

OPTIMAL SALES TO USERS WHO HOLD INVENTORY*

by

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Abstract

This paper solves the optimal inventory policy when both the supplier and the users can hold inventory: the supplier can hold the good until such a time as a buyer wishes to purchase it; a user can purchase a good before he needs it and hold it until the need arises, or delay the purchase and incur a shortage. The welfare-maximizing policy is shown to require periods of shortage in which no sales are made. We also show that the profit-maximizing policy will involve shorter periods of no-sales than does the welfare-maximizing policy.

January 1990

Economics Letters* **34 (1990), 215-220.

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1. Introduction

This paper analyzes an inventory model that explicitly takes into consideration the needs of users, in addition to the common assumptions on the supplier: the supplier provides a service (storage of the good) and the user is free to decide how much, if any, of this service to obtain. In Glazer and Hassin (1986) the authors analyzed the behavior of a supplier who is a profit-maximizing monopolist. Here we investigate the social welfare implications of the models.

One application of the results is to transfer pricing. A firm may have a number of divisions, with a central facility which supplies other units. (We may think, for example, of General Motors which has the Fisher Body Division supplying chassis to different assembly plants). Rather than directly controlling the supply of parts, corporate management may prefer to decentralize decisions: the supplying division sells the parts to other divisions at a fixed price. We ask here what prices and delivery times corporate management should set to maximize corporate profits, and we ask whether the supplying division would choose the same price if it were instructed to maximize its divisional profits.

We assume that the direct costs of the supplier are a fixed cost of inventory replenishment and inventory holding costs. A user may obtain the good earlier than he needs it and bear his own inventory holding costs; he can also obtain it later than at the instant at which he values the good most highly, and incur shortage costs proportional to the gap between these points of time.

Most of the literature on inventories ignores the preferences of users. Important exceptions are the works of Bucovetsky (1983), which considers random prices and storage of the goods by users, of Sobel (1984), which assumes that a user confronting a shortage may postpone his purchase to the next sales time, and Eppen and Lieberman (1984) which considers a seller who reduces his price to induce users to hold inventory.

It is not difficult to determine the first best solution. It can be supported by setting prices which vary over the course of each inventory cycle for each good. The price should increase within each cycle by at a rate equal to the supplier's inventory holding cost. Here we focus on the more interesting question of determining the second best optimal policy, where the supplier charges a fixed unit price of the good, independent of the time of delivery.

The fixed price assumption is realistic, since time-varying prices are rarely used, a phenomenon that has been explained by Wilson (1989) as follows: "The apparent explanations are technological limitations and pervasive transaction costs. For example, it is difficult or expensive to inform a customer continually about prices and to monitor the time pattern of purchases, even if one were to know the right prices ... Similarly, it is expensive for customers to monitor prices continually

and adapt their purchases. As a result, one commonly sees fixed prices...". It is also interesting to note that the ability to hold inventory reduces the supplier's incentives to change prices (see Amihud and Mendelson 1983).

The main results in our analysis are that planned shortage periods are required for maximization of both profit and social welfare, and that the profit-maximizing policy involves shorter periods with no deliveries to users than does the welfare maximizing policy. These conclusions have a direct bearing on transfer prices.

Section 2 describes our model, and Section 3 derives the welfare maximizing policy. The profit-maximizing policy is described in Section 4. Section 5 compares the welfare-maximizing and profit maximizing policies; in Section 6 we give some intuitive explanations for the results.

2. The Model

Our assumptions concerning the seller are conventional (see Hadley and Whitin (1963)). We consider a sole supplier (for example, the corporation's central warehouse, or alternatively a retailer) who receives batches of goods and then sells individual units to users. The cost to the central warehouse of placing an order of any positive size is A . Inventory carrying costs are linear, h per unit of inventory per unit time. The model is stationary so that inventory fluctuates in a cyclic pattern, where a cycle is determined by two successive placements of orders. We denote the length of a cycle by T and refer to a typical cycle that begins at time 0 and ends at time T .

