Information and uncertainty in a queueing system

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Abstract

This paper deals with the effect of information and uncertainty on profits in an unobservable single server queueing system. We consider scenarios in which the service rate, or the service quality, or the waiting conditions are random variables that are known to the server but not to the customers. We ask whether the server is motivated to reveal these parameters. We investigate the structure of the profit function, and its sensitivity to the variance of the random variable. We consider and compare variations of the model according to whether or not the server can modify the service price after observing the realization of the random variable.

1 Introduction

There is a growing interest in queueing models where servers and customers react to each other's action, such as prices and queue discipline set by the server, and arrival process generated in a non-cooperative way by the customers (see [9] for a survey). Much of the literature on this subject assumes that the system's parameters are fixed and known to the participating agents. In other cases, customers own private information on the individual parameters, and one asks whether an adequate pricing system can be used to motivate them to reveal this information to the queue operator. In contrast, this paper deals with the common situation where some parameters are random variables whose realization at a particular time is known to the queue operator but not to the potential customers. For example, the rate of service may vary due to various reasons, the quality of the service may be subject to changes, and even the conditions of waiting in the queue which may be of importance to a customer when deciding on whether or not to join, may vary. The subject of this paper is the effect that this uncertainty has on the decisions made by the server and the customers, and in particular on the expected rate of profit. We ask whether or not the server is motivated to reveal the information, what is the effect of increased uncertainty, and what happens when the server is restricted to a fixed price independent of the realization of the random variable.

Several papers have dealt with the value of information in congested systems. Some have proved that information may reduce customers' welfare. This phenomenon can be broadly interpreted as a result of the non-cooperative way in which customers behave.

Hassin [6] asked whether forcing a profit maximizing server to inform customers about the length of the queue is necessarily beneficial, given that with (or without) this information, customers' behavior is not necessarily optimal. It turns out that, depending on the input parameters, it may or it may not be desired to reveal the length of the queue in front of a profit maximizing server to customers who consider whether or not to join it.

Guo and Zipkin [5] considered customers with time values uniformly distributed on [0,1], and three levels of delay information: no information, length of queue, and exact waiting time.

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They show how to compute the performance measures in the three systems. Their model has no entry pricing, and welfare increases when more accurate delay information is available.

Debo, Parlour, and Rajan [4] consider a firm that knows the quality of the service it provides but cannot credibly communicate it to its potential customers. There are two types of customers, characterized by their private signal on the quality of the service. Customers update their beliefs after observing the length of the queue and decide whether or not to join. The authors show that in general, when waiting in the queue is costly, the equilibrium behavior is not of the threshold type. Other results on the value of information in queueing systems are discussed in [1, 7, 8, 10].

Arnott, de Palma, and Lindsey [2, 3] analyzed effects of information on participation and time-of-use decisions in congestible systems, when capacity and demand fluctuate. Other papers dealing with related questions are mentioned in these references. They conclude that with appropriate price regulation of the system, added information can improve the efficiency of the system. Otherwise, the effect of information may be negative.

In this paper, we consider a structured model of a congested system - that of a Markovian queue managed by a single server who aims to maximize profits. To simplify the derivations we assume that the random variable in question may only obtain two values. For this model, we are able to obtain explicit answers to the effect of information about the system's parameters on the servers profits and system's overall welfare.

We consider an $M/M/1$ system with a large potential demand of risk neutral customers, service rate μ, waiting cost of C per unit time, entry fee T, and value of service R. The queue length is unobservable to the customers while making their decision of whether or not to join it. We consider three types of uncertainty, defined by assuming that a random parameter takes on one of two values. In the first, the capacity of the server, measured by μ is random. In the second, the cost of spending time in the system, as reflected by C, is random. In the third, the quality of the service as reflected by R is a random variable For example, the service rate and quality may depend on the particular server on duty, and the waiting cost may be affected by local time-variable conditions or quality of accommodation.

In each case we consider three sub-models. In the first, the customers are not informed about the realized value of the random parameter. In particular, this means that the server sets a single price independent of the realized value of the parameter, because price differentiation may serve as a signal to the customers about the value of the parameter.[1] In the other two cases, the realized value is revealed to the customers. In the second case, the server may set a different price for each realized value, charging a higher fee for better quality or for faster service. In the third case this is not possible, for example, because the server cannot observe the realized value, or because it is technically impossible or too costly to modify the prices.

2 Main results

For a given set of parameters and information, customers join the system according to a (Nash) equilibrium rate. By our assumption of large potential demand, this rate is such that a customer is indifferent between joining and not joining. This means that both in the case of uninformed customers and in the case of informed customers and two prices, the expected net benefit of a customer, after deduction of the admission fee and waiting cost, is 0. It is well known (see for example [9]) that since in this case the server extracts all the customer surplus, the server's objective coincides with the social objective, and therefore the server's decisions conform with welfare maximization. Note that with informed customers and a single price, customers may have positive surplus and the profit maximizing solution is in general not socially optimal.

Let Π^{un}, Π_1^{in}, and Π_2^{in} denote the server's maximum profit when customers are uninformed or informed, and in the latter case when a single price or two prices are set by the server, respectively. Since customers arrival rate in equilibrium under the profit maximizing policy is socially optimal, additional information to the customers can only increase social welfare. Therefore, $\Pi_2^{in} \geq \Pi^{un}$. Obviously, $\Pi_2^{in} \geq \Pi_1^{in}$. However, when the server is restricted to a single

[1] See [11] for how price may serve as a signal for quality.

price, there is no straightforward answer to whether or not informing the customers increases profits (and social welfare).

We find that with uncertain service rate or waiting cost, $\Pi_1^{in} \geq \Pi^{un}$. However, with uncertain value of service this inequality does not hold in general. In particular, when the waiting cost is sufficiently small we get the reverse inequality.

- For uncertain service rate, we prove (Theorem 4.3) a result which is stronger than the obvious $\Pi_2^{in} \geq \Pi^{un}$: Informing customers is desired even if the price is exogenously fixed, and is not optimally set by the server. Moreover, the difference between the profits obtained with informed and uninformed customers increases with the amount of uncertainty. Note that in such a case customers enjoy a positive consumer surplus and the server's objective differs from the social objective.
- With uncertain waiting cost, for prices that induce positive demand in the case with no information, the same level of social welfare is attained no matter whether customers are informed or not, but informing customers may be beneficial when one of the possible values of the waiting cost is so high that without information no customers will show up.
- With uncertain quality of service and a single price, when the difference between the possible values of service is large, the optimal price is so high that no customer will choose to arrive if the low value is realized. It follows that, for small values of C, the server is motivated to conceal the realized value of service from the customers. For large values of C the opposite is true.

The effect of increasing uncertainty on the three profit functions is different from case to case:

- With informed customers and two prices, increasing the variance has a positive effect on profits and welfare in all three types of uncertainty that we consider.
- With informed customers and a single price, the effect is positive except for when the variance is small and the random variable is the quality of service.
- With uniformed customers, profits are independent of the variance when the random variable is the waiting cost or quality of service. However, with random service rate, profits decrease as the variance increases.

