Scheduling maintenance services to three machines

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We study a discrete problem of scheduling activities of three types under the constraint that at most a single activity can be scheduled to any one period. Applications of such a model are the scheduling of maintenance service to machines and multi-item replenishment of stock. We assume that the cost associated with any given type of activity increases linearly with the number of periods since the last execution of this type. The problem is to specify at which periods to execute each of the activity types in order to minimize the long-run average cost per period. We analyze various forms of optimal solution which may occur, relating them to the combination of the three machine cost constants. Some cases remain unsolved by this method and for these we develop a heuristic whose worst case performance is no more than 3.33% from the optimal.

Keywords: scheduling, maintenance

1. Introduction

We consider a situation in which the performance of a machine deteriorates over time until it receives a maintenance service, and just one machine may be serviced at a time. There are three machines, M_1, M_2 and M_3, to be serviced and each with the same service time of one time unit. We consider a linear structure for the cost of operating each machine: Associated with each machine, M_i, there is a cost constant, a_i. The cost of operating M_i during a period in which it is serviced is 0, and the cost of operating it in the jth period after its last service is ja_i. There is no cost associated with maintenance service itself. The problem is to find an optimal policy specifying at which periods to service each of the machines in order to minimize the long-run average operating cost.

We stated the model in terms of "machines" and "servicing" for convenience, but it may be applied equally well in other contexts. One such context is that of multi-item stock replenishment, in which at most one item may be replenished in any one time period. Demand is assumed to be constant, q_i say for the ith item, and the only variable cost is a linear holding cost for each of the items. Let h_i denote the unit holding cost per time period for the ith item. The above model then holds for the infinite horizon, discrete time case by considering time to run in the opposite direction and substituting a_i by $h_i q_i$.

The general case of m machines was investigated in an earlier paper by the authors [2]. There, it was shown that there is always a cyclic optimal solution, consisting of repetitions of a subsequence, and a transformation of the problem into one of computing a minimum mean length cycle in a graph is also given. However, the size of the graph grows with the size of the data in a non-polynomial way. The earlier paper also developed an approximation algorithm with a bounded performance guarantee and another heuristic based on a greedy rule that works well in practice. To date it is not known whether the problem considered in [2] is NP-hard.

In [3], Bar Noy et al. obtain a better performance guarantee for the problem considered in [2]. They also consider a more general problem in which several machines can be serviced during any given period and there is a machine dependent service cost.

A variation of our problem is that in which the cost of operating a M_i during its maintenance is a_i and at the jth period after the last maintenance this cost is $(j+1)a_i$. There is no difference between this variation and the one we consider in terms of optimal solutions, since it merely means increasing the cost per period by $\sum a_i$. However, this increase in cost decreases the ratio between the cost of any pair of solutions and thus better performance guarantees can be obtained (see remark 6.4 in [2] and the broadcast disk application in [3]).

Other generalizations of the model include a general convex cost function, different lengths of times to service different machines and service time being dependent upon time since last service The properties of an optimal policy for the case of two machines are described in [1]. Related problems are treated in [4–6,8,9,11,13]. In [9], there are bounds on the length of intervals between consecutive services to each machine. In [11,13], the service intervals are fixed and the problem is to determine the number of servers needed to form a feasible schedule.

In this paper, our aim is to solve some cases of the three-machine problem to optimality and to deal with the remainder by means of a heuristic for which we present a worst-case performance bound. For convenience, we number the machines so that $a_1 \geq a_2 \geq a_3$.

This paper is organized as follows. In section 2, we present some general properties of an optimal solution for any number of machines and the form of an optimal solution to the two machine problem, which will be required later in the paper. The structure of an optimal solution for the three-machine problem in which the two machines with smallest cost constants, M_2 and M_3, are serviced consecutively, is described in section 3. Such optimal solutions occur when a_1/a_2 is small, in particular less than 2. The other cases which arise when $a_1/a_2 < 6$ are then identified in section 4 and an optimal solution prescribed for each case. This leaves the remaining case of $a_1/a_2 \geq 6$, for which a heuristic is presented in section 5. We show that the worst-case performance of the heuristic is no more than 3.33% from the optimal.

2. Preliminaries and properties of optimal solutions

Our problem consists of three machines, M_1, M_2 and M_3. The cost of operating M_i in the jth period after the last maintenance of that machine is ja_i, for $i = 1, 2, 3$ and $j \geq 0$.

A *policy* P to the m-machine problem is a sequence, $i_1, i_2, \ldots$, where $i_k \in \{1, \ldots, m\}$ for $k = 1, 2, \ldots$ denotes the machine scheduled for service during the kth period. A policy is said to be *cyclic* if it consists of repetitions of a finite sequence $i_1, \ldots, i_T$. Such a sequence is said to *generate* the policy. The minimum length of a generating sequence is denoted by $T(P)$, and any set of $T(P)$ consecutive elements of the sequence is called a *basic cycle* of P. A basic cycle of an optimal cyclic solution is referred to as an *optimal basic cycle*. For a given machine M_i, we shall refer to the time between consecutive services to M_i as the *length* of this i-interval.

Consider for example a cyclic service sequence with a basic cycle 1123. During a basic cycle, M_1 is associated with a single interval of length 1 and an interval of length 3. M_2 and M_3 are associated with intervals of length 4. Therefore, the average cost of the policy is

$$\frac{(a_1 + 2a_1) + (a_2 + 2a_2 + 3a_2) + (a_3 - 2a_3 + 3a_3)}{4} = \frac{3a_1 + 6a_2 + 6a_3}{4}.$$

Without loss of generality, we assume that $a_1 \geq a_2 \geq \ldots \geq a_m$. Moreover, we scale the a_i values so that $a_m = 1$. For a policy P, let $C(t, P)$ denote the average cost over periods $1, \ldots, t$. Clearly, we can restrict ourselves to policies with bounded costs and therefore we can define for each such policy P the lim sup of its sequence of average costs,

$$C(P) = \overline{\lim_{t \to \infty}} C(t, P).$$

A policy is *optimal* if it minimizes $C(P)$. We let C^* denote the average cost of an optimal policy.

