0.605	3.730	3.730	3.630	3.430	3.136	2.747	2.268
3	0.682	2.729	2.831	2.831	2.731	2.534	2.241	1.855
4	0.761	1.722	1.928	2.030	2.030	1.930	1.733	1.442
5	0.839	0.708	1.017	1.222	1.323	1.323	1.224	1.028
6	0.919	0	0.101	0.409	0.613	0.714	0.714	0.615
7	0.999	0	0	0	0	0.101	0.202	0.202

Table 1: Minimum $\rho \geq 0.5$ such that k no-sale intervals are optimal, and the values $V_k^{(9)}$ $k = 1, \ldots, 7$ at this ρ. $(w_1, w_2, \lambda_1, \lambda_2, Kh_f, KH_c, c) = (8.1, 7.7, 19.3, 1.6, 635, 345, 0)$.

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