The paper is organized as follows. In Section 3 we describe the well-known outcome of the model when all parameters are known with certainty. Sections 4,5, and 6 deal with uncertain service rate, waiting cost, and quality of service, respectively. Each section is divided into five subsections. The first three deal with the cases of uninformed customers, informed customers with two prices, and informed customers with a single price, respectively. The fourth subsection compares the results and draws conclusions on the effect of the assumptions on the server's expected profit. The last subsection analyzes the effect of uncertainty, as expressed by the variance of the random variable, in each of the three models.

3 Background

For given values of the parameter R, C, μ, and T, if $R - T < \frac{C}{\mu}$ then the equilibrium arrival rate is 0. Otherwise, it satisfies $R - T = \frac{C}{\mu - \lambda}$, or $\lambda = \mu - \frac{C}{R-T}$.

Suppose that R, C, and μ are given, and that T is a decision variable. The rate of profit is $\Pi = \lambda T$. If $R < \frac{C}{\mu}$ then it is not possible to gain any profits. Otherwise, Π is optimized by setting

$$T = R - \sqrt{\frac{RC}{\mu}}. \tag{1}$$

Thus, the maximum rate of profit is

$$\Pi^* = \begin{cases} \left(\sqrt{R\mu} - \sqrt{C}\right)^2 & R \geq \frac{C}{\mu} \\ 0 & R < \frac{C}{\mu} \end{cases} \tag{2}$$

(see, for example, Table 3.2 in [9]).

In the following we measure uncertainty by the variance of the random parameter. Suppose that the uncertain parameter is q and it obtains the values q_1 and q_2 with probabilities p and $1 - p$ respectively. Denote its expected value by $\bar{q}$, then, $Var(q) = p(q_1 - \bar{q})^2 + (1 - p)(\bar{q} - q_2)^2$. We consider two ways of controlling the variance. In one we fix q_1 and obtain desired values of $Var(q)$ and $\bar{q}$ by adjusting p and q_2. In the other we fix p and adjust q_1 and q_2. In both types we maintain a constant value of $\bar{q} = pq_1 + (1 - p)q_2$. Thus, $q_2 = \frac{\bar{q} - pq_1}{1-p}$ and

$$Var(q) = \frac{p}{1-p}(q_1 - \bar{q})^2. \tag{3}$$

4 Uncertain service rate

Assume that $\mu = \mu_1$ with probability p, and $\mu = \mu_2$ with probability $1 - p$, where $\mu_1 > \mu_2$. Let

$$v = \frac{R - T}{C},$$

and

$$r = \frac{R}{C}.$$

Thus, v and r represent the values of service and net gain from service after deduction of service fee, respectively, normalized with respect to the time value.

4.1 Uniformed customers

Without loss of generality,

$$R - T \geq C\left(\frac{p}{\mu_1} + \frac{1-p}{\mu_2}\right), \tag{4}$$

since otherwise $\lambda = 0$ and $\Pi = 0$. The proof of the following lemma is given in the appendix:

Lemma 4.1 *The equilibrium value of v is a root of*

$$\begin{aligned} & (\mu_1 - \mu_2)^2 \left[(\mu_1 + \mu_2)^2 - (\mu_1 - \mu_2)^2\right] v^4 \\ + \quad & 2(1 - 2p)(\mu_1 - \mu_2)\left[(\mu_1 + \mu_2)^2 - (\mu_1 - \mu_2)^2\right] v^3 \\ + \quad & \left[(\mu_1 + \mu_2)^2 - 2r(\mu_1 - \mu_2)^2\left[\mu_1 + \mu_2 + (1 - 2p)(\mu_1 - \mu_2)\right] - (1 - 2p)^2(\mu_1 - \mu_2)^2\right] v^2 \\ + \quad & 2r(\mu_1 - \mu_2)\left[-2(1 - 2p)(\mu_1 + \mu_2) - (1 - 2p)^2(\mu_1 - \mu_2) - (\mu_1 - \mu_2)\right] v \\ + \quad & r\left[r(\mu_1 - \mu_2)^2 - 2(\mu_1 + \mu_2) - r(1 - 2p)^2(\mu_1 - \mu_2)^2 - 2(1 - 2p)(\mu_1 - \mu_2)\right] = 0. \end{aligned}$$

We solve the above polynomial for v and then compute $\Pi^{un} = \lambda T = \lambda(R - Cv)$.[2]

4.2 Informed customers - two prices

Suppose that customers are informed about the service rate, and the server charges different prices depending on the realization of μ. In this case we use (2) to obtain that the server's rate of profit is

$$\Pi_2^{in} = \begin{cases} pC\left(\sqrt{r\mu_1} - 1\right)^2 + (1-p)C\left(\sqrt{r\mu_2} - 1\right)^2 & r \geq \frac{1}{\mu_2} \\ pC\left(\sqrt{r\mu_1} - 1\right)^2 & \frac{1}{\mu_1} < r < \frac{1}{\mu_2} \\ 0 & r \leq \frac{1}{\mu_1}\ . \end{cases} \tag{5}$$

4.3 Informed customers - single price

Assume that the firm must set a fixed price, and cannot change it when the value of μ is revealed. Customers are informed however about the realized value of μ.

Assume first that the price T is not too large, so that $\lambda > 0$ for both values of μ. The arrival rate, given $\mu = \mu_i$, is $\lambda_i = \mu_i - \frac{C}{R-T}$. The expected arrival rate, given T, is

$$\begin{aligned} \lambda &= p\left(\mu_1 - \frac{C}{R-T}\right) + (1-p)\left(\mu_2 - \frac{C}{R-T}\right) \\ &= p\mu_1 + (1-p)\mu_2 - \frac{C}{R-T}. \end{aligned} \tag{6}$$

Define $\bar{\mu} = p\mu_1 + (1-p)\mu_2$. The expected rate of profit, given T, is $\Pi(T) = T\lambda$. Its derivative with respect to T is

$$\bar{\mu} - \frac{C}{R-T} - \frac{TC}{(R-T)^2}.$$

The first-order optimality condition is $\frac{\bar{\mu}}{C}(R-T)^2 - R = 0$, which gives

$$T = R \pm \sqrt{\frac{RC}{\bar{\mu}}}.$$

Since $T < R$ is required, only the root associated with the minus is relevant, and $\lambda = \bar{\mu} - \sqrt{\frac{C\bar{\mu}}{R}}$. Therefore,

$$\Pi = \left(R - \sqrt{\frac{RC}{\bar{\mu}}}\right)\left(\bar{\mu} - \sqrt{\frac{C\bar{\mu}}{R}}\right) = C\left(\sqrt{\frac{\bar{\mu}R}{C}} - 1\right)^2. \tag{7}$$

The firm may choose however a high price which doesn't attract any customers if $\mu = \mu_2$. In this situation, the firm will set the price to gain maximum profit when the high rate is realized. Thus, $T = R - \sqrt{\frac{CR}{\mu_1}}$, as in (1), and

$$\Pi = pC\left(\sqrt{\frac{R\mu_1}{C}} - 1\right)^2. \tag{8}$$

The firm will choose the solution giving the higher value between (7) and (8). The two values are equal if

$$\left(\sqrt{\frac{\bar{\mu}R}{C}} - 1\right)^2 = p\left(\sqrt{\frac{R\mu_1}{C}} - 1\right)^2,$$

[2] In our numerical solution we obtained at most two real roots that satisfy (4), only one of which corresponds to an equilibrium, see Remark6.1.

or

$$\sqrt{\bar{\mu}r} - 1 = \sqrt{p}\left(\sqrt{r\mu_1} - 1\right).$$

Equivalently, $r = \eta$, where

$$\eta = \left(\frac{1 - \sqrt{p}}{\sqrt{\bar{\mu}} - \sqrt{\mu_1 p}}\right)^2.$$

Remark 4.2 By concavity of the square root function, $p\sqrt{\mu_1} + (1-p)\sqrt{\mu_2} > \sqrt{\bar{\mu}}$. This inequality can be used to show that $\sqrt{\mu_1 p} + \sqrt{\mu_2} - \sqrt{\mu_2 p} > \sqrt{\bar{\mu}}$ or $\eta > \frac{1}{\mu_2}$, for any $p \in (0,1)$.