We now quote some results from [2]. The first is a fundamental property of optimal solutions to this problem, which will enable us to restrict our attention to cyclic sequences, the second describes the solution for the two machine problem, and the third gives a lower bound on the solution value.

Theorem 2.1. There exists an optimal cyclic solution.

Lemma 2.2. For the two-machine problem, with $a_1 \geq a_2$, there is an optimal cyclic solution in which M_2 is serviced exactly once in a basic cycle and there exists an optimal basic cycle of length τ_2, which is the unique integer satisfying

$$(\tau_2 - 1)\tau_2 \leq 2a_1/a_2 < \tau_2(\tau_2 + 1).$$

The minimum average cost is given by

$$C_{12} = \frac{a_2(\tau_2 - 1)}{2} + \frac{a_1}{\tau_2}.$$

Lemma 2.3. A lower bound on the cost of an optimal solution for the m-machine problem is given by

$$\sum_{i=2}^{m} (i-1)a_i.$$

In particular, for the 3-machine problem,

$$C^* \geq a_2 + 2a_3.$$

Building upon theorem 2.1, we now develop some further properties of an optimal sequence for the general m-machine problem. We shall use $C(S)$ to denote the average cost of a cyclic solution with a basic cycle S and $|S|$ to denote the length of S.

Lemma 2.4. There exists an optimal basic cycle in which no subsequence which contains at least one service to each machine occurs more than once.

Proof. By contradiction. Take an optimal basic cycle S of shortest length. Suppose that S_0 is a subsequence containing at least one service to each machine, which occurs twice in S. Then S is of the form $S_0 S_1 S_0 S_2$ for some, possibly empty, subsequences S_1 and S_2. Subsequences $S_0 S_1$ and $S_0 S_2$ each contain all the machine indices and are therefore basic cycles to solutions to the problem. Moreover, the average cost of S,

$$C(S) = \frac{|S_0 S_1|\, C(S_0 S_1) + |S_0 S_2|\, C(S_0 S_2)}{|S|},$$

is a weighted average of $C(S_0 S_1)$ and $C(S_0 S_2)$, and thus

$$\min(C(S_0 S_1), C(S_0 S_2)) \leq C(S) = C^*.$$

Thus, one of $S_0 S_1$ and $S_0 S_2$ is a basic cycle shorter than S. □

Lemma 2.5. An optimal basic cycle S has the following properties:

(i) Extending S by k periods by inserting extra services to any machine increases the total cost of the basic cycle by at least kC^*.

(ii) Removing from S k services cannot reduce the total cost of the basic cycle by more than kC^*.

Proof. For a solution generated by a basic cycle P, denote by $K(P)$ the total cost during a basic cycle of this solution. Let T denote the length of the given optimal basic cycle S. Let S' denote a basic cycle of length $T+k$ derived from S as described in (i). Since S is optimal, we know that $K(S')/(T+k) \geq K(S)/T = C^*$ and hence $K(S') - K(S) \geq kC^*$, as claimed. Similarly, if S' denotes a basic cycle derived from S in (ii), we have $C^* = K(S)/T \leq K(S')/(T-k)$ and hence $K(S) - K(S') \leq kC^*$. □

From now on we focus on the three-machine problem.

Lemma 2.6. In an optimal basic cycle, a 2-interval contains at most two 3's.

Proof. By contradiction. Suppose that there is an optimal basic cycle S in which M_2 is serviced in periods t_0 and t, and M_3 is serviced in periods t_1, t_2, t_3 in between, i.e. $t_0 < t_1 < t_2 < t_3 < t$. Without loss of generality, we may take t_0 to be 0. Construct a new sequence S' by replacing the service to M_3 in period t_2 by a service to M_2. As a result of this exchange, the total cost due to M_1 is not changed, while the total cost due to M_2 during $T(S)$ periods is reduced by $a_2 t_2(t-t_2)$ and that of M_3 is increased by $a_3 t_2(t_3 - t_2)$. Comparing the average cost of S' to that of S, we obtain

$$C(S) - C(S') = \frac{a_2t_2(t-t_2) - a_3t_2(t_3-t_2)}{T(S)} \geq \frac{a_2t_2(t-t_2) - a_3t_2(t-1-t_2)}{T(S)}$$
$$= \frac{a_2t_2 + (a_2 - a_3)t_2(t-1-t_2)}{T(S)} \geq a_2t_2 > 0.$$

The first inequality follows from $t_3 \leq t-1$ and the last from $a_2 > 0$ and $t_2 > 2$. This contradicts the optimality of S. □

Lemma 2.7. An optimal basic cycle does not contain either of the following subsequences: (i) 22, (ii) 33.

Proof. (i) Suppose otherwise and let S denote an optimal basic cycle containing subsequence 22. Let S' denote the basic cycle got from S by removing one of these 2's. The length of both the 1-interval and the 3-interval spanning 22 in S is at least 3 and each is reduced by 1 in the transformation of the sequence to S'. The decrease in the total cost is therefore at least $2a_1 + 2a_3$. But this is greater than $a_1 + a_2 + a_3$, the average cost of the cyclic solution with basic cycle 123 and hence greater than the optimal average cost, C^*, a contradiction to lemma part (ii).

(ii) Follows by similar argument to (i), but with the roles of machines M_2 and M_3 interchanged. □

3. Structure of optimal solutions containing subsequence 32

In this section, we explore instances of the 3-machine problem in which it is best to leave the most costly machine, M_1, unserviced for more than one period consecutively. First we show that M_1 will not be without service for more than two consecutive periods and that during such a two-period gap, each of the two other machines will be serviced. We then explore the structure of an optimal solution which contains the subsequence 1321 (or 1231). This leads to a description of the structure and cost of an optimal solution to the 3-machine problem containing the subsequence 32 (or 23) which is given in theorems 3.2 and 3.3.

The following lemma allows us to exclude certain subsequences from consideration within an optimal cyclic solution containing a 32 (or a 23).

It will be convenient to adopt the following notation. For a given subsequence S, let l_i (r_i) denote the length of the i-interval to the left (right) of the leftmost (rightmost) service to M_i in S and let w_d denote $\sum_{i=1}^{d-1} i = d(d-1)/2$ so that the total cost of maintenance for machine M_i attributed to an i-interval of length d is $w_d a_i$.

