Closed coloring

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In an *arc-colored* tournament $D = (N, A)$, $|N| \geq 4$, the arcs in A are partitioned into *color classes* $\{\sigma_1, \ldots, \sigma_m\}$, and each class induces a directed bipartite subgraph (a directed graph with node set $A \cup B$ and arc set E such that $(u, v) \in E$ implies $u \in A$ and $v \in B$.)

Arc $(i, j) \in A$ is *closed* by $a, b \in A$ if
(i) Either $a = (i, k)$ and $b = (j, l)$ (in this case (i, j) is closed by the tails of a and b), or $a = (k, i)$ and $b = (l, j)$ (in which case (i, j) is closed by the heads of a and b), for some $k, l \in N \setminus \{i, j\}$.
(ii) a and b have the same color.

The arc-coloring of a subgraph $D' = (N, A')$ of D is *closed* if
(iii) Every arc of A' is closed by a pair of arcs in A'.
(iv) Every arc of A' is used to close other arcs exactly twice, once by its tail, and once by its head.

A subgraph whose arcs are colored by a closed coloring is also said to be closed.

Figure 1 shows some examples of subgraphs with closed colorings. The numbers indicate colors. We call the top-left graph *closed* C_4 and the top-middle graph a *closed* $K_{2,3}$. Note that a closed graph remains closed after reversing the directions of a color class. In particular we maintain the name $K_{2,3}$ closed subgraph after reversing the arcs of one of its color classes.

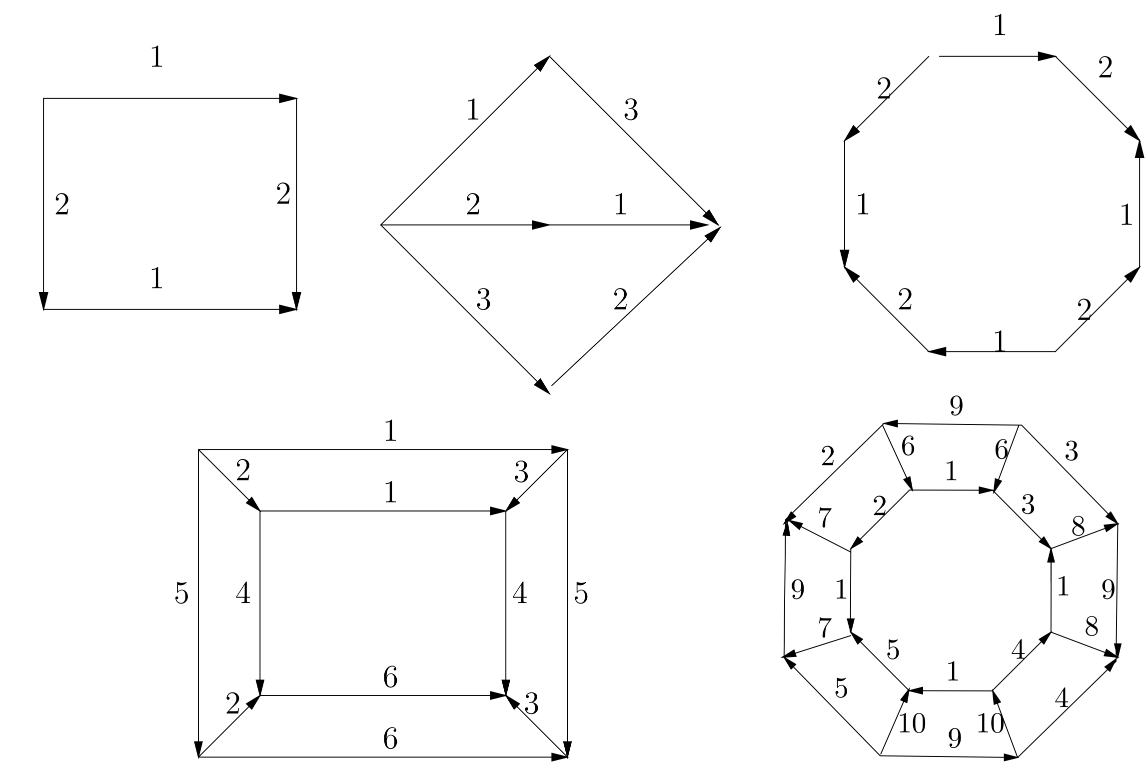

Figure 1: Closed colorings

Conjecture 1 [1] *If $m \leq |N| - 2$ then D contains a closed subgraph.*

Another way to state the conjecture is that the maximum order of an arc-colored tournament with m colors, which does not contain a closed subgraph, is $m + 1$.