Denote the resulting profit by Π_1^{in}, then

$$\Pi_1^{in} = \begin{cases} C\left(\sqrt{r\bar{\mu}} - 1\right)^2 & r \geq \eta \\ pC\left(\sqrt{r\mu_1} - 1\right)^2 & \frac{1}{\mu_1} \leq r \leq \eta \\ 0 & r \leq \frac{1}{\mu_1}\ . \end{cases} \tag{9}$$

4.4 Discussion

Denote by $\lambda^{in}(T)$ the equilibrium expected arrival rate when customers are informed and there is a single price T. Similarly, let $\lambda^{un}(T)$ be the arrival rate when customers are not informed.

Theorem 4.3 *Consider fixed input values of p, μ_1, and μ_2. For every price T, $0 < T < R - C\left(\frac{p}{\mu_1} + \frac{1-p}{\mu_2}\right)$, $\lambda^{in}(T) \geq \lambda^{un}(T)$.*

Proof: By (6) and (19), we need to prove that $\left(\bar{\mu} - \frac{1}{v}\right) - \frac{1}{2}\left(-\frac{1}{v} + \mu_1 + \mu_2 - S\right) \geq 0$, or that $S - \left[\frac{1}{v} + (\mu_1 - \mu_2)(1-2p)\right] \geq 0$. It is sufficient to show that

$$\left[\frac{1}{v^2} + (\mu_1 - \mu_2)^2 + \frac{2}{v}(1-2p)(\mu_1 - \mu_2)\right] - \left(\frac{1}{v} + (\mu_1 - \mu_2)(1-2p)\right)^2 \geq 0.$$

This is equivalent to $1 - (1-2p)^2 \geq 0$, and the claim follows. ∎

Note that the inequality in the theorem is strict except for when $p \in \{0,1\}$, the case with no uncertainty. The difference in the expected arrival rates is concave and it is maximized when $p = \frac{1}{2}$. Thus, we may say that *the difference in the expected arrival rates increases with the amount of uncertainty.*

Also note that the range of prices in the theorem is the allowed range for the uninformed customers case. For the informed customers case, the allowed range is wider, and in the added values we have $\lambda^{in}(T) > 0$ whereas $\lambda^{un}(T) = 0$, conforming with the spirit of the theorem.

Corollary 4.4 $\Pi_2^{in} \geq \Pi_1^{in} \geq \Pi^{un}$. *Therefore, the firm is motivated to reveal the realized value of μ to its customers.*

Proof: The first inequality is obvious, and the second follows from Theorem 4.3 ∎

The relative behavior of the profit functions under the three models differs in different regions of r as follows (we use here the inequality $\eta > \frac{1}{\mu_2}$ given in Remark 4.2):

- If $r \in \left(0, \frac{1}{\mu_1}\right)$ then $\Pi^{un} = \Pi_1^{in} = \Pi_2^{in} = 0$.
- If $r \in \left(\frac{1}{\mu_1}, \frac{p}{\mu_1} + \frac{1-p}{\mu_2}\right)$ then $0 = \Pi^{un} < \Pi_1^{in} = \Pi_2^{in}$.
- If $r \in \left(\frac{p}{\mu_1} + \frac{1-p}{\mu_2}, \frac{1}{\mu_2}\right)$ then $0 < \Pi^{un} < \Pi_1^{in} = \Pi_2^{in}$.
- If $r \in \left(\frac{1}{\mu_2}, \infty\right)$ then $0 < \Pi^{un} < \Pi_1^{in} < \Pi_2^{in}$. The difference $\Pi_2^{in} - \Pi_1^{in}$ is equal to $kC\sqrt{r}$ where $k = 2\left(\sqrt{\bar{\mu}} - p\sqrt{\mu_1} - (1-p)\sqrt{\mu_2}\right)$.

Figure 1 (left) gives the expected rate of profit for $p = 0.2$, $C = 30$, $\mu_1 = 1.5$ and $\mu_2 = 0.5$.[3]

Remark 4.5 Clearly, when $p = 1$ we have $\Pi^{un} = \Pi_1^{in} = \Pi_2^{in}$. However, $\lim_{p\to1} \Pi^{un}(p) < \Pi^{un}(1)$, whereas $\lim_{p\to1} \Pi_j^{in}(p) = \Pi_j^{in}(1)$, for $j = 1, 2$. The functions when $p \to 1$ are illustrated in Figure 1 (right). The reason is that when $p = 1$ we may have $\mu_2 < \lambda < \mu_1$, whereas for any $p < 1$ we must have $\lambda < \mu_2$ to guarantee a finite value for the expected waiting time.[4] Actually, the value $\lambda = \mu_1 - \sqrt{\frac{c\mu_1}{R}}$ for $p = 1$, is given by the root corresponding to the plus sign in (18).

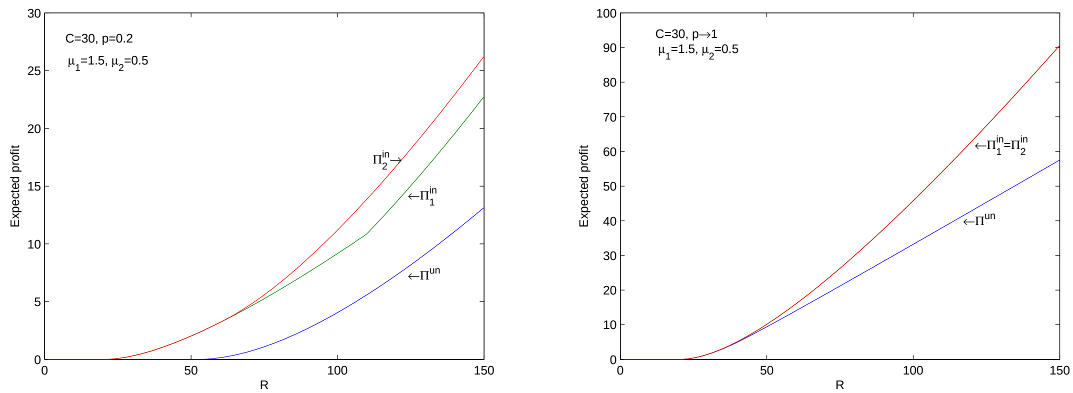

Figure 1: Profits when $C = 30$ and $\mu_1 = 0.5$ (left) and when $p \to 1$ (right)

4.5 The effect of uncertainty

We consider two types of changes in uncertainty. In both types we fix $\bar{\mu}$.