Lemma 3.1. For any problem, there exist optimal cyclic policies that do not contain the following subsequences: (i) 323, (ii) 232, (iii) 31321, (iv) 11321.

Proof. (i) Consider an optimal basic cycle S containing 323 and look for a contradiction. Compare its total cost with that of the one obtained by inserting 21, to give 32123. The cost of 323 excluding the first two periods is at least $3a_1 + a_2$, while for 32123 it is exactly $3a_1 + 2a_2 + 5a_3$. The total cost has been increased by at most $a_2 + 5a_3$, which is no more than $2(a_2 + 2a_3)$ and hence no greater than $2C^*$ by lemma 2.3. Thus, basic cycle S does not satisfy condition (i) of lemma 2.5, producing a contradiction.

(ii) Consider the case when an optimal basic cycle that does not contain 323 (see part (i) of this lemma) does contain the subsequence 232. Then it cannot extend to a 3 and according to lemma 2.7, it cannot extend to a 2, thus it must extend to 12321 within an optimal basic cycle.

Observe that we can assume that $l_1 \leq 3$ for the following reasons: if $l_1 \geq 4$, then the basic cycle extends to xyz12321 where x, y, and z each take the values 2 or 3. Since repeats of either 2 or 3 may be excluded by lemma 2.7, this leaves either 323, which must occur if $l_1 > 4$, or 232. Our assumption that the basic cycle does not contain 323 means that we are left with the case 123212321, which we may exclude since it cannot be better than 2131.

For the subsequence 12321, $l_2 \geq 2$. We first claim that $l_2 \leq 3$. Suppose that $l_2 > 3$, then swap this 2 and the adjacent 1 to get 21321. Note that this swap does not affect the cycle length and the cost due to M_3. Let l_i be defined as above for the sequence 12321. Then the total costs of the affected M_1 and M_2 cycles are: $(w_{l_1} + w_4)a_1 + (w_{l_2} + w_2)a_2$ for 12321 and $(w_{l_1+1} + w_3)a_1 + (w_{l_2-1} + w_3)a_2$ for 21321. The swap therefore costs $(l_1 - 3)a_1 + (3 - l_2)a_2$. Since $l_1 \leq 3$ and $l_2 > 3$, this gives an improvement, which is a contradiction.

If $l_2 = 3$, then we have either 12312321 or 2112321. The former subsequence contains two occurrences of subsequence 123. By lemma 2.4, we can exclude the subsequence since there exists a shorter optimal basic cycle. In the latter sequence, 2112321, swapping the second 1 by the second 2 will strictly improve the cost.

This leaves the case when $l_2 = 2$ and $l_1 = 2$ or 3, since $l_1 \neq l_2 - 1$ and $l_1 \leq 3$. When $l_2 = 2$ and $l_1 = 2$, we have subsequence 1212321 for which the cost in each period, excluding the first three, is $a_1 + (l_3 - 1)a_3$, $2a_1 + a_2$, $3a_1 + a_3$, $a_2 + 2a_3$. The corresponding costs for the sequence 121321 obtained by omitting a 2 are $a_1 + 2a_2$, $2a_1 + a_3$, $a_2 + 2a_3$. Therefore, the transformation reduces the total cost by at least $3a_1 - a_2 + 4a_3$, which is greater than $a_1 + a_2 + a_3$, an upper bound on the optimal value obtained by costing the basic cycle 123, contradicting lemma 2.5 part (ii).

If $l_2 = 2$ and $l_1 = 3$, the sequence 12321 extends either to 12212321, contradicting lemma 2.7, or to 13212321, giving a repetition of 321, contradicting lemma 2.4.

(iii) The subsequence 31321 may not be extended to the left by a 3 from lemma 2.7. We consider the two cases of the subsequence being extended to the left by a 1 and by a 2 separately.

Suppose that an optimal sequence contains 131321. Then $l_2 \geq 5$ and $r_3 \geq 3$. We may exclude the case $r_3 = 3$ (i.e. subsequence 1313213) by comparison with 1321213, since $(w_{l_2} - w_{l_2-2} - w_2)a_2 \geq (w_5 - w_3 - w_2)a_3$ is implied by $l_2 \geq 5$. Thus, $r_2 \geq l_2$ by comparison with 131231, since $(w_{l_2} + w_{r_2})a_2 + (w_2 + w_{r_3})a_3 \leq (w_{l_2-1} + w_{r_2+1})a_2 + (w_3 + w_{r_3-1})a_3$ implies $(l_2 - 1 - r_2)a_2 \leq (3 - r_3)a_3$, which combined with $r_3 > 3$ gives $(l_2 - 1 - r_2) < 0$. Note that from our subsequence 131321 and our observations that $l_2 \geq 5$, $r_2 \geq l_2$ and $r_3 > 3$, we have that the subsequence must be extended to the right by 1 and we obtain 1313211. But these properties imply that 1313121 is no more expensive than 1313211 (since $w_3 a_1 + (w_{l_2} + w_{r_2})a_2 < 2w_2 a_1 + (w_{l_2+1} + w_{r_2-1})a_2$ implies $a_1 < (l_2 - r_2 + 1)a_2 \leq a_2 \leq a_1$, which gives a contradiction). Thus, when the subsequence 131321 appears in an optimal sequence, we may replace it by 131312.

