Network Flows Exam

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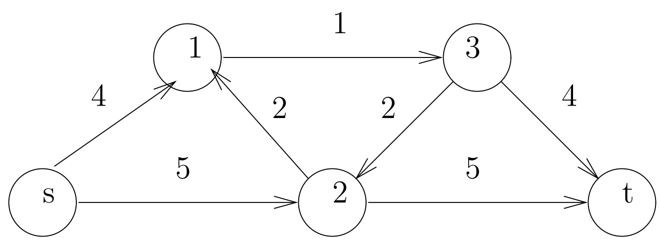
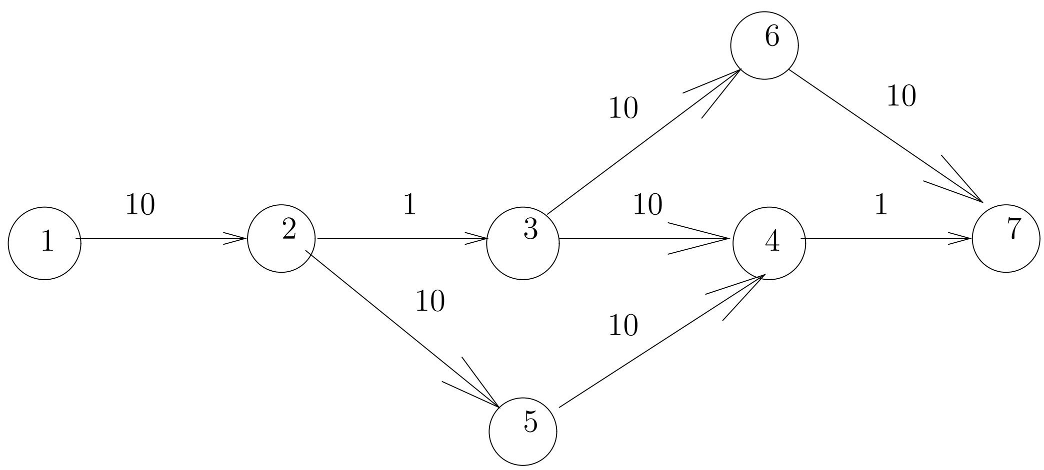
1. Given a graph $G = (N, A)$ with arc lengths $a_{ij} \geq 0$, we want to compute an $s - t$ walk of minimum average length subject to the requirement that no more than k arcs can be used. A walk is like a path but we allow to use an arc more than once. The average length is the length divided by the number of arcs in the path. Describe an algorithm and analyze its running time. Solve [by inspection, there is no need to apply an algorithm] the problem on the following graph with $k = 20$:



2. Given are a graph $G = (N, A)$ with arc capacities u_{ij}, a given pair of nodes $s, t \in N$, a given subset of arcs $A' \subseteq A$, and a constant K. Show how to compute a maximum $s - t$ flow under the constraint that the total flow in the arcs of A' is bounded by K. Note that the arc-flows in the solution need not be integers. [You may use the relation of this problem to the min cost flow problem.]

   Apply your algorithm to the following graph with $s = 1$, $t = 7$, $K = 2$. The numbers on the arcs denote capacities, and $A' = \{(2,5), (3,4), (5,4), (3,6), (6,7)\}$ (the arcs with bigger arrows).



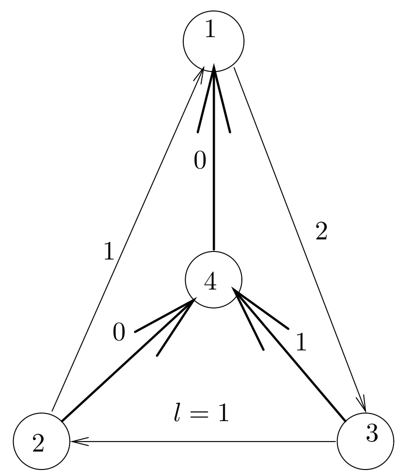
3. (Dual-Simplex Algorithm) Let $G = (N, A)$ be a directed graph. [For simplicity, suppose that G has no anti-parallel arcs.] Let $\{u_i\}$ be a dual solution for a min-cost-flow problem. Let T be a spanning tree of G. Suppose that $v_{ij} = 0$ for $(i, j) \in T$ and $v_{ij} \neq 0$ for all of the other arcs.

   Construct a circulation as follows: For every $(i, j) \notin T$, if $v_{ij} > 0$ set $x_{ij} = l_{ij}$ and if $v_{ij} < 0$ set $x_{ij} = u_{ij}$. Compute the flows on T in the unique way that satisfies flow conservation. [Start from a leaf, compute its in-flow and out-flow, then set the flow in the arc of T that is incident to it so that the flow is conserved.]

   If for every $(i, j) \in T$ $l_{ij} \leq x_{ij} \leq u_{ij}$ then the circulation is optimal. Suppose otherwise, that for some $(p, q) \in T$ $x_{pq} > u_{pq}$. Let P, Q be the partition of N induced when we remove (p, q) from T, with $p \in P$ and $q \in Q$.

   Prove that (P, Q) is a positive cut.

   Execute[1] one iteration (specifying the flow and updating the dual variables) of the algorithm that the above ideas imply on the graph below. The numbers on the arcs are the lower bounds while all upper bounds satisfy $u_{ij} = 3$ and all costs are of one unit. T consists of the three bold arcs incident with node 4. Start with $u_1 = 0$ and complete the dual solution so that $v_{ij} = 0$ for all the arcs of T (that is, $u_i = 1 + u_j$).



4. For a graph $G = (N, A)$ with $s, t \in N$, let f_{max} be the maximum $s - t$ flow satisfying $l_{ij} \leq x_{ij} \leq u_{ij}$ for all $(i, j) \in A$.

   State and prove a generalization of the max-flow min-cut theorem for this case (possibly using Hoffman's theorem).

[1] Note that you may successfully solve this part even if you failed proving the required property above.