In the first case we also fix $\mu_1 > \bar{\mu}$ (recall that $\mu_1 > \mu_2$), and simultaneously change p and μ_2 so that $\bar{\mu}$ is preserved at the same level and $Var(\mu)$ varies. In Figure 2 (left) we assumed $\bar{\mu} = 2$ and $\mu_1 = 2.5$, so that by (3) $Var(\mu) = \frac{p}{4(1-p)}$. We note that this is an increasing function of p. In the second case we fix p and modify μ_1 and μ_2, again preserving $\bar{\mu}$. In Figure 2 (right) we set $p = 0.5$ and $\bar{\mu} = 2$, giving $Var(\mu) = (\mu_1 - 2)^2$.

As expected from (9), Π_1^{in} depends only on $\bar{\mu}$ and not on the individual values of μ_1 and μ_2, as long as $\eta \le r$. For $\eta > r$ Π_1^{in} linearly increases with p in Figure 2 (left) (recall that μ_1 is fixed). We obtain that $\Pi_1^{in} = \Pi_2^{in}$ when p is large enough so that to maintain $\bar{\mu} = 2$ the value of μ_2 is smaller than $\frac{C}{R}$.[5] Figure 2 (right) gives similar results but this time $p = \frac{1}{2}$ is fixed, $\mu_1 \in (2, 3.4)$ and $\mu_2 = 4 - \mu_1 \in (\frac{C}{R} = 0.6, 2)$ so that again $\bar{\mu} = 2$ is preserved. The two figures are essentially identical except for a nonlinear change of scale.

The function $\left(\sqrt{\mu} - 1\right)^2$ is convex. Suppose that $Var(\mu)$ is increased while maintaining $\bar{\mu}$. Then, Π_2^{in} in the first two cases of (5) also increases. Therefore, Π_2^{in}, which is the maximum of these two expressions (and 0) is a nondecreasing function of the variance (and strictly increasing where the profit is positive).

Finally, Π^{un} decreases when the variance increases due to the convexity of the expected waiting time as a function of λ for any given μ.

[3] In this example, $\eta \approx 110$, corresponding to $R \approx 110$. The R values corresponding to $\frac{1}{\mu_1}$, $\frac{p}{\mu_1} + \frac{1-p}{\mu_2}$, and $\frac{1}{\mu_2}$, are 20, 52, and 60, respectively.

[4] In the other cases that we consider later, where C or R are random, we naturally have $\Pi^{un} = \Pi_1^{in} = \Pi_2^{in}$ in the limit, when $p \to \infty$.

[5] Specifically, for the parameters used for this figure, $\mu_2 \le \frac{30}{50}$ is obtained when $p \ge \frac{14}{19}$, or $Var(\mu) \ge 0.7$.

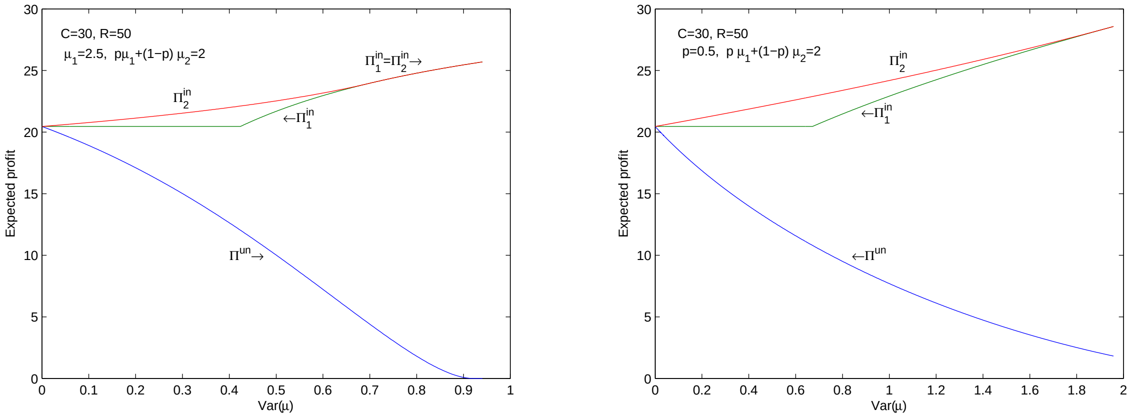

Figure 2: The effect of uncertain μ: changing (p, μ_2) (left) and (μ_1, μ_2) (right)

5 Uncertain waiting cost

Suppose that $C = C_1$ with probability p and $C = C_2$ with the complementary probability, where $C_1 < C_2$. Denote the expected value of C by $\bar{C} = pC_1 + (1-p)C_2$.

5.1 Uniformed customers

Given a price T, the equilibrium arrival rate is $\lambda = \left(\mu - \frac{\bar{C}}{R-T}\right)^+$. By (2), the maximum rate of profit is

$$\Pi^{un} = \begin{cases} \left(\sqrt{R\mu} - \sqrt{\bar{C}}\right)^2 & R\mu \geq \bar{C} \\ 0 & R\mu < \bar{C} \ . \end{cases} \tag{10}$$

5.2 Informed customers - two prices

Suppose that customers are informed about the service rate, and the server charges different prices depending on the realization of C. Then,

$$\Pi_2^{in} = \begin{cases} p\left(\sqrt{R\mu} - \sqrt{C_1}\right)^2 + (1-p)\left(\sqrt{R\mu} - \sqrt{C_2}\right)^2 & R\mu \geq C_2 \\ p\left(\sqrt{R\mu} - \sqrt{C_1}\right)^2 & C_1 < R\mu < C_2 \\ 0 & R\mu \leq C_1 \ . \end{cases} \tag{11}$$

5.3 Informed customers - single price

Given the price T and the information that C_i is realized, the equilibrium arrival rate is $\lambda_i = \left(\mu - \frac{C_i}{R-T}\right)^+$.

For $T < R - \frac{C_1}{\mu}$ we have $\lambda_1, \lambda_2 > 0$. The expected rate of profit is then

$$\begin{aligned} \Pi &= T[p\lambda_1 + (1-p)\lambda_2] \\ &= T\left\{p\mu - p\frac{C_1}{R-T} + (1-p)\mu - (1-p)\frac{C_2}{R-T}\right\} \\ &= T\mu - \frac{\bar{C}T}{R-T}. \end{aligned}$$

This is the same expression as in the uniformed customers case and the maximum profit is

$$\Pi = \left(\sqrt{R\mu} - \sqrt{\bar{C}}\right)^2 . \tag{12}$$

Expression (12) is correct if the maximizing price satisfies $T < R - \frac{C_1}{\mu}$. Otherwise, it underestimates the profit by assuming a negative λ_2, whereas the true value is 0. For $T \in (R - \frac{C_2}{\mu}, R - \frac{C_1}{\mu})$ we have $\lambda_1 > 0$ and $\lambda_2 = 0$. In this range, the profit maximizing price is $R - \sqrt{\frac{RC_1}{\mu}}$ and the profit is

$$\Pi = p\left(\sqrt{R\mu} - \sqrt{C_1}\right)^2 . \tag{13}$$

We compare (13) with (12) to compute the maximum possible profit given R and μ. Thus,

$$\Pi_1^{in} = \max\left\{\left(\sqrt{R\mu} - \sqrt{\bar{C}}\right)^2, p\left(\sqrt{R\mu} - \sqrt{C_1}\right)^2\right\}.$$