This leaves the case 231321. Consider how this subsequence may extend to the right. Sequence 2313212 cannot be optimal by comparison with 2312312. Consider the sequence 2313211 and compare it to 2312131. The total cost of the intervals which are affected is $w_3a_1 + (w_4 + w_{r_2})a_2 + (w_2 + w_{r_3})a_3$ for 2313211, while 2312131 has total cost $2w_2a_1 + (w_3 + w_{r_2+1})a_2 + (w_4 + w_{r_3-2})a_3$. Changing to 2312131 therefore gives savings of $a_1 + (3 - r_2)a_2 + (2r_3 - 3 - 5)a_3$ which is $\geq a_1 + (3 - r_2)a_2$ as $r_3 > 3$, i.e. a saving of at least $(a_1 - a_2) + (4 - r_2)a_2$. On the other hand, the sequence 2313121 gives savings of at least $(a_1 - a_2) + (r_2 - 4)a_2$ as it has a total cost of $2w_2a_1 + (w_5 + w_{r_2-1})a_2$ associated with M_1 and M_2. We may therefore exclude all but the case of $a_1 = a_2$ and $r_2 = 4$. But then we could renumber the machines by interchanging 1 and 2 and apply lemma 2.7 to exclude this case. Now all cases have been excluded other than 2313213. But the service immediately to the left of this sequence must be a 1 and therefore the sequence must extend to 312313213 applying the above argument to the sequence in reverse. 312313213 cannot be extended to the left or right by 3 according to lemma 3.1. Also, it cannot be extended to the left (right) by 2 to 2312313213 (3123132132) as the subsequence 231 (132) repeats itself, see lemma 2.4. Thus, the extension is 13123132131. This case may be excluded by comparison with 13212312131. To see this, observe that $C(13123132131) < C(13212312131)$ implies $(2w_3 + 2w_2)a_1 + (w_{l_2} + w_4)a_2 + (2w_3 + w_2)a_3 < (2w_3 + 2w_2)a_1 + (w_{l_2-1} + w_2 + w_3)a_2 + 2w_4a_3$, which in turn implies $(l_2 - 1 + 6 - 1 - 3)a_2 < (12 - 7)a_3$ so that $(l_2 + 1)a_2 < 5a_3$, and in view of the fact that $l_2 \geq 4$, we get a contradiction.

(iv) Suppose that an optimal sequence contains the subsequence 11321. The 2- and 3-intervals have the following properties: $l_2 \geq 4$, $l_3 \geq 3$, and $r_3 > l_3$ by comparison with the subsequence 13121 (since $C(11321) < C(13121)$ implies $w_3a_1 + (w_{l_3} + w_{r_3})a_3 < w_2a_1 + (w_{l_3-1} + w_{r_3+1})a_3$). Thus, the next machine to be serviced in the sequence cannot be 3. Moreover, it cannot be machine 2 (i.e. 113212) as the decrease in cost obtained by swapping the adjacent 3 and 2 to get 112312 is at least $(l_2 - 3)a_2 + (r_3 - l_3 - 1)a_3$, which is positive (since $l_2 \geq 4$, $r_3 > l_3$, $a_2 > 0$ and $a_3 > 0$), and hence the swap gives an improvement. This leaves the case 113211.

For 113211 by comparison with costs for 112311, we obtain $(l_2 - 1 - r_2)a_2 \leq (l_3 - r_3 + 1)a_3$ from $(w_{l_2} + w_{r_2})a_2 + (w_{l_3} + w_{r_3})a_3 \leq (w_{l_2-1} + w_{r_2+1})a_2 + (w_{l_3+1} + w_{r_3-1})a_3$. Now using the property $r_3 > l_3$, we get $l_2 \leq r_2 + 1$ (since $(l_2 - 1 - r_2)a_2 \leq (l_3 - r_3 + 1)$ $a_3 < a_3 \leq a_2 \Rightarrow l_2 - 1 - r_2 < 1$). But comparison with costs for 113121 reveals that $r_2 < l_2$ (as $(w_1 + w_3)a_1 + (w_{l_2} + w_{r_2})a_2 < 2w_2a_1 + (w_{l_2+1} + w_{r_2-1})a_2$ implies that $(l_2 - r_2 + 1)a_2 - a_1 > 0$). Thus, $l_2 = r_2 + 1$ and $2a_2 > a_1$. Now look at the extension of the subsequence 113211 to the left. $l_1 \leq 3$ by lemma 2.7 and parts (i) and (ii) of this lemma. If $l_1 = 3$, we have either 132113211 or 123113211. The former cannot be optimal since it represents the sequence generated by 3211 (from part (i) of lemma 2.4) which is always more costly than the one generated by 1312. For the latter sequence 123113211, $l_2 = 5$, thus $r_2 = 4$. Therefore, this sequence cannot be extended to the right by 2 as there will be two consecutive 2's. Also, it cannot be extended to the right by 3 as there will be a repetition of 1132, see lemma 2.4. Thus, this sequence must be extended as follows: 12311321112. In this sequence, $r_3 \geq 6$. It is easy to check that 12311231112 is less expensive.

This leaves the case when $l_1 \leq 2$. To get a contradiction as required, it is therefore sufficient to show that the improvement obtained by changing to the subsequence 213121 is at least $(1 - l_1)a_1 + 2a_2$ (since $a_1 < 2a_2$). Recall that $r_2 + 1 = l_2$. The immediately relevant costs of M_1 and M_2 in the original sequence are $a_1(w_{l_1} + w_3)$ and $a_2(w_{l_2} + w_{r_2})$, while in the other sequence, they are $a_1(w_{l_1+1} + w_2 + w_2)$ and $a_2(w_{l_2-3} + w_4 + w_{r_2-1})$. The decrease in cost due to the change is therefore $(1 - l_1)a_1 + (4l_2 - 14)a_2$ since the difference for a_2 is $l_2 - 1 + l_2 - 2 + l_2 - 3 - 6 + r_2 - 1 = 4l_2 - 14$ and the difference for a_1 is $-l_1 - 1 - 1 + 3 = -l_1 + 1$. The fact that $l_2 \geq 4$ now completes the proof of our claim. □

Theorem 3.2. Machines M_2 and M_3 need be serviced consecutively in an optimal basic cycle only when there are as many services to machine M_1 as to M_2.

Moreover, in this case machine M_3 is serviced at regular intervals, τ_3^* say, and an optimal basic cycle has one of two forms depending on whether τ_3^* is even or odd, namely:

$$321\ldots21$$

or its inverse or

$$3212\ldots123121\ldots21$$

or its inverse.