Users are identical in all but the time they would most prefer to obtain the good. The value of a good to a user at his most preferred time is denoted by w , and the point of time where he would most prefer to obtain the good is called his w -time. Consider a user with w -time t_p . In obtaining a good before then, he incurs a holding cost of i per unit of time, so that his reservation price at time t , $v(t)$, is $w - i(t_p - t)$ for $t < t_p$. In obtaining a good after time t_p , the user incurs a shortage cost of s per unit of time, so that $v(t)$ is $w - s(t - t_p)$ for $t > t_p$. The number of users with w -times in any interval $(t, t + dt)$ is λdt .

The welfare-maximizing solution maximizes the aggregate value of user surpluses plus the supplier's profits per unit time. The profit-maximizing solution for a corporation maximizes the aggregate benefit to the users minus the costs of the central warehouse. The corporation views payments from users to the central warehouse as having no effect on corporate profits. The corporation's objective is therefore mathematically identical to the objective of maximizing social welfare. For succinctness, we will henceforth call the problem studied to be the maximization of social welfare. The values of λ, w, s, h, i and A are exogenously fixed. Under a second-best, fixed-price, system, each user decides when, if at all, to purchase the good. The supplier's choice variables

are the length of an inventory cycle T , the price of the good p , and the times within each cycle at which the good will be delivered to users (which we denote by S). We denote optimal values by S^* , T^* , and so forth.

3. Welfare Maximization

This section describes the welfare maximizing policy. We assume that it is socially desirable to deliver the good to at least some users, so that $p \leq w$.

We start by arguing that under a welfare maximizing policy, each potential user has a point of time $t \in S$ such that $v(t) \geq p$. Therefore, all potential users will obtain the good.

Suppose, contrary to the above assertion, that there exists a nonempty interval (t_1, t_2) such that for all users with w -time in this interval $v(t) < p$ for all $t \in S$. Since $p \leq w$ the definition of the interval implies that it does not intersect with S . Let S' be a new delivery policy obtained from S by "cutting" from each cycle the interval (t_1, t_2) . More rigorously, $S' = \{t | t \in S, t < t_1\} \cup \{t - (t_2 - t_1) | t \in S, t > t_2\}$. Actual deliveries before t_1 are identical under the two policies, while deliveries after t_1 under S' are identical to deliveries after t_2 under S . Aggregate user welfare per cycle is therefore identical under the two policies, while the supplier incurs lower inventory holding costs under S' . Thus, aggregate welfare per cycle is greater under S' than under S , and since the cycle under S' is shorter, welfare per unit time is greater.

We conclude that a policy which makes deliveries to some, but not all users, is not socially optimal. We turn now to determine the optimal delivery pattern. Since all users obtain the good, maximization of welfare reduces to minimization of costs. We note that S^* must contain time zero, otherwise the inventory renewal could be postponed to reduce the supplier's inventory costs without affecting users welfare.

Theorem 3.1. *Suppose $h > i$, then $S^* = \{0\}$ and $T^* = \sqrt{2A(i+s)/\lambda is}$.*

Proof: Since the supplier has greater inventory carrying costs than the user, any deliveries within the cycle entail loss of social welfare. Welfare maximization therefore requires deliveries only at the beginning of each cycle. A user with w -time t will prefer to obtain the good at the beginning of the cycle containing t rather than at the beginning of the next cycle if $it < s(T-t)$, that is if $t < sT/(i+s)$. Users with a w -time in the cycle later than $sT/(i+s)$ will postpone their purchase to T . Social costs per cycle are then

$$\lambda \int_0^{sT/(i+s)} it dt + \lambda \int_{sT/(i+s)}^T s(T-t) dt + A = \lambda is T^2 / 2(i+s) + A.$$

To maximize social welfare per unit time we first divide by T and then maximize, obtaining T^* as claimed. □