The two values are equal if $\sqrt{R\mu} - \sqrt{\bar{C}} = \sqrt{p}\left(\sqrt{R\mu} - \sqrt{C_1}\right)$, or $R\mu = \gamma$ where

$$\gamma = \left(\frac{\sqrt{\bar{C}} - \sqrt{pC_1}}{1 - \sqrt{p}}\right)^2 .$$

Note that from $C_2 > C_1$ it follows that $\gamma > C_1$. Therefore,

$$\Pi_1^{in} = \begin{cases} \left(\sqrt{R\mu} - \sqrt{\bar{C}}\right)^2 & R\mu \geq \gamma \\ p\left(\sqrt{R\mu} - \sqrt{C_1}\right)^2 & C_1 < R\mu < \gamma \\ 0 & R\mu \leq C_1 \ . \end{cases} \tag{14}$$

5.4 Discussion

The relative behavior of the profit functions under the three models differs in different regions of $R\mu$ defined by $0 \leq C_1 \leq \bar{C} \leq C_2 \leq \gamma < \infty$ as follows:

- If $R\mu \in (0, C_1)$ then $\Pi^{un} = \Pi_1^{in} = \Pi_2^{in} = 0$.
- If $R\mu \in (C_1, \bar{C})$ then $0 = \Pi^{un} < \Pi_1^{in} = \Pi_2^{in}$.
- If $R\mu \in (\bar{C}, C_2)$ then $0 < \Pi^{un} < \Pi_1^{in} = \Pi_2^{in}$.
- If $R\mu \in (C_2, \gamma)$ then $0 < \Pi^{un} < \Pi_1^{in} < \Pi_2^{in}$.
- If $R\mu \in (\gamma, \infty)$ then $0 < \Pi^{un} = \Pi_1^{in} < \Pi_2^{in}$. The difference $\Pi_2^{in} - \Pi_1^{in}$ is equal to $\kappa\sqrt{R\mu}$ where $\kappa = 2\left(\sqrt{\bar{C}} - p\sqrt{C_1} - (1-p)\sqrt{C_2}\right)$.

Figure 3 shows the profit functions for $p = 0.2$, $C_1 = 20$, and $C_2 = 80$. The variable corresponding to the x axis is $R\mu$ which is the value produced by the server per unit of time of service.[6]

Corollary 5.1 $\Pi_2^{in} \geq \Pi_1^{in} \geq \Pi^{un}$. *Therefore, the firm is motivated to reveal the realized value of C to its customers.*

[6]With this data, $\bar{C} = 68$, $\gamma \approx 127.7$, and $\kappa \approx 0.39$.

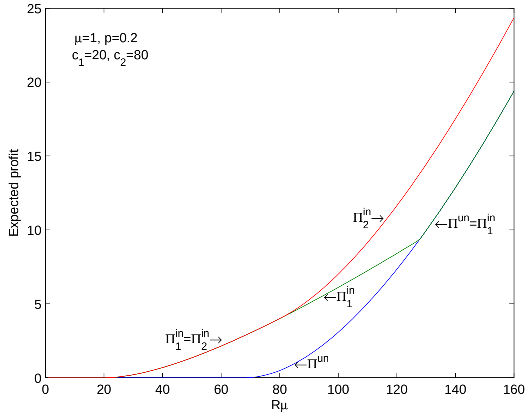

Figure 3: Random waiting cost example

5.5 The effect of uncertainty

We consider two types of changes in uncertainty. In both types we fix $\bar{C}$. In the first case we also fix $C_1 \leq \bar{C}$ (recall that $C_1 \leq C_2$), and simultaneously change p and C_2 so that $\bar{C}$ is preserved at the same level. In Figure 4 (left) we assumed $\bar{C} = 30$ and $C_1 = 20$ giving by (3, $Var(\mu) = 100\frac{p}{1-p}$. In the second case we fix p and modify C_1 and C_2, again preserving $\bar{C}$. An increase in C_1 comes with an adequate decrease of C_2. In Figure 4 (right) we assumed $p = 0.5$ and $\bar{C} = 30$ giving, by (2), $Var(\mu) = (C_1 - 30)^2$.[7]

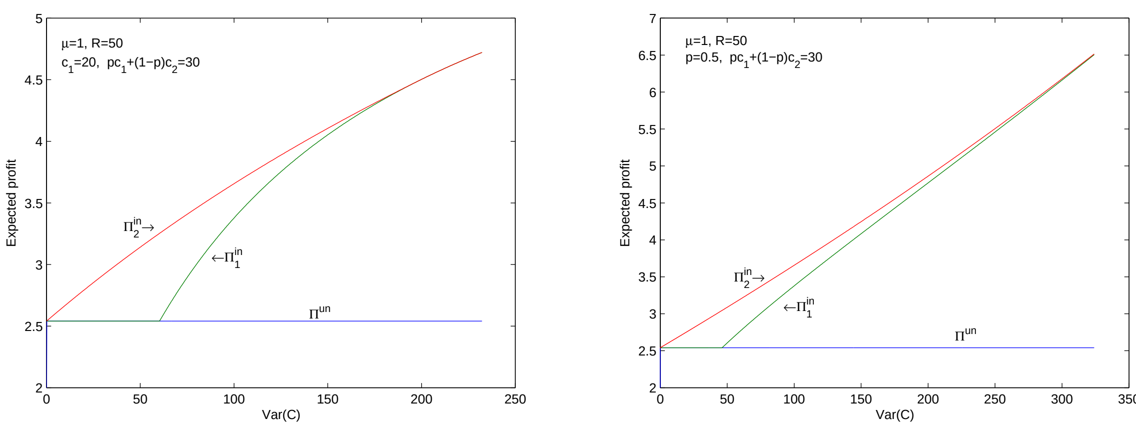

Figure 4: The effect of uncertain C: changing (p, C_2) (left) and (C_1, C_2) (right)

[7] $\Pi_1^{in} = \Pi_2^{in}$ when $R\mu \leq C_2$, see (11) and (14). With the parameters used for Figure 4 (right) this means $C_2 > 50$, $p \geq \frac{2}{3}$, and equivalently $Var(C) > 200$. In Figure 4 (left) the analogous condition is $C_1 \leq 10$ or $Var(C) \leq 400$. Similarly, from (10) and (14), $\Pi_1^{in} = \Pi^{un}$ if $R\mu \geq \rho$. In Figure 4 (left) this means $p \leq \left(\frac{\sqrt{50}-\sqrt{30}}{\sqrt{50}-\sqrt{20}}\right)^2$, or approximately $Var(C) \leq 60$. In Figure 4 (right) this means $C_1 \leq \left(\sqrt{60} + \sqrt{50} - 10\right)^2$ given approximately $Var(C) \leq 46$.

6 Uncertain quality of service

Let $R = R_1$ with probability p, and $R = R_2$ with the complementary probability. Assume $R_1 > R_2$ and that μ and C are fixed.