Proof. Take an optimal basic cycle with 32 in it, ... $\delta\gamma\alpha\,32\beta\varepsilon\ldots$ say. Then both α and β are 1, since all other possibilities are excluded by lemma 2.7 and parts (i) and (ii) of lemma 3.1. Moreover, γ cannot be 3 or 1 from part (iii) and (iv) of lemma 3.1. Thus, γ is 2 and hence δ is either 3 or 1 by lemma 2.7. If δ is 3, then the sequence is 321321ε and 321 is a basic cycle, by lemma 2.4, satisfying the theorem. By a similar argument, the theorem holds with basic cycle 213 if $\varepsilon = 3$ and we therefore need only examine those solutions which contain a subsequence of the form $121321S_r3$ for some subsequence S_r of length at least 1 containing no 3.

Note also that S_r cannot contain a 22. Subsequence $1S_r$ does not contain a 11, since otherwise the whole sequence could be rearranged by moving one of the 1's to between the 3 and the 2 to produce a cheaper (or same cost) schedule. Thus, $1S_r$ must consist of alternating 1's and 2's.

If S_r ends with a 1, then there are two occurrences of 213 in the sequence and hence the sequence $321S_r$ is itself the basic cycle of the sequence by lemma 2.4. It is also of the first form described in the theorem. If this is not the case, then S_r ends in a 2 and we may depict the sequence as $3S_l13212\ldots123$, where S_l is a subsequence containing no 3 and ending with 12. Consider the lengths of the 3-intervals, l_3 and r_3, between consecutive 3's. If they are not equal, then it would incur no extra cost to swap the middle 3 with either the 1 to its left if $l_3 > r_3 \geq 4$, or the 2 to its right if $l_3 < r_3$, implying two occurrences of 123 in the sequence, contradicting lemma 2.4. Thus, we may assume that $l_3 = r_3$.

If S_l starts with a 2, then there are two occurrences of 321 in the sequence, contradicting lemma 2.4. Thus, S_l must start with a 1. We show that S_l does not contain two consecutive 1's. If there are any 1 triples, 111, then we may swap a 1112 with a 1212 in the right-hand part of the cycle without altering the cost of the schedule. But we have just shown that no such subsequence may occur on the right-hand side, giving the required contradiction. Similarly, if there are two 2112's, then we may put them together and exchange them with 2121212 on the right at no extra cost, again giving a contradiction. This leaves the case of just one occurrence of a 11. But in this case, l_3 must be odd, which gives the required contradiction since l_3 and r_3 are equal. It follows that S_l must contain as many 2's as 1's and must begin with a 12. Consider the sequence $3S_l132\ldots(12)3$: lemma 2.7 and lemma 3.1 part (ii) imply that this sequence continues on the right with 1. Lemma 3.1 parts (iii) and (iv) applied on the reverse subsequence of 1231 at the end of $3S_l132\ldots(12)31$ imply that this sequence continues on the right with 2. Therefore, we know that the sequence $3S_l132\ldots(12)3$ continues on the right with 12. The subsequence 312 at the beginning and at the end of our sequence indicate that we have a complete basic cycle. Moreover, this sequence is of the second form, completing the proof. □

The description of an optimal solution for the case when M_2 and M_3 are scheduled consecutively is completed by theorem 3.3. We first need to establish an observation that will be used in the proof of the theorem. Consider the function $g = (\theta/n) + n$ defined for positive integers n. Let $\tilde{g}$ be the extension of g to non-negative real numbers. Thus, $\tilde{g} = (\theta/x) + x$ for $x \in \mathbb{R}^+$. Note that $\tilde{g}$ is convex with a minimum at $x^* = \sqrt{\theta}$. From [12], the unique integer n^* satisfying the following inequalities is an integer optimizer of g: $\sqrt{n^*(n^*-1)} \leq x^* < \sqrt{n^*(n^*+1)}$.

Recall that τ_3^*, the optimal regular interval length for M_3, exists from theorem 3.2.

Theorem 3.3. The optimal solution which contains 32 has basic cycle of length $T = \tau_3^*$ or $T = 2\tau_3^*$ as τ_3^* is odd or even, respectively, where the value of τ_3^* is given by the unique integer t which satisfies

$$t(t-1) \leq 3(a_1 + a_2)/a_3 < t(t+1).$$

The optimal average cost is $(a_1 + a_2 - a_3)/2 + 3(a_1 + a_2)/2t + ta_3/2$.

Proof. If τ_3^* is odd, then a basic cycle is of the form 321...21, the basic cycle is of length $k = \tau_3^*$ and has average cost $(a_1 + a_2)(w_2(k-3)/2 + w_3)/k + a_3 w_k/k$. If τ_3^* is even, then a basic cycle is of form 321...2312...13 and is of length $\tau = 2\tau_3^*$ and has average cost $(a_1 + a_2)(w_2(\tau - 6)/2 + 2w_3)/\tau + 2a_3 w_{\tau/2}/\tau$. This expression is the same as the one above with $\tau = 2k$, and is equal to $(a_1 + a_2 - a_3)/2 + 3(a_1 + a_2)/2k + ka_3/2$. The continuous minimizer of the above function is $\sqrt{3(a_1 + a_2)/a_3}$. □

4. Optimal solutions for $a_1/a_2 < 6$

In the previous section, we analyzed the special case in which the subsequence 32 (or 23) appears in a basic cycle. We now extend the range of basic cycles for which we solve the problem to optimality. In particular, we enumerate all the cases which may arise for relatively small values of a_1/a_2, namely $a_1/a_2 < 6$.

Let d_i denote the length of the interval in between two specific occurrences of i in a sequence.

Theorem 4.1. If $a_1/a_2 < 6$, then an optimal basic cycle is of one of the following forms:

(i) it contains subsequence 32 (or 23) and is of a form described in theorem 3.2; or else

(ii) it has precisely one occurrence of 3, which occurs in subsequence 21312, and intervals 21...12 in one of the following combinations:

(a) all 212;
(b) one 2112 and the rest 212;
(c) two 212 and the rest 2112;
(d) one 212 and the rest 2112;
(e) all 2112;
(f) one 21112 and the rest 2112; or
(g) two 21112 and the rest 2112.

Proof. Suppose that there is no 32 (or 23) in a basic cycle, then we should consider basic cycles with a 131...121 (or equivalently 121...131) with at least one 1 in between the 2 and 3.