Theorem 3.2. *If $i > h$ then $S^* = [0, T_D]$ for some $0 \leq T_D \leq T^*$.*

Proof: Consider first a policy under which there exist times t_1, t_2 , with $0 < t_1 < t_2 < T^*$, such that deliveries are made at times t_1 and t_2 , but not inside the interval (t_1, t_2) . Whether or not deliveries are made during this interval will affect only the behavior of users with w -times inside the interval, so we need consider only such users. A user will prefer to purchase the good at t_1 if $v(t_1) = w - i(t - t_1) > w - s(t_2 - t) = v(t_2)$, that is, if $t < t_1 + (t_2 - t_1)s/(i + s)$. Otherwise, it will purchase the good at t_2 . Aggregate user costs are therefore

$$\lambda \int_{t_1}^{t_1 + (t_2 - t_1)s/(i+s)} i(t - t_1) dt + \lambda \int_{t_1 + (t_2 - t_1)s/(i+s)}^{t_2} s(t_2 - t) dt = \lambda i s (t_2 - t_1)^2 / 2(i + s). \quad (3.1)$$

The supplier's inventory holding costs are

$$\lambda(t_2 - t_1)[ht_1 + ih(t_2 - t_1)/(i + s)]. \quad (3.2)$$

Under an alternative policy of continuous deliveries in (t_1, t_2) , each user obtains the good at its w -time and incurs no shortage or holding costs. The supplier's inventory carrying costs are

$$\lambda(t_2 - t_1)[ht_1 + h(t_2 - t_1)/2]. \quad (3.3)$$

Continuous deliveries inside the interval are preferable to deliveries only at its endpoints because when $i > h$, the value of (3.3) is less than the sum of (3.1) and (3.2). This proves the theorem. \square

We derive next the optimal values of T_D and T under the assumption that $i > h$. Users with w -times in the interval $(0, T_D)$ obtain the good at their w -times. They are responsible for an inventory carrying cost to the supplier of $\lambda h T_D^2 / 2$. Users with w -times in the interval (T_D, T) obtain the good either at time T_D or at time T . The number doing so at the former point is $\lambda s(T - T_D)/(i + s)$ and the number at the latter point is $\lambda i(T - T_D)/(i + s)$. The supplier's total holding cost for the goods delivered at time T_D is $\lambda s h (T - T_D) T_D / (i + s)$; that cost is zero for a good delivered at the beginning of a new cycle. A total cost of $\lambda i s^2 (T - T_D)^2 / 2(i + s)^2$ is incurred by users who purchase the good at T_D , and $\lambda s i^2 (T - T_D)^2 / 2(i + s)^2$ by those who purchase it at T . The aggregate costs for the supplier and the users *per unit time* are

$$C(T) = \frac{\lambda h T_D^2}{2T} + \frac{\lambda h s (T - T_D) T_D}{(s + i)T}.$$

For a given value of T this cost is minimized when

$$T_D = \frac{s(i - h)}{s(i - h) + hi} T \equiv \eta T. \quad (3.5)$$

For $i > h > 0$ we obtain from (3.5) that $0 < T_D < T$, and thus that welfare-maximization requires that no deliveries be made in some non-degenerate interval at the end of each cycle.

Substituting (3.5) in (3.4) we obtain

$$C(T) = \xi T - \frac{A}{T},$$

where $\xi = \lambda h \eta^2 / 2 + \lambda h s (1 - \eta) \eta / (s + i) + \lambda i s (1 - \eta)^2 / 2(i + s)$. The welfare maximizing cycle length is then

$$T^* = \sqrt{\frac{A}{\xi}}. \quad (3.6)$$

4. Profit Maximization

Managing a firm with many divisions is greatly simplified if each division can be simply instructed to maximize the division's profits. Under some circumstances, however, if each division is allowed to determine the transfer price for the goods it supplies, then maximizing each divisions profits does not lead to the same outcome as maximizing aggregate corporate profits. Accordingly, this section determines the profit-maximizing policy of a supplier who can choose the price to charge. The main qualitative difference between these results and those of Section 3 is that a profit maximizing supplier may adopt a policy of continuous deliveries throughout the cycle.