6.1 Uninformed customers

Given a price T, the equilibrium arrival rate is $\lambda = \left(\mu - \frac{C}{\bar{R}-T}\right)^+$. By (2), he maximum rate of profit is

$$\Pi^{un} = \begin{cases} \left(\sqrt{\bar{R}\mu} - \sqrt{C}\right)^2 & \bar{R}\mu \geq C \\ 0 & \bar{R}\mu < C\ . \end{cases} \tag{15}$$

6.2 Informed customers - two prices

Suppose that customers are informed about the service quality, and the server charges different prices depending on the realization of R. Then,

$$\Pi_2^{in} = \begin{cases} p\left(\sqrt{R_1\mu} - \sqrt{C}\right)^2 + (1-p)\left(\sqrt{R_2\mu} - \sqrt{C}\right)^2 & R_2 \geq \frac{C}{\mu} \\ p\left(\sqrt{R_1\mu} - \sqrt{C}\right)^2 & R_2 < \frac{C}{\mu} < R_1 \\ 0 & R_1 \leq \frac{C}{\mu}\ . \end{cases} \tag{16}$$

6.3 Informed customers - single price

For a given price T, the arrival rate is $\lambda_i = \left(\mu - \frac{C}{R_i-T}\right)^+$ if R_i is realized.

To have positive gain, the price cannot exceed $R_1 - \frac{C}{\mu}$. We distinguish the two cases according to whether or not $T < R_2 - \frac{C}{\mu}$.

- Consider first the case $T < R_2 - \frac{C}{\mu}$:
  The expected profit is

$$\begin{aligned} \Pi(T) &= T[p\lambda_1 + (1-p)\lambda_2] \\ &= T\left(p\mu - \frac{pC}{R_1 - T} + (1-p)\mu - \frac{(1-p)C}{R_2 - T}\right) \\ &= T\mu - CT\left(\frac{p}{R_1 - T} + \frac{1-p}{R_2 - T}\right). \end{aligned}$$

  The first-order optimality conditions are

$$\Pi' = \mu - C\left(\frac{p}{R_1 - T} + \frac{1-p}{R_2 - T}\right) - CT\left(\frac{p}{(R_1 - T)^2} + \frac{1-p}{(R_2 - T)^2}\right) = 0.$$

  Multiplication by $\frac{(R_1-T)^2(R_2-T)^2}{C}$ gives

$$\begin{aligned} \frac{\mu}{C}(R_1 - T)^2(R_2 - T)^2 &- \left(p(R_1 - T)(R_2 - T)^2 + (1-p)(R_1 - T)^2(R_2 - T)\right) \\ &- T\left(p(R_2 - T)^2 - (1-p)(R_1 - T)^2\right) = 0, \end{aligned}$$

  or

$$\frac{\mu}{C}(R_1 - T)^2(R_2 - T)^2 - pR_1(R_2 - T)^2 - (1-p)(R_1 - T)^2 R_2 = 0.$$

  This gives the following polynomial

$$\frac{\mu}{C}T^4 - 2\frac{\mu}{C}(R_1+R_2)T^3 + \left(\frac{\mu}{C}(R_1^2+R_2^2+4R_1R_2) - \bar{R}\right)T^2 + 2R_1R_2\left(1-\frac{\mu}{C}(R_1+R_2)\right)T + R_1R_2\left(\frac{\mu}{C}R_1R_2 - pR_2 - (1-p)R_1\right) = 0, \quad (17)$$

where $\bar{R} = pR_1 + (1-p)R_2$.

- Suppose now that $T \geq R_2 - \frac{C}{\mu}$. In this case $\lambda_2 = 0$ and the optimal price gives profit $p\left(\sqrt{R_1\mu} - \sqrt{C}\right)^2$.

Figure 5 (left) shows the profit as a function of the price for $C = 20$, $p = 0.2$, $R_1 = 70$, $\mu = 1$, and $R_2 = 30$. With these parameters $R_2 - \frac{C}{\mu} = 10$, and we see that the server gains more by raising the price to a level that induces no arrivals when R_2 is realized. Lowering R_1 to 60 and increasing R_2 to 40 (and then $R_2 - \frac{C}{\mu} = 20$) we obtain a new situation, shown in Figure 5 (right), in which it is worth lowering the price to attract customers also when R_2 is realized.

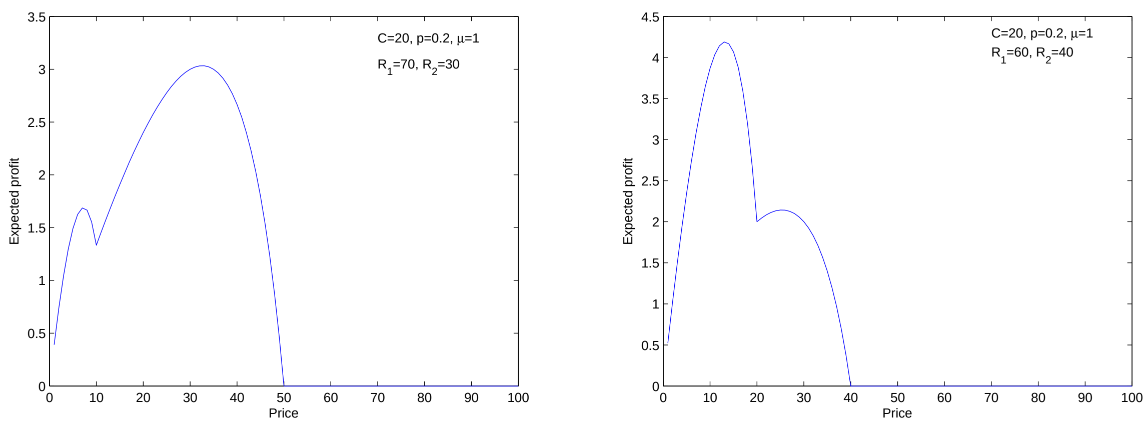

Figure 5: Optimal price is greater (left) or smaller (right) than $R_2 - \frac{C}{\mu}$

6.4 Discussion

When C is close to 0, $\Pi^{un} = \Pi_2^{in} = \mu\bar{R} > \mu[max(R_2, pR_1)] = \Pi_1^{in}$. Thus, for small values of C, if the server is restricted to a single price, the server is motivated to conceal the realized value of R from the customers. However, for large values of C the opposite is true.

Figure 6 gives the profits as a function of C, when $R_1 = 100$,$R_2 = 30$, $p = 0.2$, and $\mu = 1$. Note that these are actually functions of $\frac{C}{\mu}$.[8]

6.5 The effect of uncertainty

We fix $\bar{R}$ and thus $R_2 = \frac{\bar{R} - R_2 p}{1-p}$. In the first case we also fix $R_1 \geq \bar{R}$ (recall that $R_1 > R_2$), and simultaneously change p and R_2 so that $\bar{R}$ is preserved. In Figure 7 (left) we assumed $\bar{R} = 30$ and $R_1 = 50$ giving (by (3)) $Var(\mu) = 400\frac{p}{1-p}$. In the second case we fix p and modify

[8]For this instance, $\Pi^{un} > 0$ for $C < \mu\bar{R} < 44$, and the other curves are positive for $C < \mu R_1 = 100$.