If the cycle contains a subsequence 131...121 with at least two 1's in between the 2 and the 3, then compare it, on the one hand, with the subsequence obtained by omitting one of the 1's in between the 2 and the 3, and on the other hand, with the subsequence obtained by inserting a 2 before the 1 preceding 2. Using lemma 2.5, we get

$$a_3(r_3 - 1) + a_2(l_2 - 1) \leq C^* \leq a_1 - a_2(l_2 - 2) + a_3 r_3,$$

implying

$$a_1 \geq a_2(2l_2 - 3) - a_3 \geq a_2(2l_2 - 4) \geq 6a_2.$$

The last inequality follows from the fact that $l_2 \geq 5$ and is in contradiction to our assumption that $a_1/a_2 < 6$. Thus, there is no need for more than one 1 in between the 3 and the 2. Applying the same argument to the reverse subsequence 121...131 implies that for $a_1/a_2 < 6$, 2 and 3 are separated by at most a single 1. Thus, any 2-interval that contains a single 3 must be of the form 21312 and any 2-interval containing more than one 3 must be of the form 12131...3121.

Suppose that an optimal solution contains a 2-interval 1213...13121. By lemma 2.6, the interval does not contain a third 3. Let d_3 (d_2) denote the length of the 3-interval (2-interval) in the sequence. The total savings due to omitting the subsequence starting at the 1 following the first 3 and ending at the second 3, i.e. a subsequence of length d_3 of the form 1...3 is

$$\begin{aligned} &a_1 + a_2(w_{d_2} - w_{d_2 - d_6}) + a_3 w_{d_3} \\ &= a_1 + a_2(w_{d_3+4} - w_4) + a_3 w_{d_3} \\ &= a_1 + a_2 d_3(d_3 + 7)/2 + a_3 d_3(d_3 - 1)/2. \end{aligned}$$

By lemma 2.5, this is no greater than $d_3 C^*$ and hence no greater than $d_3 C(2131) = d_3(2a_1 + 6a_2 + 6a_3)/4$. Thus,

$$a_1(d_3 - 2) \geq a_2 d_3(d_3 + 4) + a_3 d_3(d_3 - 4) > a_2 d_3^2.$$

But this inequality has no solutions for $a_1/a_2 < 6$, giving a contradiction.

Therefore, when $a_1/a_2 < 6$, all occurrences of 3 occur either next to a 2 or within a subsequence 21312. The former case, corresponding to part (i) of the theorem, is covered by theorem 3.2. We therefore concentrate on the latter case. That there is only one occurrence of 3 in a basic cycle as claimed in part (ii) now follows from lemma 2.4. It is left to consider the form of such a basic cycle. Observe that 2-intervals within a basic cycle may occur in any order without affecting the cost of the schedule.

Suppose that an optimal solution contains the subsequence 21...12 with at least two 1's and no 3's in between the two occurrences of 2. Then by comparison with the sequences 21...212 and 21...2 obtained by inserting a 2 and deleting a 1, respectively, we get $(d_2-1)a_2+(d_3-1)a_3 \le C^* \le a_1-(d_2-2)a_2+d_3a_3$, which implies $a_1/a_2 \ge 2d_2-4$.

Thus, when $a_1/a_2 < 2$, we have $d_2 \le 2$ and hence $d_2 = 2$ for intervals which do not contain a 3. This is the combination described in part (ii) (a).

When $a_1/a_2 < 3$, we have $d_2 \le 3$ for 2-intervals which do not contain a 3. But we have no more than one such interval with $d_2 = 3$ since comparing 2112112 and 2121212 gives a saving, as $2w_3a_2 + w_2a_1 > 3w_2a_2 + 2w_2a_1$ when $a_1 < 3a_2$, corresponding to cases (a) or (b).

For $3 \le a_1/a_2 < 4$, 2-intervals with no 3 have $d_2 \le 3$. Moreover, since two 2-intervals of length 3 are not more costly than three of length 2, there are at most two 2-intervals with $d_2 = 2$, i.e. 212, and the rest are 2112 (which demonstrates items (c), (d), (e) of part (ii) of the theorem).

Moreover, when $a_1/a_2 \ge 4$, a basic cycle cannot contain two 2-intervals of length 2 as a single 2-interval of length 4 is as cheap, since in this case $w_4a_2 \le 2w_2a_2 + a_1$ and hence $C(21112) \le C(21212)$.

A basic cycle will not contain one 2-interval of length 2 and another with no 3, of length 4, since two 2-intervals of length 3 are always cheaper. Moreover, since $a_1/a_2 < 6$, $4w_3a_2 + 3a_1 < 3w_4a_2 + 2a_1$ and hence $C(2112112112112) < C(2111211121112)$. It follows that four 2-intervals of length 3 are cheaper than three 2-intervals with no 3, each of length 4. This demonstrates when the cases in items (f) and (g) of part (ii) of the theorem arise. □

Remark 4.2. The categories in theorem 4.1 are not mutually exclusive.

Remark 4.3. The above proof gives more information about the relationship between the value of a_1/a_2 and each type of basic cycle that appears in the statement of the theorem.

Recall the function $g = (\theta/n) + n$ defined for positive integers n. As before, we let $\tilde{g}$ be the extension of g to non-negative real numbers. Thus, $\tilde{g} = (\theta/x) + x$ for $x \in \mathbb{R}^+$ is minimized by $x^* = \sqrt{\theta}$. Also, for any $\gamma \ge 0$,

$$\tilde{g}(\gamma x^*) = \tilde{g}(x^*/\gamma) = \tilde{g}(x^*)/e(\gamma),$$

where $e(\gamma) = 2/(\gamma + \gamma^{-1})$ is quasi-concave with a maximum at $\gamma = 1$. Now, suppose that one wants to minimize $g(n)$ under the constraint that $n \in L$, where L is a subset of the positive integer numbers. By using the above propoerties of the function e, the above problem can be solved by finding the member of l which is the closer to x^* to the left of x^* (if any) and to the right of x^* (if any). If l is non-empty, then we get one or two values which are candidates as minimizers of the given problem. If there is only one candidate, it means that either x^* is larger (smaller) than all the numbers in L or $x^* \in L$. In either of these cases, the candidate is the minimizer. Otherwise, we have two values, one to each side of x^*. Denote these values by n_1 and n_2. The best of the two candidates is the one that maximizes $e(x^*/n)$. From [12], the unique integer n^* satisfying the following inequalities is an integer optimizer of g: $\sqrt{n^*(n^*-1)} \le x^* < \sqrt{n^*(n^*+1)}$. We are now ready to prove the following theorem.