In this section we summarize the results derived by Glazer and Hassin (1986) that are relevant to the present discussion. Arguments identical to those presented in Section 3 were used to prove that profit maximization requires that the good be sold to all users, regardless of their w -times. The profit-maximizing policy as a function of the exogenous parameters is obtained as follows:

Theorem 4.1. *Profit is maximized by the following policy:*

(i) *If $i \leq h$ and $h(1/i + 1/s) \geq 2$, then deliveries are made only at the beginning of each cycle.*

The optimal cycle length is

$$T^* = \sqrt{A(i + s) / \lambda i s} \quad (4.4)$$

(ii) *If $s \leq h \leq i$, then deliveries are made continuously during an interval $[0, T_D^*]$. In this case*

$$T^* = \sqrt{2A / \lambda h \gamma}, \quad (4.2)$$

where

$$\gamma = 1 - \frac{[(1/s) - (1/h)]^2}{(1/s)^2 - (1/i)^2},$$

and

$$T_D^* = \frac{s(i - h)}{h(i - s)} T^*. \quad (4.3)$$

(iii) If $s \geq h$ and $h(1/i + 1/s) \leq 2$, then deliveries are made throughout a cycle of length

$$T^* = \sqrt{2A/\lambda h}. \quad (4.4)$$

5. Profit Maximization vs. Welfare Maximization

This section compares profit maximization (by a monopolist or by a division to welfare maximization (or equivalently to maximizing corporate profits across the divisions). We have already seen that a profit maximizer may adopt a policy of deliveries throughout the cycle, though such a policy is never socially optimal. In fact, checking the conditions in Theorems 3.1, 3.2,

and 4.1 we observe that in both cases deliveries are made continuously during part of the cycle only if $i \geq h$; deliveries are made only at the beginning of the cycle only if $i \leq h$. For the socially optimal policy the above conditions are also sufficient, while under certain conditions the profit maximizer prefers to sell throughout the cycle. We now compare the size and frequency of deliveries in the two cases.

Suppose first that i , s , and h are such that both profit-maximization and welfare-maximization require continuous deliveries within an interval $[0, T_D]$ where $0 < T_D < 1$, and no deliveries afterwards. Comparing Equations (3.5) and (4.3) we find that the ratio T_D^*/T^* is $s(i-h)/[h(i-s) + si]$ for a welfare maximizing solution and $s(i-h)/h(i-s)$ for a profit maximizing solution. Thus the profit maximizer makes deliveries over a longer fraction of the cycle.

Suppose next that both solutions call for deliveries at the beginning of each cycle only. Welfare maximization requires that $T^* = \sqrt{2A(i+s)/\lambda is}$, while profit maximization requires that $T^* = \sqrt{A(i+s)/\lambda is}$. Thus the profit maximizer sells too often.

To summarize, for $i > h$ welfare maximization requires continuous deliveries with a no-deliveries interval at the end of the cycle, while a profit maximizer either delivers throughout the cycle or has a no-deliveries interval with relative length smaller than the socially desired one. For $i < h$ welfare maximization requires deliveries only at the beginning of the cycle, while a profit maximizer either delivers continuously throughout the cycle, or adopts the above policy but delivers too often. In any case the shortages users face are less apparent under a profit maximizing policy.

6. Concluding Remarks

We have analyzed an inventory model that explicitly considers the preferences of users with regard to the time of supply. Most models, such as those of Hadley and Whitin (1963) assume

that either users insist on a particular point in time when they want to obtain the good (the *lost-deliveries* case), or that they agree to wait to the beginning of the next sale-cycle and the supplier bears a "penalty" for a delay (the *backorders* case). The penalty is commonly assumed to be linear in the delay time. In this respect, the contribution of our model is to generate implicitly the penalty function from assumptions on the preferences of users.

The main result of our analysis is that planned shortages, in the sense that no deliveries are made during some interval, can be explained even without recourse to uncertainty. Moreover, we showed that shortages are desirable. Our result follows since the marginal cost for the supplier of providing the good at the beginning of a new cycle is zero, whereas the cost of providing it dt time units earlier is greater than zero by a non-infinitesimal amount. On the other hand, the decrease in user welfare caused by a no-deliveries period of length dt is infinitesimal. Deliveries towards the end of the cycle are therefore inefficient.

Our second result is that a profit maximizer generates, under our model, less shortage than is socially desirable. If users obtain the good at some time or other, the maximum price the supplier can charge is a decreasing function of the length of the longest no-deliveries interval. A supplier that decreases the length of this interval can increase the price to *all* users – not only to those with w -times in the no-deliveries interval. It therefore has too great an inducement to shorten a no-deliveries interval.

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