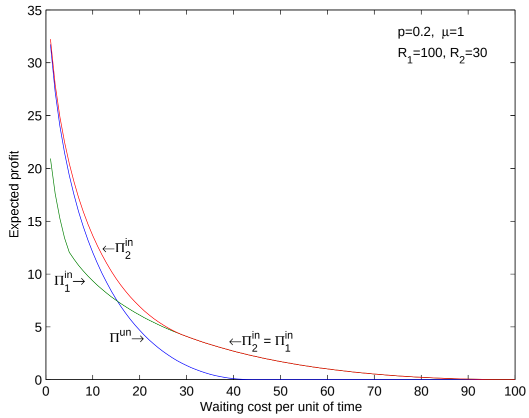

Figure 6: Random value of service

R_1 and R_2, again preserving $\bar{R}$. In Figure 7 (right) we assumed $p = 0.5$ and $\bar{R} = 30$ giving $Var(\mu) = (R_1 - 30)^2$.

We see that Π^{un} only depends on $\bar{R}$ and is not affected by the change in uncertainty, as expected from (15). We also see that in the case of informed customers and two prices, profits increase with uncertainty. However, in the case of a single price, profits initially decrease as a function of the variance, and only for larger variance there is an increase.[9]

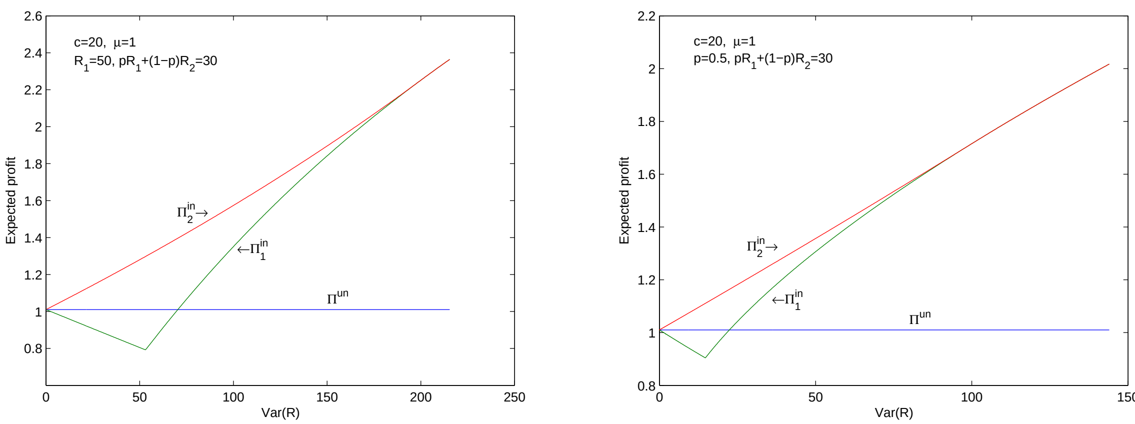

Figure 7: The effect of uncertain R: changing (p, R_2) (left) and (R_1, R_2) (right)

[9]We have $\Pi_1^{in} = \Pi_2^{in}$ if $R_2 \leq \frac{C}{\mu}$. With the parameters used to create the figures, this means $R_2 \leq 20$. In the left figure this means $p \geq \frac{1}{3}$, and thus $Var(R) \geq 200$. In the right figure it means $R_1 \geq 40$, and thus $Var(R) \geq 100$.

References

[1] E. Altman, T. Jiménez, R. Núñez-Queija, and U. Yechiali (2004) "Optimal routing among $\cdot/M/1$ queues with partial information," *Stochastic Models* **20** (2004) 149-171.

[2] R. Arnott, A. de Palma, and R. Lindsey, "Information and usage of free-access congestible facilities with stochastic capacity and demand," *International Economic Review* **37** (1996) 181-203.

[3] R. Arnott, A. de Palma, and R. Lindsey, "Information and time-of-usage decisions in the bottleneck model with stochastic capacity and demand," *European Economic Review* **43** (1999) 525-548.

[4] L. G. Debo, C. A. Parlour, and U. Rajan , "The value of Congestion," 2005.

[5] P. Guo and P. Zipkin, "Analysis and comparison of queues with different levels of delay information," 2004.

[6] R. Hassin, "Consumer information in markets with random products quality: The case of queues and balking," *Econometrica* **54** (1986) 1185-1195.

[7] R. Hassin, "On the advantage of being the first server," *Management Science* **42** (1996) 618-623.

[8] R. Hassin and M. Haviv "Equilibrium strategies and the value of information in a two line queueing system with threshold jockeying," *Communications in Statistics - Stochastic Models* **10** (1994) 415-436.

[9] R. Hassin and M. Haviv *To Queue or not to Queue: Equilibrium Behavior in Queueing Systems,* Kluwer International Series, 2003.

[10] C. Larsen, "Investigating sensitivity and the impact of information on pricing decisions in an $M/M/1/\infty$ queueing model," *International Journal of Production Economics* **56-57** (1998) 365-377.

[11] P. Milgrom and J. Roberts, "Price and advertising signals of product quality," *Journal of Political Economy* **94** (1986) 796-821.

Appendix

Proof of Lemma 4.1: In equilibrium,

$$v = \frac{p}{\mu_1 - \lambda} + \frac{1-p}{\mu_2 - \lambda},$$

or

$$v(\mu_1 - \lambda)(\mu_2 - \lambda) = p\mu_2 + (1-p)\mu_1 - \lambda,$$

$$v\lambda^2 + \lambda(1 - v\mu_1 - v\mu_2) + v\mu_1\mu_2 - p\mu_2 - (1-p)\mu_1 = 0,$$

and finally,

$$\begin{aligned} \lambda &= \frac{-1 + v(\mu_1 + \mu_2)}{2v} \pm \frac{\sqrt{(1 - v\mu_1 - v\mu_2)^2 - 4v[v\mu_1\mu_2 - p\mu_2 - (1-p)\mu_1]}}{2v} \\ &= \frac{1}{2}\left\{ -\frac{1}{v} + (\mu_1 + \mu_2) \pm \sqrt{\frac{1}{v^2} + (\mu_1 - \mu_2)^2 + \frac{2}{v}(1-2p)(\mu_1 - \mu_2)} \right\}. \end{aligned} \quad (18)$$

Assuming $\mu_1 > \mu_2$, the square root is minimized when $p = 1$, and then it is equal to $v - (\mu_1 - \mu_2)$. Therefore, the root corresponding to the plus sign is greater than μ_2. We conclude that the equilibrium is defined by the root corresponding to the minus sign. Hence,

$$\lambda = \frac{1}{2}\left\{ -\frac{1}{v} + (\mu_1 + \mu_2) - S \right\}, \quad (19)$$

where

$$S = \sqrt{\frac{1}{v^2} + (\mu_1 - \mu_2)^2 + \frac{2(1-2p)(\mu_1 - \mu_2)}{v}} = \sqrt{\left(\frac{1}{v} - (\mu_1 - \mu_2)\right)^2 + 4(1-p)\frac{\mu_1 - \mu_2}{v}}.$$

Note that by our assumptions that $p \in (0,1)$ and $\mu_1 > \mu_2$, it follows that S is strictly positive.