The next two theorems confirm Conjecture 1 for $N = 4, 5$ (longer proofs can be found in [1]). The proofs assume that we have for each color class σ_i a directed cut (S_i, T_i) such that the σ_i-colored arcs are in (S_i, T_i). (This cut need not be unique.) Every arc belongs to at least one of these cuts (corresponding to its color).

Theorem 2 *A tournament with bicolored arcs on four nodes contains a closed C_4.*

Proof: In the case of four nodes, since every arc belongs to a cut, these cuts must intersect, for example $S_1 = \{1, 2\}$ and $S_2 = \{1, 4\}$. This means $(1, 3), (2, 4) \in \sigma_1$ and $(1, 2), (3, 4) \in \sigma_2$, thus inducing a closed C_4. ■

Theorem 3 *A tournament on five nodes colored with three colors contains a closed C_4 or a closed $K_{2,3}$.*

Proof: W.l.o.g assume $|S_i| < |T_i|$ $i = 1, 2, 3$ (if $|S_i| > |T_i|$ reverse the orientation of σ_i). If $|S_i| = 1$ then all σ_i-colored arcs leave the same node and by removing this node we obtain a 4-nodes 2-colored tournament that contains a closed subgraph by Theorem 2. Therefore, assume $|S_i| = 2$ $i = 1, 2, 3$ There are two cases to consider:

$S_1 = \{1, 2\}$, $S_2 = \{3, 4\}$, $S_3 = \{1, 3\}$. This means arcs $(1, 2), (3, 4) \in \sigma_3$ and $(4, 5) \in \sigma_2$ (as each of these arcs is covered by a single cut). By symmetry there is no loss of generality assuming $(3, 1) \in \sigma_2$ (the alternative is $(1.3) \in \sigma_1$). If $(4, 2) \in \sigma_2$ then with $(3, 1) \in \sigma_2$ and $(1, 2), (3, 4) \in \sigma_3$ we obtain a closed C_4. Assume therefore the alternative option $(2, 4) \in \sigma_1$. Similarly, $(1, 5) \in \sigma_3$ would create a closed C_4 with $(3, 4) \in \sigma_3$ and $(3, 1), (4, 5) \in \sigma_2$. Therefore assume $(1, 5) \in \sigma_1$. We now obtained a closed $K_{2,3}$ with terminal nodes 1 and 4.

The other case has $S_1 = \{1, 2\}$, $S_2 = \{1, 3\}$, $S_3 = \{1, 4\}$. Arcs incident to 5 are covered by a unique cut and therefore $(2, 5) \in \sigma_1$, $(3, 5) \in \sigma_2$, and $(4, 5) \in \sigma_3$. W.l.o.g $(1, 2) \in \sigma_2$ (the alternative is $(1, 2) \in \sigma_3$). To avoid a closed C_4 on 1,2,3,5 we must have $(1, 3) \in \sigma_3$, and now to avoid a closed C_4 on 1,3,4,5 we must have $(1, 4) \in \sigma_1$. We now have a closed $K_{2,3}$ with terminals 1 and 5. ■

A closed C_4 is equivalently a closed $K_{2,2}$ and therefore one could be led from Theorems 2 and 3 to conjecture that a tournament with N nodes and $C = N - 2$ colors contains a closed $K_{2,r}$ for some $2 \le r \le N - 2$. However it is possible to refute this possibility already for $N = 6$.

References

[1] N. Guttmann-Beck and R. Hassin , "On coloring the arcs of a tournament, covering shortest paths, and reducing the diameter of a graph ," *Discrete Optimization* **8** (2011) 302-314.