Theorem 4.4. The optimal basic cycles of the types described in part (ii) of theorem 4.1 are of length t, where t is the closest integer to τ^* in the set L, where τ^* and L are defined below:

(a) $\tau^* = \sqrt{8a_2/a_3}$ and $L = \{4 + 2k : k = 0,1,2,\ldots\}$;

(b) $\tau^* = \sqrt{(11a_2 - a_1)/a_3}$ and $L = \{7 + 2k : k = 0,1,2,..\}$;

(c) $\tau^* = \sqrt{8a_1/3a_3}$ and $L = \{8 + 3k : k = 0,1,2,..\}$;

(d) $\tau^* = \sqrt{2(a_1 + a_2)/a_3}$ and $L = \{6 + 3k : k = 0,1,2,\ldots\}$;

(e) $\tau^* = \sqrt{4(a_1 + 3a_2)/3a_3}$ and $L = \{4 + 3k : k = 0,1,2,\ldots\}$;

(f) $\tau^* = \sqrt{2(a_1 + 12a_2)/(3a_3)}$ and $L = \{8 + 3k : k = 0,1,2,\ldots\}$; or

(g) $\tau^* = \sqrt{12a_2/a_3}$ and $L = \{12 + 3k : k = 0,1,2\ldots\}$.

Proof. The average cost functions, $C(\tau)$, for the cases are, respectively:

(a) $(a_1 + a_2 - a_3)/2 + 4a_2/\tau + \tau a_3/2$;

(b) $(a_1 + a_2 - a_3)/2 + (11a_2 - a_1)/2\tau + \tau a_3/2$;

(c) $(a_1/3 + a_2 - a_3/2) + 4a_1/3\tau + \tau a_3/2$;

(d) $(a_1/3 + a_2 - a_3/2) + (a_1 + a_2)/\tau + \tau a_3/2$;

(e) $(a_1/3 + a_2 - a_3/2) + (a_1 + 3a_2)2/3\tau + \tau a_3/2$;

(f) $(a_1/3 + a_2 - a_3/2) + (a_1 + 12a_2)/3\tau + \tau a_3/2$; and

(g) $(a_1/3 + a_2 - a_3/2) + 6a_2/\tau + \tau a_3/2$.

These costs are minimized by the values τ^* given in the theorem. The feasible lengths in each case are computed by considering the structure of the respective sequence.

Since each of the cost functions is convex the minimum feasible cost will be given by one of the integers in L on either side of τ^* and the proof is completed by the observation preceding this theorem. □

Corollary 4.5. For $a_1/a_2 < 6$, we may determine the optimal solution in constant time.

5. A heuristic and its performance

We now describe a heuristic for finding a solution whose value is within 3.33% of the optimal. In many cases, we may use the results proved above to find the optimal solution itself. For the other cases, we develop a heuristic based on the following relaxation of the problem into independent 2-machine problems.

Let $R2$ denote the relaxation of the problem in which we allow machines M_2 and M_3 to be serviced simultaneously. However, we do not relax the condition on the total number of services and therefore every time machines M_2 and M_3 are serviced simultaneously, there must be a corresponding gap with no service somewhere in the schedule. We cost maintenance for machines M_2 and M_3 in the usual way but for machine M_1, a cost of a_1 is incurred with each service to machines M_2 and M_3. In this way, the positioning of the gaps in the maintenance schedule becomes immaterial and consecutive services of machines M_2 and M_3 incur no more cost for maintenance of machine M_1 than separate services.

Let $C(R2)$ denote the average reduced cost of an optimal solution to the relaxed problem $R2$. This cost provides a lower bound to the average cost of the original problem, i.e. $C(R2) \le C^*$. Let $\tilde{\tau}_i$ and C_{1i} denote the basic cycle length and the average cost of an optimal solution to the 2-machine problem involving M_i and M_1.

Lemma 5.1. An optimal solution to $R2$ involves regular services to machine M_i at interval $\tilde{\tau}_i$, for $i = 2, 3$, and $C(R2) = C_{12} + C_{13}$.

Proof. Under $R2$, M_2 and M_3 are independent of each other. For a given M_i, it must be decided at what periods to schedule it, taking into account a fixed cost of a_1 per scheduled period plus the maintenance costs associated with it. However, we have seen that the 2-machine problem involving M_1 and M_i has a solution in which M_i is never served in two consecutive periods. Hence, the actual cost structure is as in the relaxed problem and the minimum average cost is again C_{1i}. $R2$ is therefore equivalent to the amalgamation of 2 independent 2-machine problems for M_i and M_1 with average minimum cost $C_{12} + C_{13}$. □

Algorithm 5.2 (3-machine heuristic, *H*3)

A. If $a_1/a_2 < 6$.
Find the optimal solution as described in theorems 4.1 and 4.4.

B. If $a_1/a_2 \geq 6$.

Step 1. Construct a basic cycle (possibly not feasible), S_2, which is optimal for the relaxation $R2$:

- Schedule services to M_3 at regular intervals of length $\tilde{\tau}_3$ starting at period 1.
- Schedule services to M_3 at regular intervals of length $\tilde{\tau}_2$ starting at period 3.
- Schedule services to machine M_1 in all the gaps except the ones before a period in which M_2 and M_3 are serviced simultaneously.

Step 2. Modify S_2 to make a feasible schedule S_3:

- In periods where M_2 and M_3 are both serviced simultaneously, move the service to M_2 one period to the left and exchange the service to M_3 with the service to M_1 on its right.
- Where 231 occurs, swap the 3 with the 1 to its right.
- Where 132 occurs, swap the 3 with the 1 to its left.