The derivative of λ with respect to v is

$$\lambda'_v = \frac{1}{2v^2}\left\{1 - \frac{-\frac{1}{v} - (1-2p)(\mu_1 - \mu_2)}{S}\right\},$$

and with respect to T

$$\lambda'_T = \lambda'_v \frac{dv}{dT} = -\frac{1}{2Cv^2}\left\{1 - \frac{-\frac{1}{v} - (1-2p)(\mu_1 - \mu_2)}{S}\right\}.$$

The profit Π^{un} is equal to λT. The first-order optimality condition is $T\lambda'_T + \lambda = 0$, or

$$-\frac{T}{2Cv^2}\left(1 + \frac{\frac{1}{v} + (1-2p)(\mu_1 - \mu_2)}{S}\right) + \frac{1}{2}\left(-\frac{1}{v} + (\mu_1 + \mu_2) - S\right) = 0.$$

Multiply by $2S$ (allowed since $S > 0$):

$$S\left(-\frac{T}{Cv^2} - \frac{1}{v} + \mu_1 + \mu_2\right) = \frac{T}{Cv^3} + T(1-2p)\frac{\mu_1 - \mu_2}{Cv^2} + \frac{1}{v^2} + (\mu_1 - \mu_2)^2 + 2(1-2p)\frac{\mu_1 - \mu_2}{v}.$$

Substitute $T = R - vC$:

$$S\left(-\frac{R}{Cv^2} + \mu_1 + \mu_2\right) = \frac{R}{Cv^3} + R(1-2p)\frac{\mu_1 - \mu_2}{Cv^2} + (1-2p)\frac{\mu_1 - \mu_2}{v} + (\mu_1 - \mu_2)^2. \qquad (20)$$

Recall that $r \equiv \frac{R}{C}$. Multiply the left-hand side of (20) by v^3 and square:

$$\begin{aligned}
LS &= \left(\frac{1}{v^2} + (\mu_1 - \mu_2)^2 + 2(1-2p)\frac{\mu_1 - \mu_2}{v}\right)\left(r^2v^2 + v^6(\mu_1 + \mu_2)^2 - 2r(\mu_1 + \mu_2)v^4\right) \\
&= r^2 + (\mu_1 + \mu_2)^2 v^4 - 2r(\mu_1 + \mu_2)v^2 + 2(1-2p)r^2(\mu_1 - \mu_2)v \\
&+ (\mu_1 - \mu_2)^2 r^2 v^2 + (\mu_1 - \mu_2)^2(\mu_1 + \mu_2)^2 v^6 - 2r(\mu_1 - \mu_2)^2(\mu_1 + \mu_2)v^4 \\
&+ 2(1-2p)(\mu_1 - \mu_2)(\mu_1 + \mu_2)^2 v^5 - 4r(1-2p)(\mu_1 - \mu_2)(\mu_1 + \mu_2)v^3.
\end{aligned}$$

Multiply the right-hand side of (20) by v^3 and square:

$$\begin{aligned}
RS &= r^2 + r^2(1-2p)^2(\mu_1 - \mu_2)^2 v^2 + (1-2p)^2(\mu_1 - \mu_2)^2 v^4 + (\mu_1 - \mu_2)^4 v^6 \\
&+2r^2(1-2p)(\mu_1 - \mu_2)v + 2r(1-2p)(\mu_1 - \mu_2)v^2 + 2r(\mu_1 - \mu_2)^2 v^3 \\
&+2r(1-2p)^2(\mu_1 - \mu_2)^2 v^3 + 2r(1-2p)(\mu_1 - \mu_2)^3 v^4 + 2(1-2p)(\mu_1 - \mu_2)^3 v^5.
\end{aligned}$$

By (20), $LS - RS = 0$. Division by v^2 gives

$$\begin{aligned}
& (\mu_1 - \mu_2)^2\left[(\mu_1 + \mu_2)^2 - (\mu_1 - \mu_2)^2\right] v^4 \\
+ \; & 2(1-2p)(\mu_1 - \mu_2)\left[(\mu_1 + \mu_2)^2 - (\mu_1 - \mu_2)^2\right] v^3 \\
+ \; & \left[(\mu_1 + \mu_2)^2 - 2r(\mu_1 - \mu_2)^2[\mu_1 + \mu_2 + (1-2p)(\mu_1 - \mu_2)] - (1-2p)^2(\mu_1 - \mu_2)^2\right] v^2 \\
+ \; & 2r(\mu_1 - \mu_2)\left[-2(1-2p)(\mu_1 + \mu_2) - (1-2p)^2(\mu_1 - \mu_2) - (\mu_1 - \mu_2)\right] v \\
+ \; & r\left[r(\mu_1 - \mu_2)^2 - 2(\mu_1 + \mu_2) - r(1-2p)^2(\mu_1 - \mu_2)^2 - 2(1-2p)(\mu_1 - \mu_2)\right] = 0.
\end{aligned}$$

(With $\mu_1 = \mu_2 = \mu$, the polynomial reduces to $(2\mu)^2 v^2 = 4r\mu$, or $v = \sqrt{\frac{r}{\mu}}$. This gives T as in (1).) ∎

Remark 6.1 Figure 8 shows the roots as a function of R, with $p = 0.2$, $\mu_1 = 1.5$, and $\mu_2 = 0.5$. The figure shows the real roots that satisfy $0 \le T \le R - C\left(\frac{p}{\mu_1} + \frac{1-p}{\mu_2}\right)$. (Recall that by (4) T must be bounded this way.[10]) For large values of R we get two such roots, marked Π_1^{un} and Π_2^{un}. For example, with $R = 3000$, the values of λ, v, T, and Π^{un} are $(0.44, 12.82, 2615.4, 1142.1)$ and $(0.20, 2.80, 2916.1, 575.49)$, respectively. However, the new root, denoted Π_2^{in} in Figure 8, does not correspond to a new equilibrium. Substituting these values in (20) we obtain that the left-hand side is negative, whereas the right-hand side is positive (and equal to the left-hand side in its absolute value). Thus, the root Π_2^{un} results from the squaring of the two sides of (20).

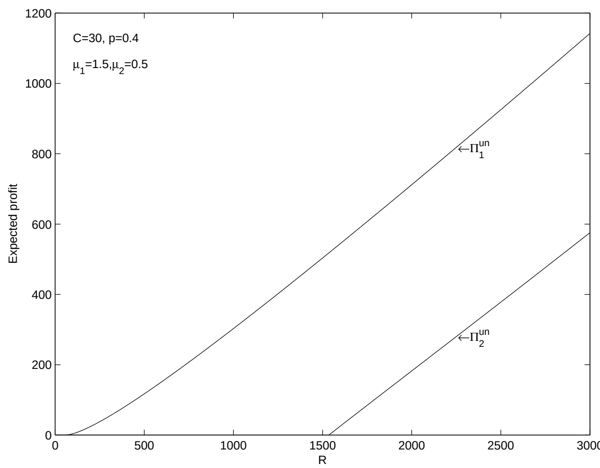

Figure 8: Profits corresponding to the two real roots

[10] $\Pi = 0$ if $R < C\left(\frac{p}{\mu_1} + \frac{1-p}{\mu_2}\right) = 44$.