Let $C(H3)$ denote the cost of the solution produced by heuristic $H3$.

Lemma 5.3. When $a_1/a_2 \geq 6$,

$$C(H3) \leq C(R2) + 2(a_2 + a_3)/\tilde{\tau}_2\tilde{\tau}_3.$$

Proof. By construction, the average cost of the basic cycle, S_2, developed in step 1 is $C(R2)$. Moreover, S_2 contains at most one occurrence of 2 and 3 overlapping, one of 12311 and one of 11321, and it is of length $T = lcm(\tilde{\tau}_2, \tilde{\tau}_3)$. Substituting $6 \leq a_1/a_2$ in lemma 2.2, we conclude that $\tilde{\tau}_2 \geq 4$. We distinguish between two cases:
(a) $lcm(\tilde{\tau}_2, \tilde{\tau}_3) = \tilde{\tau}_2\tilde{\tau}_3$. In this case, $T = \tilde{\tau}_2\tilde{\tau}_3$. The schedule may be made feasible by moving the 2 and the 3 which overlap each one place at a cost of $a_2 + a_3$. Moreover, modifying 1132 and 2311 to 1312 and 2131 adds a cost of at most $2a_3 \leq a_2 + a_3$. We have therefore produced a schedule with average cost of $2(a_2 + a_3)/\tilde{\tau}_2\tilde{\tau}_3$ above that of the relaxed problem.
(b) $lcm(\tilde{\tau}_2, \tilde{\tau}_3) < \tilde{\tau}_2, \tilde{\tau}_3$. In this case, there exist integers $p > 1$ and τ_2', τ_3' such that $\tilde{\tau}_i = p\tau_i'$ for $i = 2, 3$ and $gcd(\tau_2', \tau_3') = 1$. Thus, $T = \tilde{\tau}_2\tilde{\tau}_3/p = p\tilde{\tau}_2'\tilde{\tau}_3'$.

We first show that the basic cycle generated in step B1 cannot contain more than one of the three possible occurrences that require a modification by step B2. If 2 and 3 overlap in the schedule, then there exist integers $k_1 \geq 1$ and $l_1 \geq 1$ such that $1 + k_1\tilde{\tau}_3 = 3 + l_1\tilde{\tau}_2$, i.e., $k_1\tilde{\tau}_3 - l_1\tilde{\tau}_2 = 2$, which implies $p = 2$. If the schedule contains 11321, then there exist integers $k_2 \geq 1$ and $l_2 \geq 1$ such that $1 + (1 + k_2\tilde{\tau}_3) = 3 + l_2\tilde{\tau}_2$, i.e., $k_2\tilde{\tau}_3 - l_2\tilde{\tau}_2 = 1$, which contradicts the assumption that $p > 1$. If the schedule contains 12311, then there exist integers $k_3 \geq 1$ and $l_3 \geq 1$ such that $-1 + (1 + k_3\tilde{\tau}_3) = 3 + l_3\tilde{\tau}_2$, i.e. $k_3\tilde{\tau}_3 - l_3\tilde{\tau}_2 = 3$, implying that $p = 3$.

Thus, the modification to the schedule in step B2 incurs an additional cost only if $p = 2$ or 3. In the former case, $T = \tilde{\tau}_2\tilde{\tau}_3/2$ and the additional cost is $a_2 + a_3$, resulting in an increase in the average cost of the schedule relative to the relaxed problem of $(a_2 + a_3)/T = 2(a_2 + a_3)/\tilde{\tau}_2\tilde{\tau}_3$. In the latter case, $T = \tilde{\tau}_2\tilde{\tau}_3/3$ and the cost associated with the modification of the schedule is a_3. Thus, the increase in the average cost of S_3 relative to S_2 is $a_3/T = 3a_3/\tilde{\tau}_2\tilde{\tau}_3 \leq 2(a_2 + a_3)/\tilde{\tau}_2\tilde{\tau}_3$. □

Theorem 5.4.

$$\frac{C(H3)}{C^*} \leq 1\frac{1}{30}.$$

Proof. When $a_1/a_2 < 6$, heuristic $H3$ determines an optimal solution.

When $a_1/a_2 \geq 6$, from the observation that $C(R2) < C^*$ and from lemma 5.3,

$$\frac{C(H3)}{C^*} \leq \frac{C(H3)}{C(R2)} \leq 1 + \frac{2(a_2 + a_3)}{C(R2)\tilde{\tau}_2\tilde{\tau}_3}.$$

Using the value of $C(R2)$ from lemma 5.1, the inequality $(\tilde{\tau}_i - 1)\tilde{\tau}_i/2 \leq a_1/a_i$ for $i = 2, 3$ from lemma 2.2, and $\tilde{\tau}_3 \geq \tilde{\tau}_2$, we obtain the following inequalities:

$$\begin{aligned}\tilde{\tau}_2\tilde{\tau}_3 C(R2) &= \tilde{\tau}_3(a_1 + \tilde{\tau}_2(\tilde{\tau}_2 - 1)a_2/2) + \tilde{\tau}_2(a_1 + \tilde{\tau}_3(\tilde{\tau}_3 - 1)a_3/2)\\ &\geq \tilde{\tau}_2\tilde{\tau}_3 a_2(\tilde{\tau}_2 - 1) + \tilde{\tau}_2\tilde{\tau}_3 a_3(\tilde{\tau}_3 - 1)\\ &\geq (a_2 + a_3)\tilde{\tau}_3(\tilde{\tau}_2 - 1)\tilde{\tau}_2.\end{aligned}$$

Therefore, $C(H3)/C^* \leq 1 + 2/(\tilde{\tau}_2 - 1)\tilde{\tau}_2\tilde{\tau}_3$. Now $\tilde{\tau}_2 \geq 4$, since $6 \leq a_1/a_2 < \tilde{\tau}_2(\tilde{\tau}_2 + 1)/2$, and we may exclude the case $\tilde{\tau}_2 = \tilde{\tau}_3 = 4$, since then $H3$ gives the optimal solution. Thus, the largest possible value of the above expression which may occur is $1 + \frac{2}{3\cdot 4\cdot 5} = 1\frac{1}{30}$. □

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